MODAL LOGICS OF SPACES

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CERTIFICATE

This is to certify that Shashank Pathak, I.PhD. student in Department of Mathematics, has completed bonafide work on the thesis entitled **'Modal Logics of Spaces'** under my supervision and guidance.

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Shashank Pathak

ABSTRACT

We describe two soundness and completeness results. The first one is the soundness and completeness of **S4** with respect to the class of all topological spaces. The second, a stronger one is the soundness and completeness of **S4** with respect to the class of all dense-in-itself separable metric spaces, called the McKinsey-Tarski Theorem (first given in [1]). The theorems give a syntactic characterization of the logic of these class of spaces.

For soundness and completeness with respect to the class of all spaces, we describe two different proofs [2]. The first one uses the soundness and completeness of **S4** with respect to the class of all reflexive transitive frames. The second one is called the *canonical topo-model* proof, and is constructive. Next, we describe a recent proof of the McKinsey-Tarski theorem which uses the fact that every formula which is not in **S4** can be falsified on the space of rational numbers with the usual topology. We conclude with how these two results together point out the limitations of the introduced topological interpretation.

LIST OF SYMBOLS OR ABBREVIATIONS

$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rationals
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive reals
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}^*$	The set $\mathbb{Z} - \{0\}$
$\mathbb{R}^+$	The set of positive reals
$\mathrm{Int}(A)$	Interior of a set A
$\mathrm{Cl}(A)$	Closure of a set A
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if

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1. INTRODUCTION

A long studied class of problems in mathematical logic is axiomatizing the logics of mathematical structures, and conversely, giving a semantic characterization of logics, which are defined syntactically. In this thesis, we tackle one such problem.

The birth of modal logic as a mathematical discipline is considered to have been in 1918 by C.I. Lewis [3]. The first treatments were syntactic rather than semantic, and in 1933, Gödel took $\Box$ as a primitive and formulated **S4** in the way that has become standard: he enriched a standard system for the classical propositional logic with the rule of generalization, the K axiom, and the additional axioms T and 4 [4].

Around 1960s, modal logic emerged as a new field when relational semantics were introduced [5], but one of the first interpretation of modal logic is topological, introduced some 20 years before the relational semantics. The topological interpretation was influenced by the work of Kuratowski, who in 1922, gave an axiomatization of topological spaces by means of the closure operator [6]. In 1944, J.C.C. McKinsey and Alfred Tarski gave the famous McKinsey-Tarski theorem which says that **S4** is the logic of the class of all dense-in-itself separable metric spaces [1].

In this project, first we will see two proofs of a result which states that **S4** is the logic of the class of all topological spaces. The first proof uses completeness of **S4** with respect to the class of all reflexive transitive frames. The second proof is more model-theoretic. Both of them can be found in [2]. Also, assuming the McKinsey-Tarski theorem, it can be proved that **S4** is sound and complete with respect to the class of all topological spaces. This is because completeness of a normal modal logic with respect to the class of dense-in-itself separable spaces implies the completeness of the logic with respect to the class of all topological spaces. Thus, the McKinsey-Tarski theorem is stronger. Finally, we will see a recent proof of the McKinsey-Tarski theorem given in [7]. The results together give us an axiomatization of the logics of the class of all topological spaces, and of the class of all dense-in-itself separable metric spaces.

2. PRELIMINARIES

2.1 Basic preliminaries

In this section, we discuss some definitions and results from [8], which will be needed later.

Definition 2.1.1. Let U be a set, $B \subseteq U$, and let $f : U \times U \to U$ and $g : U \to U$ be binary, and unary functions on U, respectively. A set $C \subseteq U$ is said to be *generated* from B by f and g, if C is the smallest set containing B, that is closed under f and g. $\dashv$

Example 2.1.2. If $B = \{a, b\}$, then C is the smallest set containing a, b, $f(a, b)$, $f(a, a)$, $g(b)$, $g(f(a, b))$, $f(g(a), g(b))$, and all such combinations. $\dashv$

Definition 2.1.3. Let U, B, C, f, and g be as in Definition 2.1.1. Then C is said to be *freely generated* from B by f and g, if C is generated from B by f and g, and the following conditions hold:

1. The restrictions f_C and g_C of f and g to C, respectively are injective.
2. The range of f_C, the range of g_C, and B are pairwise disjoint.

$\dashv$

The conditions 1 and 2 above can be thought of as implying that no element of C can be 'derived' from elements of B by applying f and g in two different ways. Let us assume we have such U, B, C, f, and g as in Definition 2.1.3. We want to see when can we define a function on C recursively. Precisely, suppose h is a function from B to a set V, and we want to extend it to the set C to form the function $\bar{h}$. Assume we have been given the following rules for $\bar{h}$:

1. rules for computing $\bar{h}(f(x,y))$, making use of $\bar{h}(x)$ and $\bar{h}(y)$, and

2. rules for computing $\bar{h}(g(x))$, making use of $\bar{h}(x)$.

Then, the recursion theorem, which will be stated next, guarantees the existence and uniqueness of such an extension, given C is freely generated from B by f and g.

Theorem 2.1.4 (Recursion theorem). *Assume that a subset C of U is freely generated from B by f and g, where*

$$\begin{aligned} f &: U \times U \to U, \\ g &: U \to U. \end{aligned}$$

Further, assume that V is a set, and F, G, and h are functions such that

$$\begin{aligned} h &: B \to V, \\ F &: V \times V \to V, \\ G &: V \to V. \end{aligned}$$

Then, there is a unique function $\bar{h} : C \to V$, such that

1. *for $x \in B$, we have $\bar{h}(x) = h(x)$, and*

2. *for $x, y \in C$, we have*

$$\begin{aligned} \bar{h}(f(x,y)) &= F(\bar{h}(x), \bar{h}(y), \text{ and} \\ \bar{h}(g(x)) &= G(\bar{h}(x)) \end{aligned}$$

Proof of Theorem 2.1.4 can be found in [8, Chapter 1].

Definition 2.1.5. Let X be set and R be a binary relation on X. R is called a *partial order* on X if:

1. (reflexivity) for each $x \in X$, we have xRx,
2. (anti-symmetry) for each $x, y \in X$, if xRy and yRx, then $x = y$, and
3. (transitivity) for each $x, y, z \in X$, if xRy and yRz, then xRz.

The pair (X, R) is called a *partially ordered set* or a *poset*. $\dashv$

Example 2.1.6. For a set S, let $\mathcal{P}(S)$ denote its power set. Then $(\mathcal{P}(S), \subseteq)$ forms a poset. $\dashv$

Definition 2.1.7. Let X and X' be two sets. Let R and R' be n-ary relations on X and X', respectively. Let f be a function from X to X'. The function f is said to *preserve the relational structure* if for each $x_1, \ldots, x_n \in X$, we have $(f(x_1), \ldots, f(x_n)) \in R'$, whenever $(x_1, \ldots, x_n) \in R$. If the relations R and R' are partial orders and if f preserves the relational structure, then f is said to be an *order-preserving* function. $\dashv$

Example 2.1.8. For the posets $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$, the function $f(x) : \mathbb{Z} \to \mathbb{R}$ given by $f(x) = e^x$ is an order-preserving function, as for integers z_1 and z_2, we have $e^{z_1} \leq e^{z_2}$, whenever $z_1 \leq z_2$. $\dashv$

Later, we will define relations which are called *linear orders*. Functions preserving linear orders will also be called order-preserving.

Definition 2.1.9. Let X and X' be sets, and R and R' be binary relations on X and X', respectively. Let $f : X \to X'$ be a function which preserves the relational structure. Then f is said to be a *p-morphism*, if for each $a \in X$ and $b' \in X'$, there exists $b \in X$ such that $f(b) = b'$ and aRb, whenever $f(a)R'b'$. $\dashv$

Example 2.1.10. Consider the set $\mathbb{Z}$ with the relation

$$\{(x, x+1) \mid x \in \mathbb{Z}\},$$

and the set $\{o, e\}$ with the relation

$$R = \{(o, e), (e, o)\}.$$

Consider the function $f : \mathbb{Z} \to \{o, e\}$ which maps odd integers to o and even ones to e, as shown in Figure 2.1.

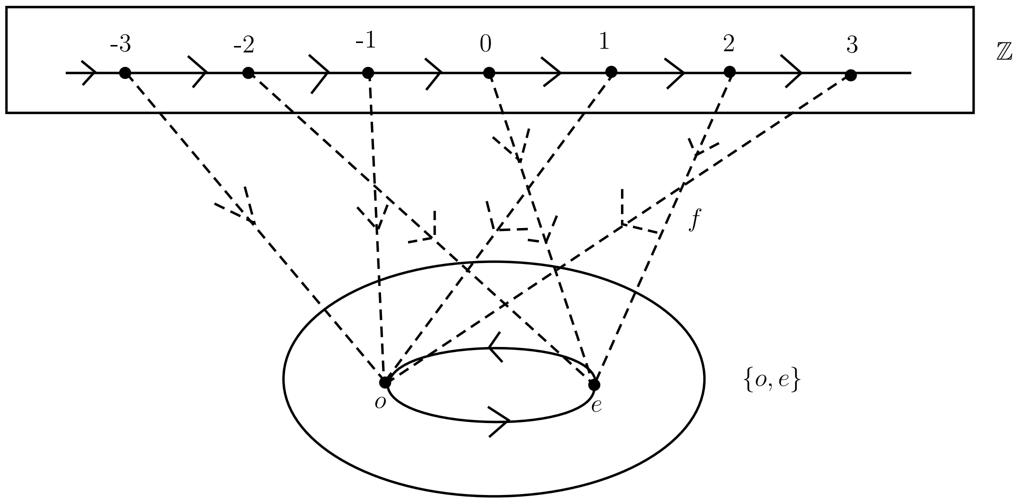

Fig. 2.1: f forms a p-morphism.

Then, as the successor of an odd number is even, and vice-versa, it follows that f preserves the relational structure. Also, for $a \in \mathbb{Z}$, if $f(a)Ro$ holds, then a is even. Consequently, $a + 1$ is odd, and so $f(a + 1) = o$. Thus, there exists $a + 1 \in \mathbb{Z}$ such that $f(a + 1) = o$, and a is related to $a + 1$. Reasoning in a similar way for the case $f(a)Re$, we get that f is a p-morphism. $\dashv$

Definition 2.1.11. A *partial function* f from a set A to a set B is a function from a set C to B such that $C \subseteq B$. $\dashv$

Example 2.1.12. The function which maps a non-negative real to its positive square root is a partial function from $\mathbb{R}$ to $\mathbb{R}$. $\dashv$

2.2 Classical Propositional Logic preliminaries

Just like we use natural languages in our everyday life to communicate, we can use formal languages to describe formal theories. The advantage of using a formal language is that the languages can be handled by computers, and the processes that we use can be mechanized. In this section, we describe a formal language, called the *language of classical propositional logic* (or *propositional logic*). We can also call it the *classical propositional language* or the *sentential language*. The language is named so - because the smallest independent entities in the language correspond to sentences (or propositions). This language roughly models the sentences which are related to each other by connectives (and, or, so, only if, neither-nor, etc.)

A natural language (for example, English) has basic building blocks as the letters in its alphabet. The letters are put together to form words, which in turn are put together according to some rules to produce meaningful sentences. We have a similar construction for formal languages. All of this can be found in detail in [8, Chapter 1].

2.2.1 The Syntax of the Classical Propositional Language

The syntax of a language means the arrangement of words and phrases to create well-formed sentences in a language. First, we define the symbols we will be using to construct the classical propositional language. The symbols that we will be using can be grouped into the following subcollections:

- countably many propositional variables $\{p, q, r, \ldots\}$,
- the logical constant $\perp$,
- logical connectives $\{\neg, \wedge\}$, and
- parentheses $\{(,)\}$.

Thus, our whole set of symbols becomes

$$\{p, q, r, \ldots, \bot, \neg, \wedge, (,)\}.$$

We further assume that none of these symbols is a finite sequence of other symbols. The significance of the names of the sub-collections will be clear later. This collection of symbols forms the equivalent of the letters of the 'alphabet' for the classical propositional language.

Now, we give a process to form the 'sentence' equivalents of the formal language. We call the 'sentence' equivalents as well-formed formulas or simply, formulas. Just as, in a natural language not every finite sequence of words is a sentence, so it is natural to expect that the formulas will also have some restrictions. We will now formalize these notions.

Definition 2.2.1. *Formulas* are finite sequences of symbols with the following properties:

1. Each propositional variable is a formula.
2. $\bot$ is a formula.
3. If α and β are formulas, then so are $\neg\alpha$ and $(\alpha \wedge \beta)$.
4. Nothing else is a formula.

$\dashv$

The above definition is sometimes written in the following compact form.

$$\boxed{\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \wedge \psi).}$$

Let α and β be formulas. We define the following abbreviations:

1. $(\alpha \vee \beta)$ is an abbreviation of $\neg(\neg\alpha \wedge \neg\beta)$,
2. $(\alpha \rightarrow \beta)$ is an abbreviation of $(\neg\alpha \vee \beta)$,
3. $(\alpha \leftrightarrow \beta)$ is an abbreviation of $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$, and

4. $\top$ is an abbreviation of $\neg\bot$.

Note that, using the symbols $\bot, \vee, \rightarrow, \leftrightarrow$ for abbreviating does not contradict the fact that our set of symbols for the language is just $\{p, q, r, \ldots, \bot, \neg, \wedge, (,)\}$. In fact, the abbreviations should be thought of as 'shorthand' notations, and the original formulas should be kept in mind while working with the language.

Whenever ambiguity can be avoided, we skip writing "(" and ")". For example, sometimes we write $p \wedge q$ instead of $(p \wedge q)$, and $p \rightarrow q$ instead of $(p \rightarrow q)$.

Example 2.2.2. Some formulas are $p, \neg r, (\neg q \vee \bot), \neg\neg(p \leftrightarrow s)$, and $\neg(\neg p \wedge \neg r)$. Some finite sequences which are not formulas are $\neg$, $($, $p((\wedge$, $\wedge qrs(\neg$, and $\neg p \neg \top$. ⊣

Remark 2.2.3. We now make the fourth property from Definition 2.2.1 precise. Let $\mathcal{S}$ denote the set of all finite sequences on our set of symbols. We can define two operators $\mathcal{E}_\neg$ and $\mathcal{E}_\wedge$ on $\mathcal{S}$, as follows:

$$\mathcal{E}_\neg(\alpha) = \neg\alpha, \text{ and}$$

$$\mathcal{E}_\wedge(\alpha, \beta) = (\alpha \wedge \beta).$$

Then, formulas are exactly the finite sequences which can be built up from propositional variables, and $\bot$ by repeatedly applying $\mathcal{E}_\neg$ and $\mathcal{E}_\wedge$. ⊣

Using Remark 2.2.3, we describe a result which would help us in proving many results for formal languages.

Theorem 2.2.4 (Induction principle). *If A is a set of formulas which contains all propositional variables and $\bot$, and is closed under $\mathcal{E}_\neg$ and $\mathcal{E}_\wedge$, then A is the set of all formulas.*

Two proofs of Theorem 2.2.4 are available in [8].

Example 2.2.5. Using the induction principle, we can prove that each formula has an equal number of left and right parentheses. The idea is as follows. As

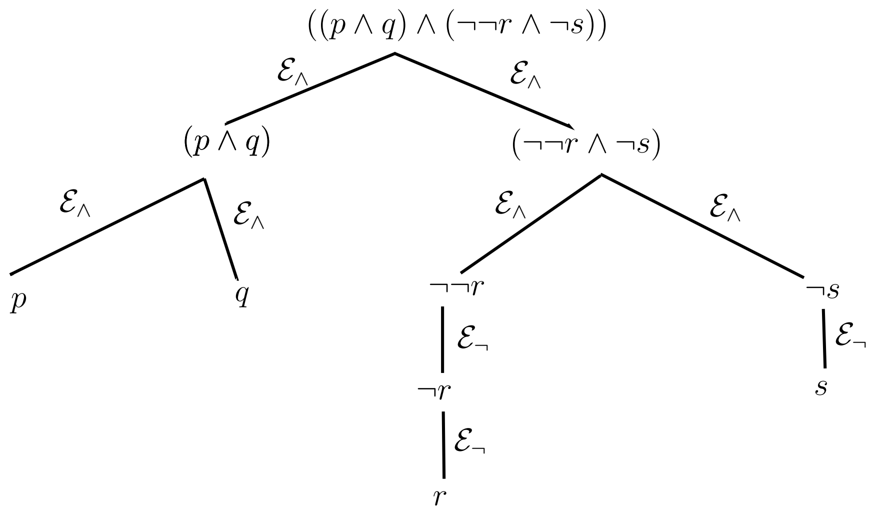

Fig. 2.2: Construction tree of $((p \wedge q) \wedge (\neg\neg r \wedge \neg s))$.

- propositional variables and $\perp$ have no parentheses, so they have an equal number of left and right parentheses,
- $\mathcal{E}_\neg$ doesn't introduce new left or right parentheses, and
- $\mathcal{E}_\wedge$ introduces one new left and one new right parentheses,

so the set of all formulas with equal number of left and right parentheses contains all propositional variables and $\perp$, and is closed under $\mathcal{E}_\neg$ and $\mathcal{E}_\wedge$. $\dashv$

Till now, we have defined what our 'sentences' in the formal language look like. Next, we associate meanings to the different symbols, and as a result, to the formulas that we are using.

2.2.2 The Semantics of the Classical Propositional Language

We fix, once and for all, a two-point set $\{F, T\}$, where

F is called *falsity*, and

T is called *truth*.

Let Φ denote the set of propositional variables, that is,

$$\Phi = \{p, q, r, \ldots\}.$$

Definition 2.2.6. Any function from Φ to $\{F, T\}$ is called a *truth assignment.* $\dashv$

Let v be a truth assignment. We want to extend v from Φ to the set of all the formulas in such a way, that the symbols $\neg$ and $\wedge$ mimic the action of *negation* and *joining by and*, respectively. Specifically, calling the extension as $\overline{v}$, we want $\overline{v}$ to have the following properties:

1. For any $z \in \Phi$, $\overline{v}(z) = v(z)$. (Thus, $\overline{v}$ is an extension of v.)

2. $\overline{v}(\perp) = F$.

3. $\overline{v}(\neg\alpha) = \begin{cases} T, & \text{if } \overline{v}(\alpha) = F, \text{ and} \\ F, & \text{otherwise.} \end{cases}$

4. $\overline{v}(\alpha \wedge \beta) = \begin{cases} T, & \text{if } \overline{v}(\alpha) = T \text{ and } \overline{v}(\beta) = T, \text{ and} \\ F, & \text{otherwise.} \end{cases}$

The existence of such a unique extension is guaranteed by the *Recursion theorem* (2.1.4). This happens because the set of formulas is freely generated from the set of propositional variables and $\perp$. As the extension is unique, sometimes v is written in place of $\bar{v}$ for ease of expression.

Remark 2.2.7. We now see how does $\overline{v}(\phi \vee \psi)$ depend upon $\overline{v}(\phi)$ and $\overline{v}(\psi)$.

We have

$$\begin{aligned}\overline{v}(\phi \vee \psi) &= \overline{v}(\neg(\neg\phi \wedge \neg\psi)) \\ &= \begin{cases} T, & \text{if } \overline{v}(\neg\phi \wedge \neg\psi) = F, \text{ and} \\ F, & \text{otherwise.} \end{cases} \\ &= \begin{cases} T, & \text{if } \overline{v}(\neg\phi) = F \text{ or } \overline{v}(\neg\psi) = F, \text{ and} \\ F, & \text{otherwise.} \end{cases} \\ &= \begin{cases} T, & \text{if } \overline{v}(\phi) = T \text{ or } \overline{v}(\psi) = T, \text{ and} \\ F, & \text{otherwise.} \end{cases}\end{aligned}$$

Similarly, we get the following:

$$\begin{aligned}\overline{v}(\phi \to \psi) &= \begin{cases} F, & \text{if } \overline{v}(\phi) = T \text{ and } \overline{v}(\psi) = F, \text{and} \\ T, & \text{otherwise.} \end{cases} \\ \text{and } \ \overline{v}(\phi \leftrightarrow \psi) &= \begin{cases} T, & \text{if } \overline{v}(\phi) = \overline{v}(\psi), \text{and} \\ F, & \text{otherwise.} \end{cases}\end{aligned}$$

$\dashv$

The next example is an application of Remark 2.2.7.

Example 2.2.8. Let $v : \Phi \to \{F, T\}$ be such that $v(p) = T$, and $v(q) = F$. Then $v(p \to q) = F$, and $v(\neg p) = F$. $\dashv$

It can be shown that for a formula φ, $v(\varphi)$ depends only upon the value of the truth assignment of the propositional variables appearing in the formula. The next example captures this fact.

Example 2.2.9. For two truth assignments v_1 and v_2, if $v_1(p) = v_2(p) = T$, $v_1(q) = v_2(q) = F$, $v_1(r) = T$, and $v_2(r) = F$, then $v_1(p \to q) = v_2(p \to q) = F$. $\dashv$

Definition 2.2.10. For a formula ϕ, we say a truth assignment v *satisfies* ϕ if $v(\phi) = T$. $\dashv$

Example 2.2.11. Both the truth assignments in Table 2.1 satisfy $\neg(p \vee q) \to \neg r$.

	p	q	r	$p \vee q$	$\neg r$	$\neg(p \vee q)$	$\neg(p \vee q) \to \neg r$
v_1	T	F	T	T	F	F	T
v_2	F	F	F	F	T	T	T

Tab. 2.1: A truth table for $\neg(p \vee q) \to \neg r$ for two valuations.

$\dashv$

Definition 2.2.12 (Tautologies). A formula φ is said to be a *tautology* if for every truth assignment v, we have $v(\phi) = T$. $\dashv$

Thus, if a formula is a tautology, then every truth assignment satisfies it.

Example 2.2.13. The formula $p \vee \neg p$ is a tautology. This can be seen in Table 2.2. As for any truth assignment, there are only two possibilities for the value of p, and as p is the only propositional variable occurring in $p \vee \neg p$, so the two truth assignments shown in the Table 2.2 exhaust every possible case for the formula $p \vee \neg p$.

	p	$\neg p$	$p \vee \neg p$
v_1	T	F	T
v_2	F	T	T

Tab. 2.2: The truth table for $p \vee \neg p$.

$\dashv$

Sometimes, we will refer to the tautologies as propositional tautologies. They will play an important role later when we define *normal modal logics*.

2.3 Modal logic preliminaries

Having known propositional language, we enrich our formal language by adding more symbols to it. The formal language that we will be discussing in this section is called the *basic modal language*. The language is an extension of the propositional language that we have seen. It comes from the family called the *general modal language*. The general modal language helps us to formalize concepts like necessity - possibility, knowledge - belief, obligation - permission - prohibition, and time [9], among others.

2.3.1 Syntax

The set of symbols that we will be using can be divided into four types:

- propositional variables: $p, q, r, \ldots,$
- logical symbols: $\bot, \wedge, \neg,$
- modal operator: $\Diamond$, and
- parentheses: $(,)$.

Thus, the set of symbols is

$$\mathcal{S} = \{p, q, r, \ldots, \bot, \wedge, \neg, \Diamond, (,)\}.$$

The formulas in the basic modal language are finite sequences of symbols with the following properties:

1. Each propositional variable is a formula.
2. $\bot$ is a formula.
3. If α and β are formulas, then so are $\neg\alpha$ and $(\alpha \wedge \beta)$.
4. If α is a formula, then so is $\Diamond\alpha$.
5. Nothing else is a formula.

Thus,

$$\boxed{\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \wedge \psi) \mid \Diamond\phi.}$$

As in the case of the propositional language, here too we have some natural abbreviations. Let α and β be finite sequences of symbols. We have the following abbreviations:

1. $(\alpha \vee \beta)$ is an abbreviation of $\neg(\neg\alpha \wedge \neg\beta)$,
2. $(\alpha \to \beta)$ is an abbreviation of $(\neg\alpha \vee \beta)$,
3. $(\alpha \leftrightarrow \beta)$ is an abbreviation of $((\alpha \to \beta) \wedge (\beta \to \alpha))$,
4. $\Box\alpha$ is an abbreviation of $\neg\Diamond\neg\alpha$, and
5. $\top$ is an abbreviation of $\neg\bot$.

Example 2.3.1. Some formulas are $p, \neg r, (\Diamond q \vee \bot), \neg\neg(p \leftrightarrow s)$, and $\neg(\neg p \wedge \Box r)$. Some finite sequences which are not formulas are $\neg$, (, $p\Diamond(\wedge$, $\wedge q\Box s(\neg$, and $\neg p\neg\top$. ⊣

Again, whenever ambiguity is avoidable, we skip writing "(" and ")". For example, sometimes we write $p \wedge q$ instead of $(p \wedge q)$ and $p \to \Box q$ instead of $(p \to \Box q)$. Moreover, as in the case of propositional language, here too the last property can be made precise by defining the operators $\mathcal{E}_\neg, \mathcal{E}_\wedge$, and $\mathcal{E}_\Diamond$, and for every formula, we get a corresponding construction tree.

The induction principle for the basic modal language looks like the following.

Theorem 2.3.2 (Induction principle). *If A is a set of formulas which contains all propositional variables and $\bot$, and is closed under $\mathcal{E}_\neg$, $\mathcal{E}_\wedge$, and $\mathcal{E}_\Diamond$, then A is the set of all modal formulas.*

Before going to the semantics, we formalize the notion of uniformly substituting a formula for a propositional variable. Let Φ denote the set of propositional variables.

Definition 2.3.3. Any map from Φ to the set of formulas is called a *substitution.* ⊣

For a substitution σ, we want to extend σ to the set of all formulas such that it mimics the action of uniformly substituting formulas for propositional variables. Let us call the extension as $(\cdot)^\sigma$. We want $(\cdot)^\sigma$ to satisfy the following properties:

$$\begin{aligned}
\bot^\sigma &= \bot, \\
p^\sigma &= \sigma(p), \\
(\neg\phi)^\sigma &= \neg\phi^\sigma, \\
(\phi \wedge \psi)^\sigma &= (\phi^\sigma \wedge \psi^\sigma), \text{ and} \\
(\Diamond\phi)^\sigma &= \Diamond\phi^\sigma.
\end{aligned}$$

For a given substitution σ, there exists a unique extension $(\cdot)^\sigma$ satisfying the above properties by the *Recursion theorem* (2.1.4). This happens because the set of formulas is freely generated from the set of propositional variables and $\bot$, by using the operators $\mathcal{E}_\neg, \mathcal{E}_\wedge$, and $\mathcal{E}_\Diamond$.

Definition 2.3.4. A formula ϕ is said to be a *substitution instance* of a formula ψ if there is a substitution σ such that $\psi^\sigma = \phi$. $\dashv$

Example 2.3.5. If σ is a substitution that maps p to $\Box p \vee q$, q to $\neg r \to \Box q$, and leaves all the other propositional variables untouched, then we have $((p \wedge q) \vee s)^\sigma = ((\Box p \vee q) \wedge (\neg r \to \Box q)) \vee s$. $\dashv$

2.3.2 Semantics

The basic modal language is interpreted over frames, which are sets accompanied with relations.

Definition 2.3.6. A *frame* for the basic modal language is a pair $\mathfrak{F} = (W, R)$, where

1. W is a non-empty set, and

2. R is a binary relation on W.

Elements of W are also called the *states* of W. $\dashv$

Example 2.3.7. $(\mathbb{N}, \leq)$, $(\{x\}, \{(x, x)\})$, and $(\{x\}, \emptyset)$ are all examples of frames. $\dashv$

Let Φ denote the set of propositional variables, that is, $\Phi = \{p, q, r, \ldots\}$.

Definition 2.3.8. A *model* $\mathfrak{M}$ is a tuple $(\mathfrak{F}, V)$, where

1. $\mathfrak{F} = (W, R)$ is a frame, and
2. V is a function from Φ to the powerset of W (denoted by $\mathcal{P}(W)$).

For a model $\mathfrak{M} = (\mathfrak{F}, V)$, $\mathfrak{F}$ is called the *underlying frame*, and V is said to be a *valuation* on $\mathfrak{F}$. ⊣

For a propositional variable p, by definition we have $V(p) \subseteq W$. $V(p)$ should be seen as points in W where p is 'true'.

Example 2.3.9. Consider the frame $\mathfrak{F} = (W, R)$, where

$$W = \{1, 2, 3, 4\}, \text{ and } R = \{(1,2), (2,3), (3,4), (4,2)\}.$$

Figure 2.3 shows two models based on the same frame.

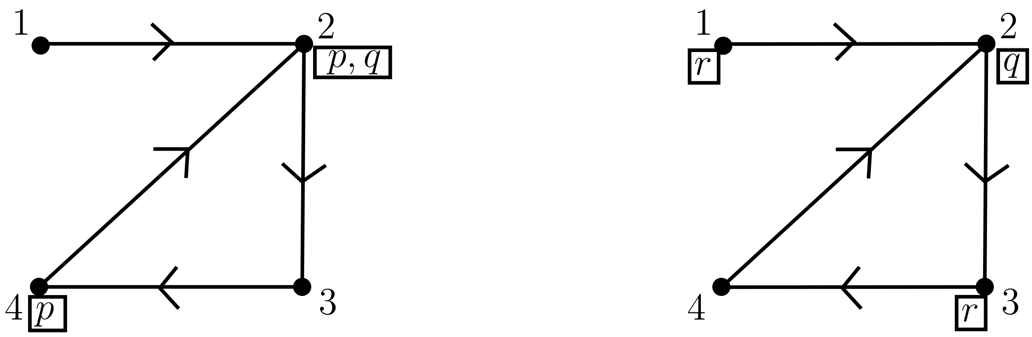

Fig. 2.3: Two models based on the same frame $\mathfrak{F}$.

⊣

We recall that the set of propositional variables $(p, q, r, \ldots)$ is denoted by Φ. Next, we define what we mean by a formula being 'true' at a state.

Definition 2.3.10. Let w be a state in a model $\mathfrak{M} = (W, R, V)$. Then we inductively define the notion of a formula ϕ being *satisfied* (or *true*) in $\mathfrak{M}$ at a state w as follows:

1. $\mathfrak{M}, w \vDash p$ iff $w \in V(p)$, where $p \in \Phi$,

2. $\mathfrak{M}, w \vDash \bot$ never,

3. $\mathfrak{M}, w \vDash \neg\phi$ iff it's not the case that $\mathfrak{M}, w \vDash \phi$ (denoted by $\mathfrak{M}, w \nvDash \phi$),

4. $\mathfrak{M}, w \vDash (\phi \wedge \psi)$ iff both $\mathfrak{M}, w \vDash \phi$ and $\mathfrak{M}, w \vDash \psi$, and

5. $\mathfrak{M}, w \vDash \Diamond\phi$ iff there exists a $v \in W$ such that Rwv and $\mathfrak{M}, v \vDash \phi$.

If a formula ϕ is true at all the states in a model $\mathfrak{M}$, then we say that ϕ is true in $\mathfrak{M}$ (notation: $\mathfrak{M} \vDash \phi$). For a set of formulas Σ, if for each $\phi \in \Sigma$, we have $\mathfrak{M} \vDash \phi$, then we say that Σ is true on $\mathfrak{M}$ (notation: $\mathfrak{M} \vDash \Sigma$). $\dashv$

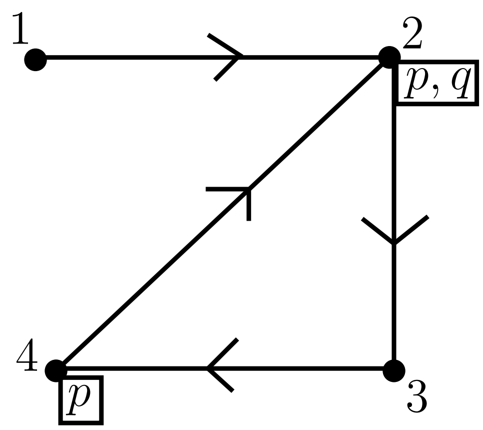

Fig. 2.4: The model $\mathfrak{M}$.

Example 2.3.11. In Figure 2.4, we have:

1. $\mathfrak{M}, 4 \vDash p$,

2. $\mathfrak{M}, 1 \nvDash \bot, \ldots, \mathfrak{M}, 4 \nvDash \bot$,

3. $\mathfrak{M}, 1 \vDash \neg r, \mathfrak{M}, 4 \vDash \neg q$,

4. $\mathfrak{M}, 2 \vDash (p \wedge q)$, $\mathfrak{M}, 4 \vDash (\neg q \wedge p)$,

5. $\mathfrak{M}, 1 \vDash \Diamond q$, $\mathfrak{M}, 3 \vDash \Diamond p$, and $\mathfrak{M}, 2 \vDash \Diamond\neg r$.

$\dashv$

Next, we see how the abbreviations get interpreted.

Remark 2.3.12. For $\vee$ we have,

$$\begin{aligned}
\mathfrak{M}, w \vDash (\phi \vee \psi) &\Leftrightarrow \mathfrak{M}, w \vDash \neg(\neg\phi \wedge \neg\psi) \\
&\Leftrightarrow \text{it's not that } \mathfrak{M}, w \vDash (\neg\phi \wedge \neg\psi) \\
&\Leftrightarrow \text{it's not that both } \mathfrak{M}, w \vDash \neg\phi \text{ and } \mathfrak{M}, w \vDash \neg\psi \\
&\Leftrightarrow \text{at least one of } \mathfrak{M}, w \vDash \neg\phi \text{ or } \mathfrak{M}, w \vDash \neg\psi \text{ doesn't hold} \\
&\Leftrightarrow \mathfrak{M}, w \vDash \phi \text{ or } \mathfrak{M}, w \vDash \psi.
\end{aligned}$$

Similarly, we get the following:

1. $\mathfrak{M}, w \vDash (\phi \rightarrow \psi)$ iff $\mathfrak{M}, w \vDash \psi$, whenever $\mathfrak{M}, w \vDash \phi$.
2. $\mathfrak{M}, w \vDash (\phi \leftrightarrow \psi)$ iff both $\mathfrak{M}, w \vDash \phi$ and $\mathfrak{M}, w \vDash \psi$, or $\mathfrak{M}, w \nvDash \phi$ and $\mathfrak{M}, w \nvDash \psi$ hold.
3. $\mathfrak{M}, w \vDash \top$ always.

For $\Box\phi$ we have:

$$\begin{aligned}
&\mathfrak{M}, w \vDash \Box\phi \\
\Leftrightarrow\ &\mathfrak{M}, w \vDash \neg\Diamond\neg\phi \\
\Leftrightarrow\ &\mathfrak{M}, w \nvDash \Diamond\neg\phi \\
\Leftrightarrow\ &\text{it's not the case that there exists a } v \in W, \text{ such that } Rwv \text{ and} \\
&\mathfrak{M}, v \vDash \neg\phi \\
\Leftrightarrow\ &\text{it's not the case that there exists a } v \in W, \text{ such that } Rwv \text{ and} \\
&\mathfrak{M}, v \nvDash \phi \\
\Leftrightarrow\ &\text{for each } v \in W, \text{ if } Rwv \text{ holds, then } \mathfrak{M}, v \vDash \phi.
\end{aligned}$$

Thus, $\mathfrak{M}, w \vDash \Diamond\phi$ means that ϕ is true at *at least one* 'R-neighbor' of w, whereas $\mathfrak{M}, w \vDash \Box\phi$ means ϕ is true at *all* 'R-neighbors' of w. $\dashv$

Example 2.3.13. Consider the model $\mathfrak{M}$, as shown in Figure 2.5. It should be noted that here the underlying frame has infinite branching, and an infinitely long branch. Here we have:

1. $\mathfrak{M}, 0 \not\models z$, for any propositional variable z,
2. $\mathfrak{M}, 1 \not\models \Diamond\top$,
3. $\mathfrak{M}, 6 \models (p \to s)$, $\mathfrak{M}, 2 \models (r \to s)$,
4. $\mathfrak{M}, -2 \models (p \leftrightarrow s)$,
5. $\mathfrak{M}, 0 \models \Box p$, $\mathfrak{M}, 0 \models \Diamond\Box r$, and $\mathfrak{M}, 1 \models \Box\bot$.

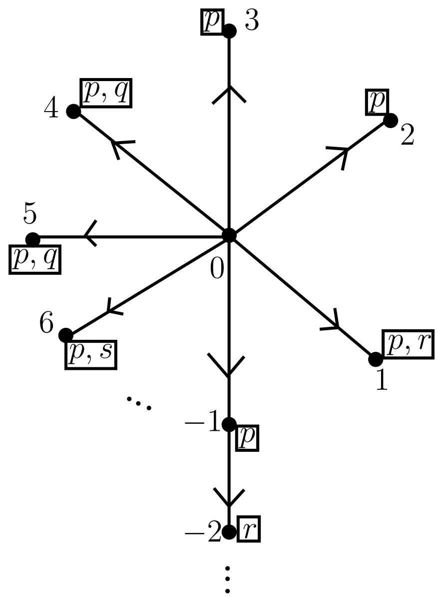

Fig. 2.5: A model with infinitely many branches.

$\dashv$

Definition 2.3.14. A formula ϕ is *valid* on a frame $\mathfrak{F} = (W, R)$ (notation $\mathfrak{F} \models \phi$) if for all models $\mathfrak{M}$ based on $\mathfrak{F}$ and any state $w \in W$, we have $\mathfrak{M}, w \models \phi$. For a class of frames F, we say that ϕ is valid on F (notation: $\mathsf{F} \models \phi$), if ϕ is valid on each frame contained in F. For a class of frames F, we define

$$\Lambda_{\mathsf{F}} = \{\phi \text{ is a formula} \mid \mathsf{F} \models \phi\}.$$

$\dashv$

Example 2.3.15. Propositional tautologies are valid on all frames. $\dashv$

Example 2.3.16. The formula $p \to \Diamond p$ is valid on the class of all reflexive frames. To see this, let $\mathfrak{F} = (X, R)$ be a reflexive frame, and let $\mathfrak{M} = (\mathfrak{F}, V)$ be a model based on the frame $\mathfrak{F}$. Let $x \in X$, such that $M, x \vDash p$. As R is a reflexive relation by assumption, we have xRx. Thus, $M, x \vDash \Diamond p$, and hence, $M, x \vDash p \to \Diamond p$. As $x, V, \mathfrak{F}$ are all arbitrary, we get that $p \to \Diamond p$ is valid on the class of all reflexive frames. $\dashv$

Definition 2.3.17. For a frame $\mathfrak{F} = (W, R)$, a modal formula ϕ is said to be *satisfiable* on $\mathfrak{F}$, if there exists a $w \in W$, and a valuation V on W, such that $(W, R, V), w \vDash \phi$. A set of formulas Σ is said to be *satisfiable* on a frame $\mathfrak{F}$ if there exists a $w \in W$, and a valuation V on W, such that $(W, R, V), w \vDash \Sigma$. For a model $\mathfrak{M}$, a formula ϕ is said to be satisfiable on $\mathfrak{M}$, if there exists a state w in $\mathfrak{M}$, such that $\mathfrak{M}, w \vDash \phi$. $\dashv$

Example 2.3.18. Consider the frame $\mathfrak{F} = (\{0\}, \{(0, 0)\})$. The formula $\Box p \to p$ is satisfiable on $\mathfrak{F}$, as there is a valuation $V(p) = \{0\}$, and the state 0 such that $(\mathfrak{F}, V), 0 \vDash \Box p \to p$. But, the formula $p \wedge \Diamond\neg p$ is not satisfiable on $\mathfrak{F}$. $\dashv$

2.3.3 Generated Submodels

Definition 2.3.19. For a model $\mathfrak{M} = (X, R, V)$, a model $\mathfrak{M}' = (X', R', V')$ is said to be a *submodel* of $\mathfrak{M}$ if:

1. $X' \subseteq X$,
2. R restricted to X' is R', (i.e. $R' = R \cap (X' \times X')$) and
3. V restricted to X' is V' (i.e. $V'(p) = V(p) \cap X'$ for each propositional variable p.)

$\dashv$

Example 2.3.20. In Figure 2.6, the picture on the right depicts a submodel of the model shown in the picture on the left. $\dashv$

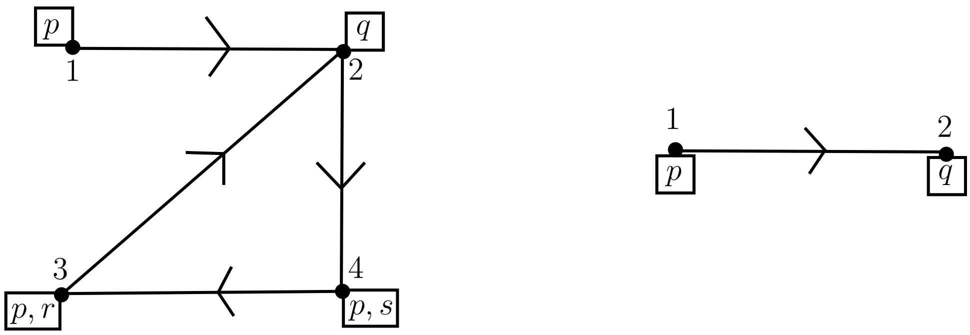

Fig. 2.6: A model and one of its submodels.

Definition 2.3.21. A submodel $\mathfrak{M}'$ of a model $\mathfrak{M}$ is said to be a *generated submodel*, if it is closed under R, that is, if $w \in X'$ and Rwv, then $v \in X'$. ⊣

Generated submodels are studied for interesting invariance properties.

Example 2.3.22. In Figure 2.7, the model shown on the right is a generated submodel of the model shown on the left. ⊣

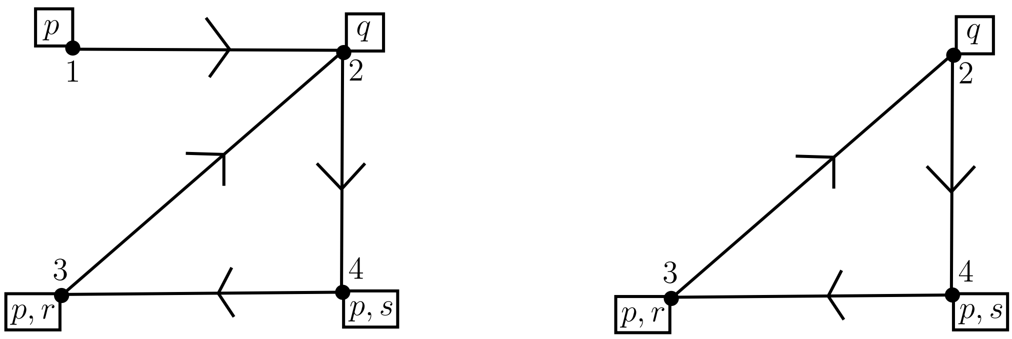

Fig. 2.7: A model and one of its generated submodels.

As the truth of a formula at a point depends upon the truth of the constituent formulas at the point and its R-neighbors, so it is natural to expect that the set of formulas which are true at any point in the generated submodel is the same as the set of formulas true at the point in the original model.

Proposition 2.3.23. *Let* $\mathfrak{M} = (X, R, V)$ *be a model and let* $\mathfrak{M}' = (X', R', V')$ *be one of its generated submodels. For* $w \in X'$*, and any formula* ϕ*, we have,* $\mathfrak{M}, w \vDash \phi$ *iff* $\mathfrak{M}', w \vDash \phi$*. Thus, modal satisfaction is invariant under generated submodels.*

The proof of Proposition 2.3.23 uses induction on the set of all formulas (see [9].) The idea behind the proof is as follows. For any propositional variable p, we have, $\mathfrak{M}, w \vDash p$ iff $\mathfrak{M}', w \vDash p$. This happens because V' is just V restricted to X'. So, the argument concerning propositional variable goes through. The argument for $\neg$ and $\wedge$ follows from the induction hypothesis and the fact that the truth of formulas $\neg\phi$ and $\phi \wedge \psi$ at a point w depends on the truth of ϕ and ψ only at the point w. For $\Diamond$, $\Diamond\phi$ is true at w iff ϕ is true at some R-neighbor of w. The definition of a generated submodel guarantees that all R-neighbors of w are present in the generated submodel, and so get that $\Diamond\phi$ is true at w in the generated submodel too.

Some other properties of models are also preserved when a generated submodel is constructed. Reflexivity is preserved because for each point $w \in X'$, we have $(w, w) \in R$ and $R' = R \cap (X' \times X')$. Thus, $(w, w) \in R'$. For transitivity, if $w, y, z \in X'$ such that $R'wy$ and $R'yz$, then as R' is just a restriction of R, we have Rwy and Ryz. As the original model is transitive, we get Rwz, and thus, we get $R'wz$. Hence, transitivity is preserved too and we get the following result.

Proposition 2.3.24. *Reflexivity and transitivity is preserved under generated submodel construction.*

2.3.4 Filtrations

In this section, we develop tools which will help us in constructing smaller models from bigger models such that the truth of certain formulas remains preserved. The results explained in this section can be found in [9].

Definition 2.3.25. A set of formulas Σ is said to be *closed under subformulas* (or: *subformula closed*) if for all formulas ϕ, ϕ': if $\phi \wedge \phi' \in \Sigma$, then so are ϕ and ϕ'; if $\neg\phi \in \Sigma$, then so is ϕ; and if $\Diamond\phi \in \Sigma$, then so is ϕ. $\dashv$

Example 2.3.26. The set $\{\neg p \vee q, \neg\neg p \wedge \neg q, \neg\neg p, \neg q, \neg p, q, p\}$ is a subformula closed set. $\dashv$

Definition 2.3.27. Let $\mathfrak{M} = (W, R, V)$ be a model, and Σ be a subformula closed set of formulas. Then the relation $\leftrightsquigarrow_\Sigma$ on the states of $\mathfrak{M}$ is defined by:

$$w \leftrightsquigarrow_\Sigma v \text{ iff for all } \phi \in \Sigma,\ \ \mathfrak{M}, w \vDash \phi \text{ iff } \mathfrak{M}, v \vDash \phi \text{ holds.}$$

It should be noted that $\leftrightsquigarrow_\Sigma$ forms an equivalence relation. For $w \in W$, let the equivalence class of w be denoted by $|w|_\Sigma$ (or $|w|$, if there is no ambiguity). The map $w \to |w|$ is said to be the *natural map*.

Let $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$. Suppose $\mathfrak{M}^f_\Sigma$ is any model (W^f, R^f, V^f) such that the following hold.

1. $W^f = W_\Sigma$.
2. If Rwv, then $R^f|w||v|$.
3. If $R^f|w||v|$, then for all $\Diamond\phi \in \Sigma$, we have $\mathfrak{M}, w \vDash \Diamond\phi$, whenever $\mathfrak{M}, v \vDash \phi$.
4. $V^f(p) = \{|w| \mid \mathfrak{M}, w \vDash p\}$, for all propositional variables $p \in \Sigma$.

then $\mathfrak{M}^f_\Sigma$ is called a *filtration of* $\mathfrak{M}$ *through* Σ; sometimes we will write $\mathfrak{M}^f$ instead of $\mathfrak{M}^f_\Sigma$. $\dashv$

It should be noted that the second condition in the above definition gives us the states in W_Σ that should be necessarily related by R^f. Meanwhile, the third condition gives us a restricting property on the states in W_Σ that are related by R^f. Thus, in a way, the second condition guarantees the presence of elements in R^f, whereas the third condition restricts the presence of all possible tuples in R^f.

Example 2.3.28. Consider a model $\mathfrak{M} = (\mathbb{N}, R, V)$ as shown in Figure 2.8, where

$$R = \{(0,1), (0,2), (1,3)\} \cup \{(n, n+1) \mid n \geq 2\},$$

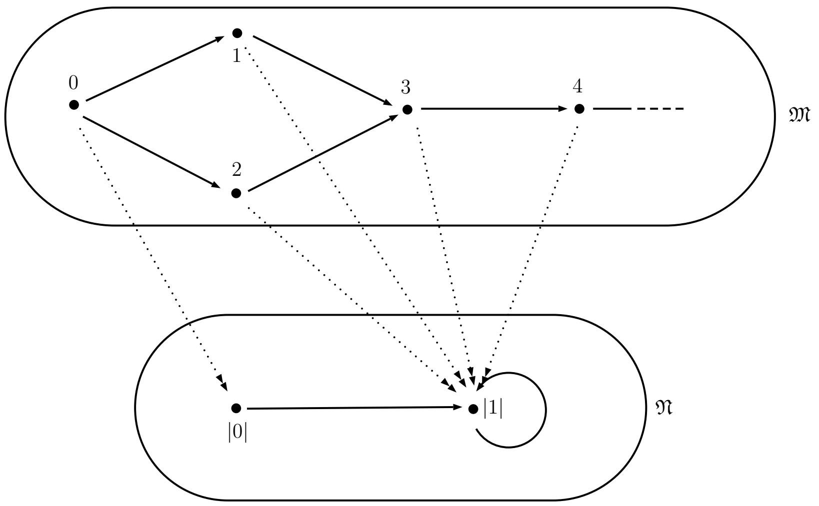

Fig. 2.8: The model $\mathfrak{M}$ and one of its filtrations.

and V has $V(p) = \mathbb{N}/\{0\}$, and $V(q) = \{2\}$. Let $\Sigma = \{\Diamond p, p\}$. Here, Σ is subformula closed. It can be checked that the model

$$\mathfrak{N} = (\{|0|, |1|\}, \{(|0|, |1|), (|1|, |1|)\}, V'),$$

where $V'(p) = \{|1|\}$ is a filtration of $\mathfrak{M}$ through Σ. $\dashv$

The next proposition states that the size of a filtration depends upon the size of the subformula closed set considered.

Proposition 2.3.29. *Let Σ be a finite subformula closed set of modal formulas. For any model $\mathfrak{M}$, if $\mathfrak{M}^f$ is a filtration of $\mathfrak{M}$ though a subformula closed set Σ, then $\mathfrak{M}^f$ contains at most 2^n states (where n denotes the size of Σ).*

The idea behind the proof is as following. To each state $|w| \in \mathfrak{M}^f$, we can associate a subset of Σ, namely, the set of formulas in Σ which are true at w. If two states $|v_1|$ and $|v_2|$ get associated with the same set of formulas, then by the definition of $\leftrightsquigarrow_\Sigma$, we get that $|v_1| = |v_2|$. As Σ is finite, the number of possible subsets of Σ is finite and is bounded by 2^n, where n is the size of Σ. Hence, there are at most 2^n distinct states in $\mathfrak{M}^f$.

Crucially, filtrations preserve satisfaction in the following sense.

Theorem 2.3.30 (Filtration Theorem). *Let $\mathfrak{M}^f$ be a filtration of $\mathfrak{M}$ through a subformula closed set Σ. Then for all formulas $\phi \in \Sigma$, and all nodes w in $\mathfrak{M}$, we have $\mathfrak{M}, w \vDash \phi$ iff $\mathfrak{M}^f, |w| \vDash \phi$.*

The proof of Theorem 2.3.30 uses induction, and conditions 2-3 facilitate the case for $\Diamond$ [9].

In fact, for each model at least one of its filtrations always exists (see [9]). Proposition 2.3.29 and Theorem 2.3.30 together yield the following theorem.

Theorem 2.3.31 (Finite Model Property - via Filtrations). *Let ϕ be a modal formula. If ϕ is satisfiable, then it is satisfiable on a finite model. Indeed it is satisfiable on a finite model containing at most 2^m nodes, where m is the number of subformulas of ϕ.*

The idea behind the proof of Theorem 2.3.31 is the following. Let Σ be the set containing all subformulas of ϕ (including ϕ). Then Σ is a subformula closed set. If ϕ is satisfiable on a model, then it is satisfiable on a filtration by Theorem 2.3.30. By Proposition 2.3.29, the filtration has at most 2^m nodes.

It is natural to ask what properties of the model are inherited by the filtrations.

Remark 2.3.32. Let $\mathfrak{M} = (W, R, V)$ be a reflexive model. Then for each $w \in W$, we have Rww. By the second condition in Definition 2.3.27, we get that in any filtration $\mathfrak{M}^f$, we have $R^f|w||w|$. Thus, reflexivity is preserved under taking filtrations. $\dashv$

Not all properties are preserved under taking filtrations. It can be checked that not all filtrations preserve transitivity. We need to 'tweak' R^f appropriately to make sure that the filtration is transitive.

Theorem 2.3.33. *Let $\mathfrak{M}$ be a model, Σ be a subformula closed set of formulas, and W_Σ be the set of equivalence classes induced on $\mathfrak{M}$ by $\leftrightsquigarrow_\Sigma$. Let R^t be the binary relation on W_Σ defined by:*

$$R^t|w||v| \text{ iff for all } \phi, \text{ if } \Diamond\phi \in \Sigma \text{ and } \mathfrak{M}, v \vDash \phi \vee \Diamond\phi, \text{ then } \mathfrak{M}, w \vDash \Diamond\phi.$$

If R is transitive then (W_Σ, R^t, V^f) is a filtration and R^t is transitive.

Thus, if a formula is satisfiable on reflexive transitive model, then it is satisfiable on a finite reflexive transitive model.

2.3.5 Normal modal logics

In this section, we describe a 'special' class of sets of modal formulas called *normal modal logics.* These sets are defined syntactically, but some have interesting properties such as having a semantic characterization, that is, these are exactly the set of formulas valid on a specific class of frames.

Definition 2.3.34. A *normal modal logic* (or normal logic) Λ is a set of modal formulas that contains:

- all propositional tautologies,
- (K) $\quad \Box(p \to q) \to (\Box p \to \Box q)$, and
- (Dual) $\quad \Diamond p \leftrightarrow \neg\Box\neg p$,

and is closed under

- modus ponens (i.e., if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$),
- uniform substitution (i.e., if φ belongs to Λ, then so do all of its substitution instances), and
- generalization (i.e., if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$).

If $\varphi \in \Lambda$, then we say φ is a *theorem* of Λ (notation: $\vdash_\Lambda \varphi$). $\dashv$

Example 2.3.35. The set of all modal formulas is a normal logic. If F is a class of frames, then the set of formulas valid on every element of F forms a normal logic. $\dashv$

For a collection of modal formulas Γ, the smallest normal logic containing Γ, denoted by $\mathbf{K}\Gamma$, is the intersection of all normal logics which contain Γ. This method of defining $\mathbf{K}\Gamma$, is called the *'top down approach'*, as $\mathbf{K}\Gamma$ is being constructed by removing elements from bigger sets.

There is another way to construct $\mathbf{K}\Gamma$ which is called the *'bottom up approach'*. This method adds necessary formulas to smaller sets to form $\mathbf{K}\Gamma$. The method is defined as follows. Let

$$C_0 = \{\text{Propositional tautologies}\} \cup \{(\text{K})\} \cup \{(\text{Dual})\} \cup \Gamma.$$

For each $n \in \mathbb{N}$, we define $C_n = C_{n-1} \cup \{$all modal formulas that can be obtained by applying the rules of modus ponens, uniform substitution, or generalization on $C_{n-1}\}$.

For example, we have $p \leftrightarrow \neg\neg p \in C_0$, so by generalisation, we get $\Box(p \leftrightarrow \neg\neg p) \in C_1$, and by uniform substitution, we obtain $(\neg r \wedge q) \leftrightarrow \neg\neg(\neg r \wedge q) \in C_1$. It can be shown that

$$\mathbf{K}\Gamma = \bigcup_{0}^{\infty} C_n.$$

Thus, the theorems of $\mathbf{K}\Gamma$ are exactly the formulas which can be obtained from C_0 by applying the rules a finite number of times. The Table 2.3 lists some formulas and their common names.

Names	Formulas
(4)	$\Box p \to \Box\Box p$
(T)	$\Box p \to p$
(B)	$p \to \Box\Diamond p$
(D)	$\Box p \to \Diamond p$
(L)	$\Box(\Box p \to p) \to \Box p$

Tab. 2.3: Some formulas and their common names.

The smallest normal logic is denoted by $\mathbf{K}$, and its corresponding set Γ is empty. The logics **KT**, **KB**, **KT4**, and **KT4B** are usually called **T**, **B**, **S4**, and **S5**, respectively.

In [9], the formulas (4) and (T) have been defined as $\Diamond\Diamond p \to \Diamond p$ and $p \to \Diamond p$, respectively. Note that, we have defined (4) and (T) differently. Let us call $\Diamond\Diamond p \to \Diamond p$ and $p \to \Diamond p$ as (4') and (T'), respectively. It can be shown that (4) is a theorem of **K4'**, and (4') is a theorem of **K4**. Similarly, (T) is a theorem of **KT'**, and (T') is a theorem of **KT**. Thus, the normal logics **K4** and **K4'** are the same. Similarly, the logics **KT** and **KT'** are the same. Thus, with no loss of generality we can call $\Diamond\Diamond p \to \Diamond p$ as (4), and $p \to \Diamond p$ as (T).

Next, we wish to understand how do these normal logics 'capture' the frame-related properties. To this end, we first give some definitions.

Definition 2.3.36. A normal logic Λ is said to be *sound* with respect to a class of frames F, if every theorem of Λ is valid on F, that is, we have $\mathsf{F} \vDash \varphi$, whenever $\vdash_\Lambda \varphi$. $\dashv$

Thus, if Λ is sound with respect to F, then $\Lambda \subseteq \Lambda_{\mathsf{F}}$.

Example 2.3.37. The logic **K** is sound with respect to the class of all frames. This happens because of the following reasons. All the propositional tautologies, and the axiom (K) and (Dual) are valid on the class of all frames. Also, the property of a formula being valid on the class of all frames is preserved under the rules of modus ponens, uniform substitution, and generalisation. That is,

- if ϕ and $\phi \to \psi$ are valid on the class of all frames, then ψ is valid on the class of all frames,
- if ϕ is valid on the class of all frames, then so are all of its substitution instances, and
- if ϕ is valid on the class of all frames, then so is $\Box\phi$.

Thus, every theorem of **K** is valid on the class of all frames. $\dashv$

In a similar way, once we prove that the axioms (T) and (4) are valid on the class of all reflexive transitive frames, and the property of a formula being valid on the class of all reflexive transitive frames is preserved under the rules of modus ponens, uniform substitution, and generalisation, we get the following result.

Proposition 2.3.38. *S4 is sound with respect to the class of all reflexive transitive frames.*

Thus, any formula that can be derived using the propositional tautologies, (K), (Dual), (T), and (4), using the rules of modus ponens, uniform substitution, and generalization, finitely many times is valid on the class of all reflexive transitive frames.

There is a notion of *completeness* which is kind of 'converse' to the definition of soundness. It is defined as the following.

Definition 2.3.39. A normal logic Λ is said to be *complete* with respect to a class of frames F, if every formula that is valid on F, is theorem of Λ, that is, we have $\vdash_\Lambda \varphi$, whenever $\mathsf{F} \vDash \varphi$. $\dashv$

Thus, if Λ is complete with respect to F, then $\Lambda_\mathsf{F} \subseteq \Lambda$, and if a normal logic Λ is both sound and complete with respect to a class of frames F, then we have $\Lambda = \Lambda_\mathsf{F}$.

Remark 2.3.40. An equivalent definition of completeness is the following. A normal logic Λ is complete with respect to a class of frames F, if every formula which is not in Λ, is not valid on F, that is, if $\varphi \notin \Lambda$, then there exists a model $\mathfrak{M}$ based on a frame in F and a state x in $\mathfrak{M}$, such that $\mathfrak{M}, x \nvDash \varphi$. $\dashv$

A well known result is that **K** is complete with respect to the class of all frames. To describe its proof, we first need a few definitions.

Definition 2.3.41. For a normal logic Λ, a set of formulas Γ is said to be Λ-*consistent* if for no finite set $\{\phi_1, \ldots, \phi_n\} \subseteq \Gamma$, we have $\vdash_\Lambda (\phi_1 \wedge \ldots \wedge \phi_n) \to \bot$, and a Λ-consistent set of formulas Γ is called a Λ-*maximally consistent set* (or MCS), if there is no Λ-consistent set of formulas properly containing Γ. A formula ϕ is said to be Λ-consistent, if the set $\{\phi\}$ is Λ-consistent. $\dashv$

MCSs behave nicely with respect to the underlying logic in the following way.

Lemma 2.3.42. *If Γ is an MCS of formulas for the normal logic Λ, then we have,*

- Γ *is closed under modus ponens: if* $\phi, \phi \to \psi \in \Gamma$, *then* $\psi \in \Gamma$,
- $\Lambda \subseteq \Gamma$,
- *for all formulas* $\phi : \phi \in \Gamma$ *or* $\neg\phi \in \Gamma$, *and*
- *for all formulas* $\phi, \psi : \phi \vee \psi \in \Gamma$ *iff* $\phi \in \Gamma$ *or* $\psi \in \Gamma$.

We have not yet established the existence of MCSs. The next lemma says that any Λ-consistent set of formulas can be extended to an MCS, and hence several examples of an MCS exist.

Lemma 2.3.43 (Lindenbaum's Lemma). *If* Σ *is a* Λ*-consistent set of formulas, then there is an MCS* Σ^+ *such that* $\Sigma \subseteq \Sigma^+$.

The key step in the proof [9] is to enumerate the set of all modal formulas, and for each formula ϕ, keep adding ϕ or $\neg\phi$ to Σ such that the consistency of Σ is preserved.

Next, we construct a model called the *canonical model* for a normal logic Λ, where the states of the model are MCSs for Λ. The model is special because every Λ-consistent set is satisfiable in this model.

Definition 2.3.44. The *canonical model* $\mathfrak{M}^\Lambda$ for a normal logic Λ is the triple $(W^\Lambda, R^\Lambda, V^\Lambda)$, where:

1. W^Λ is the set of all Λ-MCSs,
2. R^Λ is the binary relation called the *canonical relation* on W^Λ, defined by $R^\Lambda wu$ iff for all formulas ϕ, we have $\Diamond\phi \in w$, whenever $\phi \in u$, and
3. V^Λ is the valuation defined by

$$V^\Lambda(p) = \{w \in W^\Lambda \mid p \in w\}.$$

V^Λ is called the *canonical* (or *natural*) valuation.

The pair $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda)$ is called the *canonical frame* for Λ. $\dashv$

The natural valuation says that if a propositional variable is in an MCS, then it is true at the MCS. The *truth lemma*, which we will be stating next, extends this notion of truth to the set of all formulas.

Lemma 2.3.45 (Truth Lemma). *For any normal modal logic Λ, and any formula ϕ, we have, $\mathfrak{M}^{\Lambda}, w \vDash \phi$ iff $\phi \in w$.*

In the canonical model, we have all the MCSs as our states. So, if we have a Λ-consistent set, by Lindenbaum's lemma (2.3.43), the set can be extended to an MCS, and by the Truth lemma (2.3.45), the set is true on the MCS. Thus, we get the following result.

Theorem 2.3.46. *If Σ is a Λ-consistent set of formulas, then it is satisfiable on the canonical frame for Λ.*

Next, we state a proposition which bridges the gap between Λ-consistent sets being satisfiable on a class of frames, and the completeness of normal logics with respect to the class of frames.

Proposition 2.3.47. *A normal logic Λ is complete with respect to a class of frames F iff every Λ-consistent formula is satisfiable on some $\mathfrak{F} \in \mathsf{F}$.*

Thus, every normal logic Λ is complete with respect to the class of frames containing the canonical frame for Λ, and we get the following result.

Proposition 2.3.48. ***K*** *is complete with respect to the class of all frames.*

It so happens that the canonical frame for **K4** is transitive, and the canonical frame for **S4** is both reflexive and transitive [9]. Thus, we get the following result.

Proposition 2.3.49. ***K4*** *is complete with respect to the class of all transitive frames.*

This happens because every **K4**-consistent formula is satisfiable on the canonical frame for **K4**. Similarly, we get the next result.

Proposition 2.3.50. ***S4*** *is complete with respect to the class of all reflexive transitive frames.*

Thus, using the soundness and completeness of **S4** with respect to the class of all reflexive transitive frames, we can say that the theorems of **S4** are exactly the formulas which are valid on the class of all reflexive transitive frames.

We have introduced most of the logical tools that we will be needing later. The next chapter introduces another interpretation for the basic modal language, called the *topological interpretation*.

3. TOPO-SEMANTICS

In the previous chapter, we defined **S4** to be the smallest normal logic containing the following axioms:

(T) $p \to \Diamond p$,
(4) $\Diamond\Diamond p \to \Diamond p$.

Also, for an arbitrary subset Y of a topological space X, the following properties hold for the closure operator:

- $Y \subseteq \mathrm{Cl}(Y)$, and
- $\mathrm{Cl}(\mathrm{Cl}(Y)) \subseteq \mathrm{Cl}(Y)$.

Such resemblance is not a mere coincidence. Many properties of a topological space can be 'encoded' by interpreting the basic modal language on a topological space in a suitable way. We see one such interpretation which can be found in [10, Chapter 5].

3.1 Topo-models

In this section, we see an alternative interpretation of the basic modal language, based on topological spaces.

Definition 3.1.1. Consider the basic modal language, with P being the countable set of propositional variables. A *topo-model* $M = \langle X, \tau, v\rangle$ is a topological space $\langle X, \tau\rangle$ equipped with a function $v : P \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of X. The function v is called a *valuation function* on X. $\dashv$

Every topological space $\langle X, \tau \rangle$ with any such arbitrary valuation v forms a topo-model. Figure 3.1 below depicts a topo-model based on $\mathbb{R}^2$.

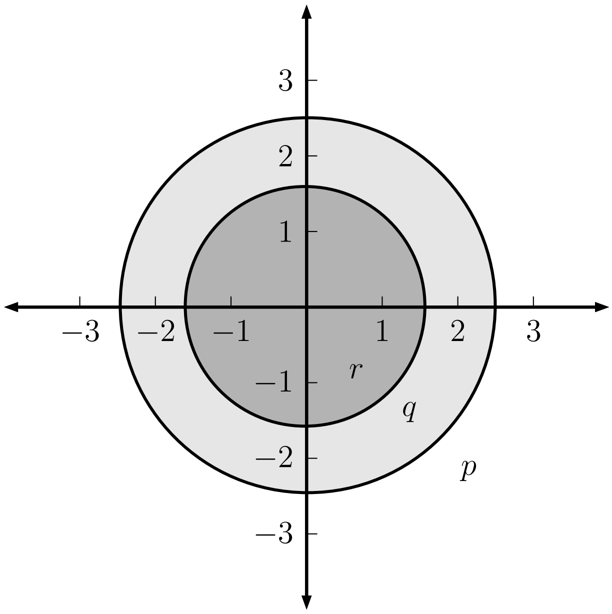

Fig. 3.1: A topo-model based on $\mathbb{R}^2$.

Next, we define the notion of a formula being 'true' at a point.

Definition 3.1.2. Truth of modal formulas is defined inductively at points x of X in a topo-model $M = \langle X, \tau, v \rangle$:

- $M, x \vDash p$ iff $x \in v(p)$, for each $p \in P$,
- $M, x \vDash \neg\phi$ iff it is not the case that $M, x \vDash \phi$,
- $M, x \vDash \phi \wedge \psi$ iff both $M, x \vDash \phi$ and $M, x \vDash \psi$, and
- $M, x \vDash \Diamond\phi$ iff for each U open in $\langle X, \tau \rangle$ containing x, there is some $y \in U$ such that $M, y \vDash \phi$.

$\dashv$

We have abbreviated $\neg(\neg\phi \wedge \neg\psi)$ as $\phi \vee \psi$, $\neg\Diamond\neg\phi$ as $\Box\phi$, $\neg\phi \vee \psi$ as $\phi \to \psi$, and $(\phi \to \psi) \wedge (\psi \to \phi)$ as $\phi \leftrightarrow \psi$. If the context is clear, we write $x \vDash \phi$ instead of $M, x \vDash \phi$.

Remark 3.1.3. We see how the truth for the abbreviations depends upon the truth of constituent formulas. We have

$$\begin{aligned} M, x \vDash \phi \vee \psi &\Leftrightarrow M, x \vDash \neg(\neg\phi \wedge \neg\psi), \\ &\Leftrightarrow M, x \nvDash \neg\phi \wedge \neg\psi, \\ &\Leftrightarrow \text{ not both } M, x \vDash \neg\phi \text{ and } M, x \vDash \neg\psi \text{ hold }, \\ &\Leftrightarrow \text{ not both } M, x \nvDash \phi \text{ and } M, x \nvDash \psi \text{ hold}, \\ &\Leftrightarrow M, x \vDash \phi \text{ or } M, x \vDash \psi. \end{aligned}$$

Similarly, we get

- $M, x \vDash \phi \to \psi$ iff we have $M, x \vDash \psi$, whenever $M, x \vDash \phi$, and
- $M, x \vDash \phi \leftrightarrow \psi$ iff both $M, x \vDash \phi$ and $M, x \vDash \psi$ hold, or both $M, x \nvDash \phi$ and $M, x \nvDash \psi$ hold.

For $\Box$, we have

$$\begin{aligned} M, x \vDash \Box\phi &\Leftrightarrow M, x \vDash \neg\Diamond\neg\phi, \\ &\Leftrightarrow \text{ it is not the case that } M, x \vDash \Diamond\neg\phi, \\ &\Leftrightarrow \text{ it is not that for each } U \in \tau \text{ containing } x, \\ &\quad\ \text{there exists a } y \in U \text{ such that } M, y \vDash \neg\phi, \\ &\Leftrightarrow \text{ there exists } U_o \in \tau \text{ containing } x, \text{ such that for each} \\ &\quad\ y \in U_o, \text{ we have } M, y \nvDash \neg\phi, \\ &\Leftrightarrow \text{ there exists } U_o \in \tau \text{ containing } x, \text{ such that for each } y \in U_o, \\ &\quad\ \text{we have } M, y \vDash \phi. \end{aligned}$$

$\dashv$

Given a valuation v, the *Recursion theorem* (2.1.4) guarantees that v can be extended to a unique function on the set of all modal formulas. As the extension is unique, with no ambiguity, we can call the extension as v. It can shown by induction on the set of modal formulas that

$$v(\phi) = \{x \in X \mid M, x \vDash \phi\}.$$

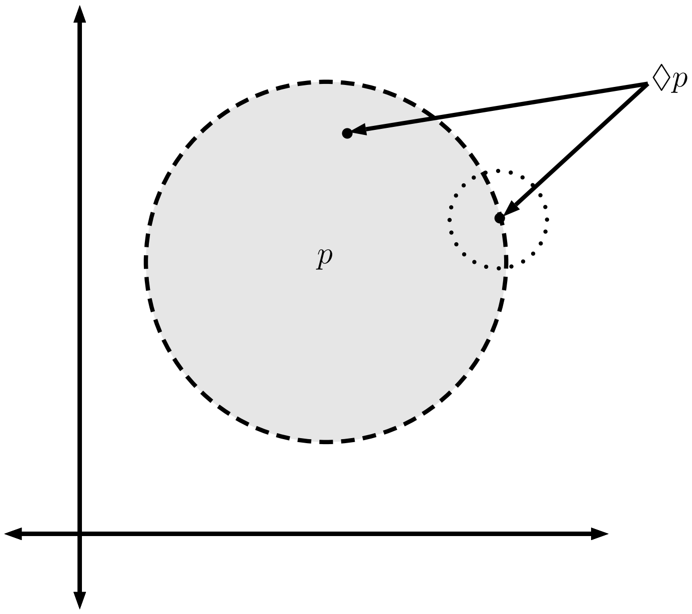

Fig. 3.2: $\Diamond$ corresponds to Closure.

Thus, $x \in v(\Box\phi)$ if and only if there exists $U_o \in \tau$ containing x such that for each $y \in U_o$, we have $M, y \vDash \phi$. So by definition of $v(\phi)$, $x \in v(\Box\phi)$ if and only if there exists $U_o \in \tau$ containing x such that $U_o \subseteq v(\phi)$. Thus, $x \in v(\Box\phi)$ if and only if $x \in \mathrm{Int}(v(\phi))$. Hence, we get $v(\Box\phi) = \mathrm{Int}(v(\phi))$. Similarly, we get $v(\Diamond\phi) = \mathrm{Cl}(v(\phi))$. Also, we get $v(\neg(\phi)) = v(\phi)^c$, $v(\phi \wedge \psi) = v(\phi) \cap v(\psi)$, and $v(\phi \vee \psi) = v(\phi) \cup v(\psi)$. So, given subsets of a set X, using modal formulas we can talk about their union, intersection, complements, closures, and interiors.

Example 3.1.4. Consider $\mathbb{R}^2$ with the usual topology, and let $v(p)$ be a spoon with boundary, as shown in Figure 3.3. Then, $\Box p$ denotes the interior, that is, the disk part, and $\Diamond p$ is the closure which is the spoon itself. So, here $M, x \vDash \Box p \leftrightarrow p$ for all x in $\mathbb{R}^2$. The formula $\neg p$ denotes the complement of the spoon. The formula $\Diamond\neg p$ denotes its closure, which is everything except the interior of the spoon. The formula $\Diamond p \wedge \Diamond\neg p$ denotes the boundary of the spoon. $\dashv$

Next, we define the notion of validity for this interpretation of the basic

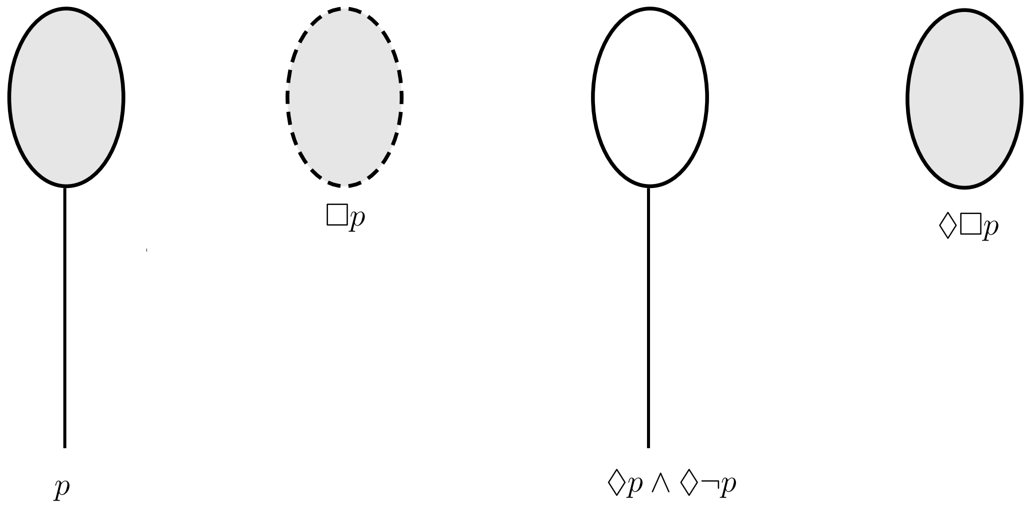

Fig. 3.3: A closed spoon in $\mathbb{R}^2$.

modal language.

Definition 3.1.5. A formula ϕ is said to be *valid* on a topological space $\langle X, \tau\rangle$ if ϕ is true at every point on every topo-model based on $\langle X, \tau\rangle$ (notation: $\langle X, \tau\rangle \vDash \phi$).

A formula ϕ is valid on a class of topological spaces S if ϕ is valid on every member of S. $\dashv$

Example 3.1.6. The formula (Dual) given by $\Diamond p \leftrightarrow \neg\Box\neg p$, which is just the abbreviation of $\Diamond p \leftrightarrow \neg\neg\Diamond\neg\neg p$, is valid on the class of all topological spaces, as for any topo-model $M = \langle X, \tau, v\rangle$, and any $x \in X$, we have

- $M, x \vDash \Diamond p$ iff $x \in v(\Diamond p)$ iff $x \in \mathrm{Cl}(v(p))$, and
- $M, x \vDash \neg\neg\Diamond\neg\neg p$ iff $x \in v(\neg\neg\Diamond\neg\neg p)$ iff $x \in \mathrm{Cl}(v(p)^{c\ c})^{c\ c}$.

As $\mathrm{Cl}(v(p)) = \mathrm{Cl}(v(p)^{c\ c})^{c\ c}$, we get that $M, x \vDash \Diamond p \leftrightarrow \neg\neg\Diamond\neg\neg p$. Since M and x are arbitrary, we get that $\Diamond p \leftrightarrow \neg\neg\Diamond\neg\neg p$ is valid on the class of all topological spaces. $\dashv$

As we will see, we can sometimes construct a relation between two topo-models such that the states related to each other posses the same logical information, that is, the same set of formulas is true at both the points. Such relations are called topological bisimulations. There is a similar concept of bisimulation for the relational interpretation of the modal language, which can be found in [9].

3.2 Topo-bisimulations

Definition 3.2.1. A *topological bisimulation* (or simply a *topo-bisimulation*) between two topo-models $M = \langle X, \tau, v \rangle$ and $M' = \langle X', \tau', v' \rangle$ is a nonempty relation $T \subseteq X \times X'$, such that if xTx', then the following conditions hold.

1. (atomic clause) For each $p \in P$, we have $x \in v(p)$ iff $x' \in v'(p)$.
2. (forth) For arbitrary $U \in \tau$, if $x \in U$, then there exists some $U' \in \tau'$ containing x' such that for each $y' \in U'$, there exists a corresponding $y \in U$ with yTy'.
3. (back) For arbitrary $U' \in \tau'$, if $x' \in U'$, then there exists some $U \in \tau$ containing x such that for each $y \in U$, there exists a corresponding $y' \in U'$ with yTy'.

A topo-bisimulation is called *total* if its domain is X, and its range is X'. That is, it is said to be total if for each $x \in X$, we have $(x, y') \in T$ for some $y' \in X'$, and for each $x' \in X'$, we have $(y, x') \in T$ for some $y \in X$.

If only the atomic clause (1) and the forth condition (2) hold, then we say that the second model *simulates* the first. We say that two topo-models M and M' are *topo-bisimilar* if there exists a topo-bisimulation between them. $\dashv$

Remark 3.2.2 (Restating the conditions). For a subset $A \subseteq X$, and a relation $T \subseteq X \times X'$, the T-image of A (denoted by $T(A)$) is defined to be

$$T(A) = \{y \in X' \mid xTy, \text{ for some } x \in A\}.$$

We now prove that the forth condition in the definition of topo-bisimulation is equivalent to the statement that T-image of every open subset of $\langle X, \tau \rangle$ is open.

Let U be open in X. Let T be a topo-bisimulation between $\langle X, \tau, v\rangle$ and $\langle X', \tau', v'\rangle$. We assume the forth condition holds for T. If $T(U) = \emptyset$, then $T(U)$ is open. Assume $T(U) \neq \emptyset$. Let $x' \in T(U)$. We claim x' is contained in an open set contained in $T(U)$.

By definition, there exists some $x \in U$ such that xTx'. By the forth condition, there exists $U' \in \tau'$ such that $x' \in U'$, and for each $y' \in U'$, there exists $y \in U$ such that yTy'. It should be noted that, for each $y \in U'$, there exists $y \in U$ such that yTy', implies $U' \subseteq T(U)$. So, U' is the required open set, and we have $x' \in U' \subseteq T(U)$. As x' is arbitrary, this implies $T(U)$ is open.

Conversely, let us assume we have a nonempty relation $T \subseteq X \times X'$ such that T-image of every open set of $\langle X, \tau\rangle$ is open in $\langle X', \tau'\rangle$. We need to prove the forth condition, that is, we need to prove that for each $x \in X$, and $x' \in X'$, if xTx' holds, then for each $U \in \tau$ containing x, we have a corresponding $U' \in \tau'$ such that $x' \in U'$, and for each $y' \in U'$, there exists a corresponding $y \in U$ such that yTy'.

Let $x \in X$, and $x' \in X'$ such that xTx', and let $x \in U$ for some $U \in \tau$. Then $x' \in T(U)$ (by definition of $T(U)$). As $T(U)$ is open (being the image of an open set under T), there exists $U' \in \tau'$ such that $x' \in U' \subseteq T(U)$. We prove U' is the required open set around x'. Let $y' \in U'$. As $y' \in U' \subseteq T(U)$, we get $y' \in T(U)$. Therefore, there exists $y \in U$ such that yTy'. Hence, the forth condition is satisfied.

A similar correspondence exists between the back condition and the inverse relation T^{-1}. Consider the inverse relation T^{-1} defined by

$$T^{-1} = \{(x', x) \mid xTx'\}.$$

As the back condition is just the forth condition if T is replaced by T^{-1}, so the back condition is equivalent to the statement that the T^{-1} image of every open set in $\langle X', \tau'\rangle$ is open in $\langle X, \tau\rangle$.

Moreover, the condition numbered 1 (the atomic clause) is equivalent to the statement that $T(v(p)) \subseteq v'(p)$, and $T^{-1}(v'(p)) \subseteq v(p)$, for all propositional variables $p \in P$. Hence, we have a formulation of all the three conditions which is point independent, and wholly in terms of the open sets and the valuations. $\dashv$

Example 3.2.3. Consider the two topo-models as shown in Figure 3.4.

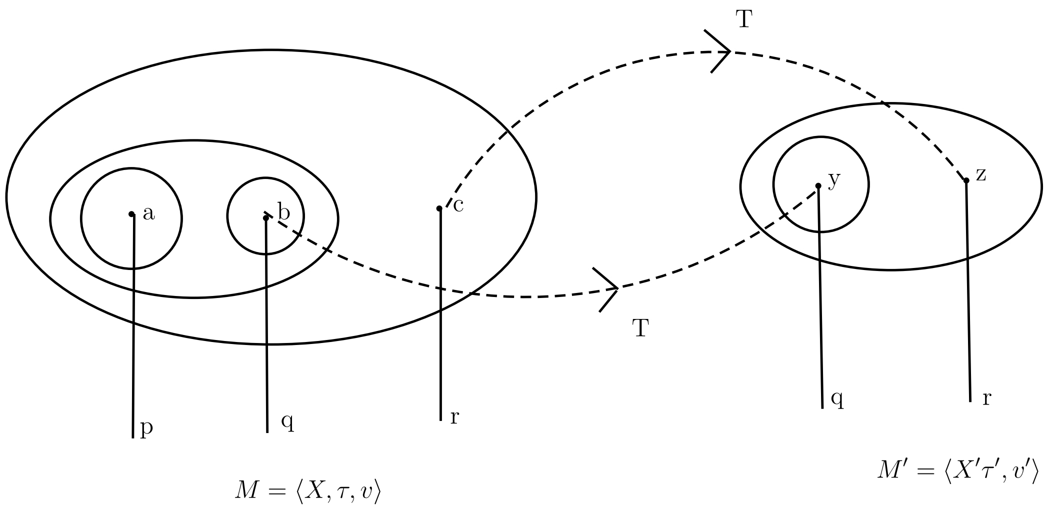

Fig. 3.4: A relation $T \subseteq X \times X'$.

Let $M = \langle X, \tau, v \rangle$, and $M' = \langle X', \tau', v' \rangle$, where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $v(p) = \{a\}$, $v(q) = \{b\}$, $v(r) = \{c\}$, $X' = \{y, z\}$, $\tau' = \{\emptyset, X', \{y\}\}$, $v'(q) = \{y\}$, $v'(r) = \{z\}$, and $v'(p) = \emptyset$. Let $T = \{(b, y), (c, z)\}$. It can be verified that the atomic clause holds for T. Also, $T(\emptyset) = \{x' \in X' \mid xTx' \text{ for some } x \in \emptyset\} = \emptyset$, $T(X) = X'$, $T(\{a\}) = \emptyset$, $T(\{b\}) = \{y\}$, and $T(\{a, b\}) = \{y\}$, all of which are open in X'. So, the forth condition is satisfied. Whereas, $T^{-1}(\{y, z\}) = \{b, c\}$, is not open in X. So, the back condition is not satisfied, and T is not a topo-bisimulation. But, here M' simulates M. Instead, if we have a model as in Figure 3.5, in the place of M and the same T, then as $T(\{b, c\}) = X'$, the forth condition is satisfied here as well, and as $T^{-1}(\emptyset) = \emptyset$, $T^{-1}(X') = \{b, c\}$, and $T^{-1}(\{y\}) = \{b\}$, all of which are open in X, we would get T to be a topo-bisimulation.

$\dashv$

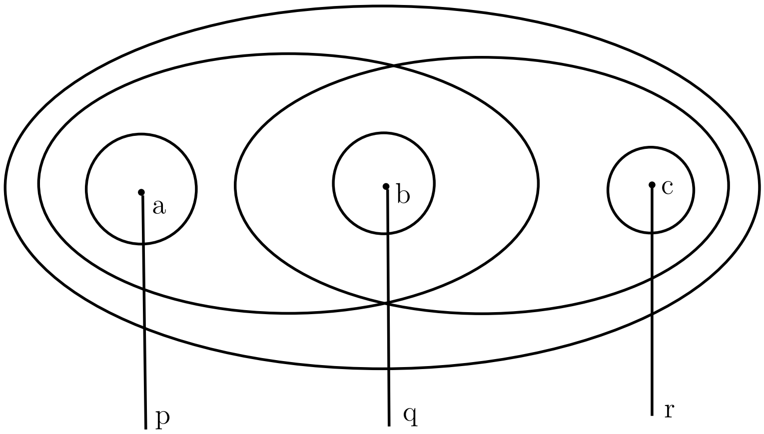

Fig. 3.5: An alternate topo-model.

Next, we see the main result of this section which asserts that for the states related under a topo-bisimulation, the set of formulas that are true at the states is the same. Thus, the states contain the same logical information.

Theorem 3.2.4. *Let $M = \langle X, \tau, v\rangle$ and $M' = \langle X', \tau', v'\rangle$ be two topo-models and $x \in X$, and $x' \in X'$ be two topo-bisimilar points. Then for each modal formula ϕ, we have, $M, x \vDash \phi$ iff $M', x' \vDash \phi$. That is, the truth of modal formulas is invariant under topo-bisimulations.*

Proof. We prove this by induction on the set of all formulas. Let $M = \langle X, \tau, v\rangle$ and $M' = \langle X', \tau', v'\rangle$ be two bisimilar topo-models. Let T be a topo-bisimulation between them, such that $T \subseteq X \times X'$. Consider the set

$$S = \left\{ \phi \text{ is a formula} \;\middle|\; \begin{array}{l} \text{For each } x \in X, \text{ and } x' \in X', \text{ we have,} \\ xTx' \text{ implies } (M, x \vDash \phi \text{ iff } M', x' \vDash \phi) \end{array} \right\}.$$

We prove by induction that S is the set of all modal formulas.

Let p be a propositional variable. Let xTx', for $x \in X$, and $x' \in X'$. By condition (1) of Definition 3.2.1, we have, $x \in v(p)$ iff $x' \in v'(p)$, that is, $M, x \vDash p$ iff $M', x' \vDash p$. So, $p \in S$, and hence, S contains all propositional variables.

Next, we move on to the case for $\wedge$ and $\neg$. Assume $\phi, \psi \in S$ as the induction hypothesis. If xTx', then we have

$$
\begin{aligned}
M, x \vDash \neg\phi &\Leftrightarrow M, x \nvDash \phi \\
&\Leftrightarrow M, x' \nvDash \phi && \text{(induction hypothesis)} \\
&\Leftrightarrow M', x' \vDash \neg\phi, \text{ and} \\
M, x \vDash \phi \wedge \psi &\Leftrightarrow M, x \vDash \phi \text{ and } M, x \vDash \psi \\
&\Leftrightarrow M', x' \vDash \phi \text{ and } M', x' \vDash \psi && \text{(induction hypothesis)} \\
&\Leftrightarrow M', x' \vDash \phi \wedge \psi.
\end{aligned}
$$

So, if $\phi, \psi \in S$, then $\neg\phi, \phi \wedge \psi \in S$.

Now, we prove the induction step for $\Diamond$. We first prove that $M, x \vDash \Diamond\phi$ implies $M', x' \vDash \Diamond\phi$. Let $M, x \vDash \Diamond\phi$. We need to prove $M', x' \vDash \Diamond\phi$, that is, if $U' \in \tau$ such that $x' \in U'$, then there exists $y' \in U'$ such that $M', y' \vDash \phi$. Let $U_o' \in \tau'$ be arbitrary such that $x' \in U_o'$. As T is a topo-bisimulation, by the back condition, for the open set $U_o' \in \tau'$, we get that there exists $U_o \in \tau$ containing x such that for each $y \in U_o$, there exists a corresponding $y' \in U_o'$ with yTy'. As $M, x \vDash \Diamond\phi$ and $x \in U_o$, by definition of satisfaction of $\Diamond\phi$ on x, there exists $y_o \in U_o$ such that $M, y_o \vDash \phi$. By the back condition, for each $y \in U_o$, there exists $y' \in U_o'$ such that yTy'. So, for $y_o \in U_o$, we get that there exists $y_o' \in U_o'$ such that y_oTy_o'. By the induction hypothesis, and the previous two statements, we get $M', y_o' \vDash \phi$. As U_o' is arbitrary, and as we have $y_o' \in U_o'$ such that $M', y_o' \vDash \phi$, so we get $M', x' \vDash \Diamond\phi$.

Similarly, assuming $M', x' \vDash \Diamond\phi$ and by using the forth condition, we get that $M, x \vDash \Diamond\phi$. So, we obtain $M, x \vDash \Diamond\phi$ iff $M', x' \vDash \Diamond\phi$, that is, $\Diamond\phi \in S$.

Hence, by induction, S is the set of all formulas. $\square$

A natural question to ask is whether the converse of Theorem 3.2.4 holds. We now see a partial converse.

Theorem 3.2.5. *Let M and M' be two finite models, and let $x \in X$, and $x' \in X'$ be such that for each ϕ, we have, $M, x \vDash \phi$ iff $M', x' \vDash \phi$. Then, there exists a topo-bisimulation between M and M' such that the points x and x' are topo-bisimilar. That is, finite modally equivalent models are topo-bisimilar.*

Proof. Let $M = \langle X, \tau, v\rangle$ and $M' = \langle X', \tau', v'\rangle$ be two finite topo-models, and x and x' be arbitrary elements in X and X', respectively, such that $M, x \vDash \phi$ iff $M', x' \vDash \phi$, for all modal formulas ϕ. We define

$$T = \left\{ (y, y') \in X \times X' \;\middle|\; \begin{array}{l} \text{for each modal formula } \phi, \\ M, y \vDash \phi \text{ iff } M', y' \vDash \phi \end{array} \right\}.$$

We claim that T forms a topo-bisimulation between M and M'.

First, we prove the atomic clause for T. Let $y \in X$, and $y' \in X'$ be such that yTy'. Then by definition of T, for any propositional variable p, we have $M, y \vDash p$ iff $M', y' \vDash p$, that is, $y \in v(p)$ iff $y' \in v'(p)$. So, the atomic clause is satisfied by T.

Now, we prove the forth condition for T. Let $y \in X$, and $y' \in X'$ be such that yTy', and let $y \in U \in \tau$. We need to prove that there exists $U'_o \in \tau'$ containing y' such that for each $z' \in U'_o$, there exists a corresponding $z \in U$ with zTz'. Assume on the contrary that this is not the case, that is, assume for each $U' \in \tau'$ containing y', there exists $z'_{U'} \in U'$ such that for no $z \in U$, we have $zTz'_{U'}$.

By definition of T, the condition zTz' not holding is equivalent to the existence of a modal formula ψ, for which $M, z \vDash \psi$ iff $M', z' \vDash \psi$ doesn't hold. So, we have that for each $U' \in \tau'$ containing y', there exists $z'_{U'} \in U'$ and modal formulas $\psi_{z,z',U'}$ for each $z \in U$, such that $M', z' \vDash \neg\psi_{z,z',U'}$, but $M, z \vDash \psi_{z,z',U'}$. (Note that, if the opposite holds, that is, if we have $M, z \nvDash \psi_{z,z',U'}$ and $M', z' \vDash \psi_{z,z',U'}$, we simply replace $\psi_{z,z',U'}$ with $\neg\psi_{z,z',U'}$.)

As M is a finite model, so in U there are finitely many points z, and hence for each $z'_{U'}$, we get finitely many formulas $\psi_{z,z',U'}$. Let $\phi_{z',U'}$ be the formula

$$\bigvee_{z \in U} \psi_{z,z',U'}.$$

Then for each $z \in U$, we have $M, z \vDash \phi_{z',U'}$. This happens because for each $z_0 \in U$, we have a constituent formula $\psi_{z_0,z',U'}$ which is true on z_0. But, $M', z' \nvDash \phi_{z',U'}$, as none of the constituent formulas $\psi_{z,z',U'}$ is true on z'. Thus, $M', z' \vDash \neg\phi_{z',U'}$.

As M' is a finite model, we get that τ' is finite, which implies that we have finitely many open sets $U' \in \tau'$, and hence finitely many corresponding $z'_{U'}$s. Let these be denoted by $z'_1, \ldots, z'_n$, and the open sets to which they correspond be $U'_1, \ldots, U'_n$, respectively. For the ease of notation, instead of writing $\phi_{z'_i, U'_i}$, we will just write $\phi_{z'_i}$, for $i \in \{1, \ldots, n\}$.

By the preceding paragraph, for all $i \in \{1, \ldots, n\}$, and each $z \in U$, we have $M, z \vDash \phi_{z'_i}$, which implies

$$M, z \vDash \phi_{z'_1} \wedge \cdots \wedge \phi_{z'_n}.$$

For $i \in \{1, \ldots, n\}$, as $M', z'_i \vDash \neg\phi_{z'_i}$, we get $M', z'_i \nvDash \phi_{z'_i}$, which implies

$$M', z'_i \nvDash \phi_{z'_1} \wedge \cdots \wedge \phi_{z'_n}.$$

As there exists $U \in \tau$ containing y, such that for each $z \in U$, we have $M, z \vDash \phi_{z'_1} \wedge \cdots \wedge \phi_{z'_n}$, we get

$$M, y \vDash \Box(\phi_{z'_1} \wedge \cdots \wedge \phi_{z'_n}).$$

But, for an arbitrary $U' \in \tau'$ which contains y', we get that there exists $z' \in \{z'_1, \ldots, z'_n\}$ and $z' \in U'$ such that $M', z' \nvDash \phi_{z'_1} \wedge \cdots \wedge \phi_{z'_n}$. So, we get

$$M', y' \nvDash \Box(\phi_{z'_1} \wedge \cdots \wedge \phi_{z'_n}),$$

which contradicts the assumption that yTy'. Hence, our assumption that the forth condition doesn't hold is wrong. So, the forth condition holds for T.

For the back condition, consider the inverse relation T^{-1}. Then $y'T^{-1}y$ iff the statement $M, y \vDash \phi$ iff $M', y' \vDash \phi$, holds for all modal formulas ϕ. Repeating the above steps for the forth condition of T for T^{-1}, we get that the forth condition holds also for T^{-1}. As the forth condition for T^{-1} is just the back condition for T, so the back condition holds for T.

Hence, T is a topo-bisimulation and as xTx' holds by assumption, so we have a topo-bisimulation between M and M' such that x and x' are topo-bisimilar points. $\square$

Theorem 3.2.4 and Theorem 3.2.5 together yield the following.

Corollary 3.2.6. *Let $M = \langle X, \tau, v\rangle$ and $M' = \langle X', \tau', v'\rangle$ be two topo-models such that both M and M' are finite. Let $x \in X$, and $x' \in X$. Then there exists a topo-bisimulation between M and M' such that x and x' are topo-bisimilar points iff for all modal formulas ϕ, we have $M, x \vDash \phi$ iff $M', x' \vDash \phi$.*

As homeomorphisms preserve the topological structure fully, and as the underlying topology decides the truth of a formula at a point, it is natural to ask how topo-bisimulations and homeomorphisms are related.

3.3 Topo-bisimulations and homeomorphisms

The next result asserts that homeomorphisms are at least as strong as topo-bisimulations.

Theorem 3.3.1. *If two topological spaces $\langle X, \tau\rangle$ and $\langle X', \tau'\rangle$ are homeomorphic, then for each valuation v on $\langle X, \tau\rangle$, there exists a corresponding valuation v' on $\langle X', \tau'\rangle$ such that the topo-models $M = \langle X, \tau, v\rangle$ and $M' = \langle X'\tau', v'\rangle$ are topo-bisimilar.*

Proof. Let $\langle X, \tau\rangle$ and $\langle X', \tau'\rangle$ be homeomorphic. Let $f : X \to X'$ be a homeomorphism. Given a valuation v on X, for all propositional variables p, we define

$$v'(p) = \{f(x) \mid x \in v(p)\}$$

which is the same as $f(v(p))$. Then v' is a valuation on X'. We define

$$T = \{(x, f(x)) \mid x \in X\}.$$

We claim T is a topo-bisimulation. By Remark 3.2.2, the atomic clause condition is equivalent to having $T(v(p)) \subseteq v'(p)$, and $T^{-1}(v'(p)) \subseteq v(p)$ for each propositional variable p. But by the definition of T and v', we have

$$T(v(p)) = f(v(p)) = v'(p),$$

and

$$T^{-1}(v'(p)) = f^{-1}(v'(p)) = v(p).$$

Hence, the atomic clause is followed.

Now, we prove the forth and back conditions for T. By Remark 3.2.2, the forth condition is equivalent to proving that $T(U)$ is open for each $U \in \tau$, and the back condition is equivalent to proving that the $T^{-1}(U')$ is open for each $U' \in \tau'$. But this holds, as f is a homeomorphism, as $T(U) = f(U)$, and as $T^{-1}(U') = f^{-1}(U')$, for each $U \in \tau$, and $U' \in \tau'$. Thus, both the back and the forth conditions are followed. Hence, T is a topo-bisimulation, and M and M' are topo-bisimilar. □

One may ask whether the converse of Theorem 3.3.1 holds. That is, whether if $\langle X, \tau\rangle$ and $\langle X', \tau'\rangle$ are topological spaces such that for each valuation v on $\langle X, \tau\rangle$, there exists a corresponding valuation v' on $\langle X', \tau'\rangle$ making the topo-models topo-bisimilar, then the two topological spaces are homeomorphic. The next example gives an answer in the negative.

Example 3.3.2. Consider two topological spaces $\langle X, \tau\rangle$ and $\langle X', \tau'\rangle$ where $X = \mathbb{R}$, $X' = \mathbb{R}^2$, and τ and τ' are euclidean topologies on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Let v be a valuation on $\langle X, \tau\rangle$. For a propositional variable p, we define

$$v'(p) = \{(x, y) \in \mathbb{R}^2 \mid x \in v(p)\}.$$

Then, v' is a valuation on $\langle X', \tau'\rangle$. Also, we define

$$T = \{(x, (x, y)) \mid x, y \in \mathbb{R}\}.$$

Then, $T \subseteq \mathbb{R} \times \mathbb{R}^2$. Note that, T associates to each real number, the vertical line passing through that real number on the x-axis (as shown in Figure 3.6).

We claim T forms a total topo-bisimulation. Let $x \in X$, and $z \in X'$ be such that xTz. Then by definition of T, we have $z = (x, y)$ for some $y \in \mathbb{R}$. Let p be a propositional variable. Then,

$$x \in v(p) \Leftrightarrow (x, y) \in v'(p) \text{ (by definition of } v') \Leftrightarrow z \in v'(p).$$

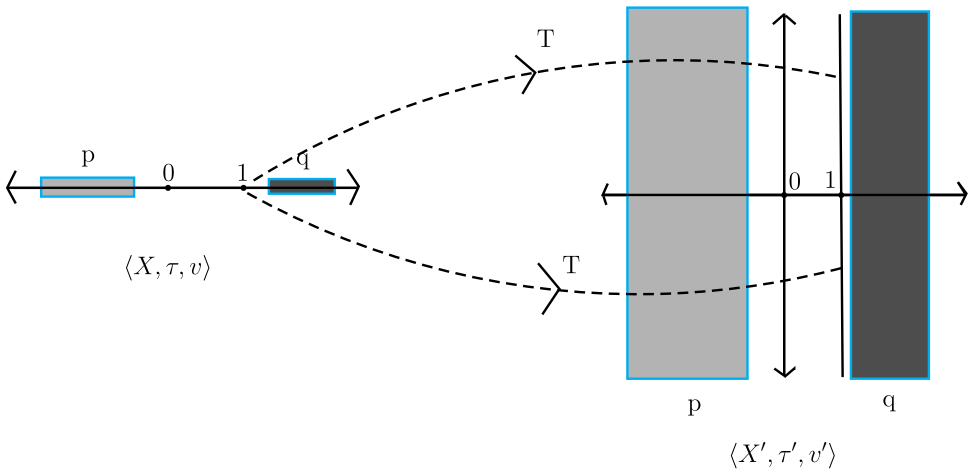

Fig. 3.6: The two topo-models.

So, the atomic clause is satisfied.

Now, we prove an assertion which will help us in proving the forth and back conditions. Let I be an arbitrary indexing set, and let $\{A_\lambda\}_{\lambda\in I}$ be an arbitrary collection of subsets of X. Then,

$$y \in T(\bigcup_{\lambda\in I} A_\lambda),$$

iff xTy for some $x \in \bigcup_{\lambda\in I} A_\lambda$ (by the definition of the T-image of a set)

iff xTy for some $x \in A_{\lambda_o}$, and $\lambda_o \in I$
iff $y \in T(A_{\lambda_o})$ for some $\lambda_o \in I$ (by the definition of the T-image of a set)
iff

$$y \in \bigcup_{\lambda\in I} T(A_\lambda).$$

So,

$$T(\bigcup_{\lambda\in I} A_\lambda) = \bigcup_{\lambda\in I} T(A_\lambda),$$

that is, T is distributive over an arbitrary union. As no property of T is used here except that T is a relation, we get that any relation distributes over unions.

Now, we prove the forth condition by proving that the T-image of open sets is open. Let $a, r \in \mathbb{R}$ such that $r > 0$. Consider a basic open set $B_r(a)$ around a in $\langle X, \tau \rangle$, where

$$B_r(a) = \{a' \in X \mid |a - a'| < r\}.$$

Then

$$T(B_r(a)) = \{(a', y) \in \mathbb{R}^2 \mid |a - a'| < r\},$$

which is a vertical strip not containing its boundary, and is an open set. For an arbitrary open set $U \in \tau$, U is a union of basic open sets. As T distributes over unions, and as image of basic open sets under T is open, we get $T(U) \in \tau'$. So, T follows the forth condition.

For the back condition we take a similar route. Let $(a, b) \in \mathbb{R}^2$, and $r \in \mathbb{R}^+$. Consider a basic open set $B_r((a, b))$ around (a, b), given by

$$B_r((a, b)) = \{(a', b') \in \mathbb{R}^2 \mid \sqrt{(a - a')^2 + (b - b')^2} < r\}.$$

Observe that if $(z_1, z_2) \in \mathbb{R}^2$ such that $zT(z_1, z_2)$, then $z_1 = z$, that is, if $(z_1, z_2)\, T^{-1} z$, then z is a projection of (z_1, z_2) on the x-axis. So, $T^{-1}(B_r((a, b)))$ is a projection of the set $B_r((a, b))$ on the x-axis, which is just $\{a' \in \mathbb{R} \mid |a - a'| < r\}$. It should be noted that

$$\{a' \in \mathbb{R} \mid |a - a'| < r\} = B_r(a),$$

and $B_r(a)$ is open in X. Thus, $T^{-1}(B_r((a, b)))$ is open in X. Also T^{-1} distributes over unions. So by a similar reasoning as in the forth case, we get that if we have an arbitrary $U' \in \tau'$, then $T^{-1}(U') \in \tau$. So, T follows the back condition too, and hence, is a topo-bisimulation between the two topo-models. It should also be noted that T is total.

Since $\mathbb{R}$ and $\mathbb{R}^2$ with the euclidean topology are not homeomorphic, the converse of Theorem 3.3.1 does not hold. $\dashv$

This chapter dealt with defining the topological interpretation, and basic results concerning topo-bisimulations and homeomorphisms. In the next chapter, we will see the first of the two soundness and completeness results that we aim to describe.

4. SOUNDNESS AND COMPLETENESS OF S4

In the first chapter, we saw that **S4** is both sound and complete with respect to the class of all reflexive transitive frames in the relational semantics. Just as we have the notion of soundness and completeness with respect to the relational semantics, we also have analogous notions with respect to the topological semantics.

The main result of this chapter is - **S4** is both sound and complete with respect to the class of all topological spaces (from [10]). Thus, the formulas that are valid in the class of all topological spaces are exactly the theorems of **S4**. Before delving into the proof, we need some terminology.

Definition 4.0.1. Let $\mathcal{X} = \langle X, \tau \rangle$ be a topological space and let $M = \langle \mathcal{X}, v \rangle$ be a topo-model. For a modal formula ϕ,

1. we say that ϕ is *true* in $M = \langle \mathcal{X}, v \rangle$ if ϕ is true at every $x \in X$.
2. We say that ϕ is *valid* in $\mathcal{X}$ if ϕ is true in every topo-model based on $\mathcal{X}$.
3. Finally, we say that ϕ is *valid in a class of topological spaces* S if ϕ is valid in every member of S (notation: $S \vDash \phi$).

$\dashv$

Definition 4.0.2. A modal logic Λ is said to be *sound* with respect to a class of topological spaces S if each member of Λ is valid in every member of S, that is, for each modal formula ϕ, we have $S \vDash \phi$, whenever $\phi \in \Lambda$. $\dashv$

As an example, we prove later that **S4** is sound with respect to the class of all topological spaces.

Definition 4.0.3. A modal logic Λ is said to be *complete* with respect to a class of topological spaces S if each modal formula which is valid on S, is also a member of Λ, that is, for each modal formula ϕ, we have $\phi \in \Lambda$, whenever $S \models \phi$. $\dashv$

Thus, to prove that a normal logic $\mathbf{K}\Gamma$ is complete with respect to a class of topological spaces S, we should prove that every formula ϕ which is valid in S is a theorem of $\mathbf{K}\Gamma$. But there is more than one way to prove that a formula ϕ is a theorem of $\mathbf{K}\Gamma$. This happens because there is more than one way to define $\mathbf{K}\Gamma$.

One of the ways in which $\mathbf{K}\Gamma$ is defined is by the 'top down approach', that is, it is defined as the intersection of all normal logics containing Γ. If we want to use this definition to prove that ϕ is a theorem of $\mathbf{K}\Gamma$, then we should prove that ϕ is contained in every normal logic containing Γ.

The other way in which $\mathbf{K}\Gamma$ is defined is the 'bottom up approach', that is, it is defined as the set of formulas that can be derived from the set of propositional tautologies, (K), (Dual), and Γ, by applying the rules of modus ponens, uniform substitution, and generalisation, finitely many times. If we want to use this definition to prove that ϕ is a theorem of $\mathbf{K}\Gamma$, then we should prove that ϕ can be derived from the set of formulas which only contain the propositional tautologies, (K), (Dual), and Γ, by using modus ponens, uniform substitution, and generalisation, which is a constructive endeavor that is generally challenging.

The next remark gives another way to prove the completeness of a logic with respect to a class of topological spaces.

Remark 4.0.4. An equivalent statement of the definition of completeness is the following. A normal logic Λ is said to be complete with respect to a class of topological spaces S if for each modal formula ϕ, we have $S \not\models \phi$, whenever $\phi \notin \Lambda$. It should be noted that the above statement is the contra-positive of the statement given in Definition 4.0.3. A modal formula ϕ is not valid on a class of topological spaces S iff there is some member of S on which ϕ is not valid. Hence, one way of proving that a normal modal logic Λ is complete with respect to a class of topological spaces S, is by showing that for every modal formula ϕ which is not a theorem of Λ, we have a topological space $\mathcal{X} = \langle X, \tau \rangle$ in S, a topo-model $M = \langle X, \tau, v \rangle$ based on the space $\mathcal{X}$, and an $x \in X$ such that $M, x \not\models \phi$ holds. $\dashv$

In a later section of this chapter, we will use the above method to prove the completeness of **S4** with respect to the class of all topological spaces. The next section deals with the soundness.

4.1 Soundness of S4

First, we recall some facts:

- Any normal modal logic contains the formulas (K) and (Dual), and is closed under modus ponens, necessitation, and uniform substitution.
- The formulas (T), (4), (K), and (Dual) are

$$\begin{array}{cl} \text{(T)} & \Box p \to p, \\ \text{(4)} & \Box p \to \Box\Box p, \\ \text{(K)} & \Box(p \to q) \to (\Box p \to \Box q), \text{ and} \\ \text{(Dual)} & \Diamond p \leftrightarrow \neg\Box\neg p. \end{array}$$

- **S4** is the smallest normal modal logic containing the formulas (T) and (4).

We later prove that **S4** is complete with respect to the class of all topological spaces. For the present, our aim is to prove that **S4** is sound with respect to the class of all topological spaces.

For that our path is as following. We know that **S4** is the set of formulas generated from propositional tautologies, (K), (Dual), (T), and (4), by applying the rules of modus ponens, necessitation, and uniform substitution, finitely many times. We first prove that the propositional tautologies, (T), (4), (K), and (Dual) are valid in any class of topological spaces. Next, we prove that this property of being valid in any class of topological spaces is preserved under modus ponens, necessitation, and uniform substitution. As a consequence, we get that all the theorems of **S4** are valid in any class of topological spaces, and hence, on the class of all topological spaces.

Proposition 4.1.1. *The formulas (T), (4), (K), and (Dual) are valid in any class of topological spaces.*

Proof. Let S be a class of topological spaces.

We claim (T) is valid in S. (T) is given by $\Box p \to p$. Let $\mathcal{X} = \langle X, \tau \rangle \in S$. Let v be a valuation on X, and $x \in X$. We need to prove $x \vDash \Box p \to p$. Let us assume $x \vDash \Box p$. Then by definition, there exists a $U \in \tau$ containing x such that for each $y \in U$, we have $y \vDash p$. In particular as $x \in U$, we get $x \vDash p$. So, $x \vDash \Box p$ implies $x \vDash p$. Hence, $x \vDash \Box p \to p$. From the arbitrariness of x, v, and $\mathcal{X}$ we get that (T) is valid in S.

By a similar line of reasoning, we show that (4) is valid in S. Let $\langle X, \tau, v \rangle$ be a topo-model and $x \in X$. We need to show $x \vDash \Box p \to \Box\Box p$. Let $x \vDash \Box p$. So by definition, there exists a $U \in \tau$ containing x such that for each $y \in U$, we have $y \vDash p$. We claim that for each $z \in U$, we have $z \vDash \Box p$. Let $z \in U$. Then as for all $y \in U$, we have $y \vDash p$, so there exists some $U \in \tau$ such that $z \in U$ and for each $y \in U$, we have $y \vDash p$. So, we get $z \vDash \Box p$. As $z \in U$ is arbitrary, we get that for each $z \in U$, we have $z \vDash \Box p$. So, there exists a $U \in \tau$ containing x such that for each $z \in U$, we have $z \vDash \Box p$, and, we get $x \vDash \Box\Box p$. Hence, $x \vDash \Box p \to \Box\Box p$. As x and the topo-model are arbitrary, so (4) is valid in S.

Now we prove (K) is valid in S. Let $\mathcal{X}, v, x$ be arbitrary as in the previous part. We need to prove $x \vDash \Box(p \to q) \to (\Box p \to \Box q)$. Let $x \vDash \Box(p \to q)$. By definition, there exists a $U \in \tau$ containing x such that for each $y \in U$, we have $y \vDash p \to q$. Now, we need to prove $x \vDash \Box p \to \Box q$, which amounts to proving that if $x \vDash \Box p$, then $x \vDash \Box q$. Let us assume that $x \vDash \Box p$. So, there exists a $V \in \tau$ containing x such that for each $z \in V$, we have $z \vDash p$. It remains to be shown that $x \vDash \Box q$.

Consider $U \cap V$. As $U, V \in \tau$ and both contain x, and as τ is closed under finite intersections, we have $U \cap V \in \tau$, and $x \in U \cap V$. Let $y \in U \cap V$. As $U \cap V \subseteq V$, so we have $y \in V$. As for each $z \in V$, we have $z \vDash p$, so in particular we have $y \vDash p$. Similarly, as $U \cap V \subseteq U$, and as we have $v \vDash p \to q$ for each $v \in U$, so we get $y \vDash p \to q$. But, $y \vDash p \to q$ is equivalent to the statement that if $y \vDash p$, then $y \vDash q$. As $y \vDash p$ holds, so we get $y \vDash q$, and since y is arbitrary, we get that for each $z \in U \cap V$, $z \vDash q$. So, there exists a $U \cap V \in \tau$ containing x, such that for each $z \in U \cap V$, we have $z \vDash q$, and so, we have $x \vDash \Box q$. Hence, (K) is valid in S.

Next, we prove that (Dual) is valid in S. Let $\mathcal{X}, v, x$ be arbitrary as in the previous part. We need to prove $x \vDash \Diamond p \leftrightarrow \neg\Box\neg p$, which is equivalent to showing that $x \vDash \Diamond p$ iff $x \vDash \neg\Box\neg p$. We have, $x \vDash \Diamond p$
iff for each $U \in \tau$ containing x, there exists $y \in U$ with $y \vDash p$
iff it is not the case that there exists a $U \in \tau$ containing x, such that for each $y \in U$ we have $y \nvDash p$
iff it is not the case that there exists a $U \in \tau$ containing x, such that for each $y \in U$ we have $y \vDash \neg p$
iff it is not the case that $x \vDash \Box\neg p$
iff $x \nvDash \Box\neg p$
iff $x \vDash \neg\Box\neg p$.
As x is arbitrary, so we get that (Dual) is valid in S. □

Proposition 4.1.2. *Propositional tautologies are valid in any class of topological spaces.*

Proof. Let S be a class of topological spaces. Let $\mathcal{X} \in S$, and let $\langle X, \tau, v\rangle$ be a topo-model based on $\mathcal{X}$. Let $x \in X$. We define a propositional valuation function $v'_x : P \to \{T, F\}$ by

$$v'_x(p) = \begin{cases} T, & \text{if } x \vDash p, \text{ and} \\ F, & \text{otherwise} \end{cases}$$

for all propositional variables $p \in P$. Then, we extend v'_x uniquely to the set of all propositional formulas using the *Recursion theorem* (Theorem 2.1.4).

Without any ambiguity, we can call the extension as v'_x.

Let

$$A = \{\phi \text{ is propositional formula} \mid v'_x(\phi) = T \text{ iff } x \vDash \phi\}.$$

We claim that A is the set of all **propositional formulas**. We prove the claim by induction on the set A.

For an arbitrary $p \in P$, by definition of v'_x, we get $p \in A$. So, A contains all the propositional variables.

Let $\phi, \psi \in A$. This assumption is our induction hypothesis. Then, $v'_x(\neg\phi) = T$ iff $v'_x(\phi) = F$. By the induction hypothesis, $v'_x(\phi) = F$ iff $x \nvDash \phi$. Also, we have $x \nvDash \phi$ iff $x \vDash \neg\phi$. Together, we get $v'_x(\neg\phi) = T$ iff $x \vDash \neg\phi$. So, A is closed under $\neg$.

Also, $v'_x(\phi \wedge \psi) = T$ iff both $v'_x(\phi) = T$ and $v'_x(\psi) = T$. Applying the induction hypothesis, we get $v'_x(\phi \wedge \psi) = T$ iff both $x \vDash \phi$ and $x \vDash \psi$. Thus, we get $v'_x(\phi \wedge \psi) = T$ iff $x \vDash \phi \wedge \psi$. So $\phi \wedge \psi \in A$, and hence, A is closed under $\wedge$. Hence by induction, A is the set of all propositional formulas.

For a propositional tautology κ, and for the above valuation v'_x, we have $v'_x(\kappa) = T$, which is equivalent to $x \vDash \kappa$. As $\kappa, \mathcal{X}, v, x$ all are arbitrary, we get that propositional tautologies are valid in S. ☐

Next, we move on to the rules of modus ponens, necessitation, and uniform substitution. To handle uniform substitution, we first need to prove a lemma which relates the truth of a modal formula with the truth of its substitution instances.

Lemma 4.1.3. *Let $M = \langle X, \tau, v\rangle$ be a topo-model, σ be a substitution function, and v' be a valuation on X defined by $v'(p) = v(\sigma(p))$, for all propositional variables p. For a topo-model $M' = \langle X, \tau, v'\rangle$, for all modal formulas ϕ, and for each $x \in X$, we have $M, x \vDash \sigma(\phi)$ iff $M', x \vDash \phi$.*

Proof. Let M, σ, v', and M' be as given above. We prove the lemma by induction on the set of modal formulas, that is, we prove that the set

$$A = \left\{ \phi \text{ is a modal formula} \;\middle|\; \begin{array}{l} \text{for each } x \in X, \text{ we have} \\ M, x \vDash \sigma(\phi) \text{ iff } M', x \vDash \phi \end{array} \right\}$$

is the set of all modal formulas.

Let p be a propositional variable, and let $x \in X$. As

$$\begin{aligned} M, x \vDash \sigma(p) &\Leftrightarrow x \in v(\sigma(p)) \\ &\Leftrightarrow x \in v'(p) \\ &\Leftrightarrow M', x \vDash p, \end{aligned}$$

so $p \in A$. Hence, A contains all propositional variables.

Let $\phi, \psi \in A$, and $x \in X$. As $\sigma(\neg\phi) = \neg\sigma(\phi)$, we get $M, x \vDash \sigma(\neg\phi)$ iff $M, x \vDash \neg\sigma(\phi)$. As $M, x \vDash \neg\sigma(\phi)$ is equivalent to $M, x \nvDash \sigma(\phi)$, we get $M, x \vDash \sigma(\neg\phi)$ iff $M, x \nvDash \sigma(\phi)$. As $\phi \in A$, we have $M, x \nvDash \sigma(\phi)$ iff $M', x \nvDash \phi$. Thus, $M, x \vDash \sigma(\neg\phi)$ iff $M', x \nvDash \phi$. Since $M', x \nvDash \phi$ is equivalent to $M', x \vDash \neg\phi$, we get $M, x \vDash \sigma(\neg\phi)$ iff $M', x \vDash \neg\phi$. Hence, $\neg\phi \in A$.

Similarly, as $\sigma(\phi \wedge \psi) = \sigma(\phi) \wedge \sigma(\psi)$, we have, $M, x \vDash \sigma(\phi \wedge \psi)$ is equivalent to $M, x \vDash \sigma(\phi) \wedge \sigma(\psi)$, which is equivalent to the statement that both $M, x \vDash \sigma(\phi)$ and $M, x \vDash \sigma(\psi)$ hold. As both ϕ and ψ are in A, the preceding statement is equivalent to the statement that both $M', x \vDash \phi$ and $M', x \vDash \psi$ hold. Hence, $\phi \wedge \psi \in A$.

For the modal case, as $\sigma(\Diamond\phi) = \Diamond\sigma(\phi)$, we have $M, x \vDash \sigma(\Diamond\phi)$ iff $M, x \vDash \Diamond\sigma(\phi)$. $M, x \vDash \Diamond\sigma(\phi)$ means that for each $U \in \tau$ containing x, there exists $y \in U$ such that $M, y \vDash \sigma(\phi)$. As $\phi \in A$, we have for each $y \in X$, and hence particularly for each $U \in \tau$, and $y \in U$, we have $M, y \vDash \sigma(\phi)$ iff $M', y \vDash \phi$. So, $M, x \vDash \Diamond\sigma(\phi)$ is equivalent to the statement that for each $U \in \tau$ containing x, there exists a $y \in U$ such that $M', y \vDash \phi$, which in turn is equivalent to $M', x \vDash \Diamond\phi$. Hence, $\Diamond\phi \in A$, and by induction, A is the whole set of modal formulas. $\square$

An upshot of the above lemma is that if $v'(p) = v(\sigma(p))$ for all propositional variables p, then

$$v'(\phi) = v(\sigma(\phi)),$$

for each modal formula ϕ. Thus, v and v' extend in a coherent manner to the set of all modal formulas.

The next result proves that validity in an arbitrary class of topological spaces remains preserved under the applications of the rules described earlier.

Proposition 4.1.4. *The rules modus ponens, uniform substitution, and necessitation preserve validity of formulas in any class of topological spaces.*

Proof. Let S be a class of topological spaces. We first prove that modus ponens preserves validity in S, that is, if ϕ and $\phi \to \psi$ are valid in S, then ψ is also valid in S. Let us assume that ϕ and $\phi \to \psi$ are valid in S. Let $\mathcal{X} \in S$, let $\langle X, \tau, v\rangle$ be a model based on $\mathcal{X}$, and let $x \in X$. We need to prove $x \vDash \psi$. As $\phi \to \psi$ is valid in S, we get $x \vDash \phi \to \psi$, that is, if $x \vDash \phi$ holds, then $x \vDash \psi$ holds. As ϕ is valid, we have $x \vDash \phi$. From the previous two statements, we get $x \vDash \psi$. As $\mathcal{X}, v, x$ are all arbitrary, we get that ψ is valid in S. Thus, modus ponens preserves validity.

Next, using Lemma 4.1.3 we prove uniform substitution preserves validity in S. Let ϕ be valid in S, and let ψ be a substitution instance of ϕ. This means that there is some substitution function σ from the set of propositional variables to the set of modal formulas whose extension σ satisfies $\sigma(\phi) = \psi$. The extension can be called σ as well because of the uniqueness of extension by the *Recursion theorem* (2.1.4). Let $\mathcal{X}, v, x$ be arbitrary as in the above part. We need to prove that for the topo-model $M = \langle X, \tau, v\rangle$, we have $M, x \vDash \psi$, which is equivalent to $M, x \vDash \sigma(\phi)$. We define a valuation v' on X by $v'(p) = v(\sigma(p))$ for all propositional variables p. Let M' be the topo-model $M' = \langle X, \tau, v'\rangle$. Then as ϕ is valid in S, we have $M', x \vDash \phi$. By Lemma 4.1.3, this is equivalent to $M, x \vDash \sigma(\phi)$. Hence, our claim is true, that is, uniform substitution preserves validity in S.

Next, we prove necessitation preserves validity. Let $\mathcal{X} \in S$, $\langle X, \tau, v\rangle$ be a topo-model based on $\mathcal{X}$, and $x \in X$. Given a ϕ valid in S, we need to prove $\Box\phi$ is valid in S. As all $\mathcal{X}, v, x$ are arbitrary, it suffices to prove $x \vDash \Box\phi$, which means we need to prove that there exists $U \in \tau$ containing x, such that for each $y \in U$, we have $y \vDash \phi$. We choose U to be X. We already have $X \in \tau$ and $x \in X$. Also, as ϕ is valid in S, for each $y \in X$, we have $y \vDash \phi$. So, X is a candidate for the required open set U, and we have $x \vDash \Box\phi$. Hence, necessitation preserves validity in S. As S is arbitrary, so the result holds for all classes of topological spaces. $\square$

Let **Top** denote the class of all topological spaces. As (K), (Dual), (T), (4), and the propositional tautologies are valid in an arbitrary class of topological spaces, so they are valid in **Top**. Similarly, as the rules of modus ponens, uniform substitution, and necessitation preserve validity in an arbitrary class of topological spaces, so they preserve validity in **Top**. As a consequence, we get the following result.

Corollary 4.1.5. ***S4*** *is sound with respect to* ***Top****.*

The remaining part of this chapter deals with proving that **S4** is complete with respect to **Top**.

4.2 The Alexandroff Topology

We define a special class of topological spaces which will help us in proving the completeness result.

Definition 4.2.1. A topological space $\mathcal{X}$ is called an *Alexandroff space* if the intersection of any family of open subsets is open. $\dashv$

Example 4.2.2. Any discrete topological space is an Alexandroff space. $\dashv$

The next proposition gives us an alternate way to define an Alexandroff space.

Proposition 4.2.3. *$\mathcal{X}$ is an Alexandroff space iff every $x \in X$ has a least open neighborhood.*

Proof. We first prove that if $\mathcal{X}$ is an Alexandroff space, then each $x \in X$ has a least open neighborhood. Let $\mathcal{X}$ be an Alexandroff space. We prove each $x \in X$ has a least open neighborhood. For $x \in X$, consider the family of open sets

$$\mathcal{B} = \{U \in \tau \mid x \in U\}.$$

As $\mathcal{X}$ is Alexandroff, so $\bigcap\limits_{U \in \mathcal{B}} U$ is open. Also

$$x \in \bigcap_{U \in \mathcal{B}} U,$$

as for each $U \in \mathcal{B}$, we have $x \in U$. For any $V \in \tau$ containing x, we have $V \in \mathcal{B}$, by definition of $\mathcal{B}$. So for any neighborhood V of x, we get

$$\bigcap_{U \in \mathcal{B}} U \subseteq V.$$

Hence, $\bigcap\limits_{U \in \mathcal{B}} U$ is the least open neighborhood of x. As x is arbitrary, we get every $x \in X$ has a least open neighborhood.

Conversely, let us assume that each $x \in X$ has a least open neighborhood. We want to prove $\mathcal{X}$ is Alexandroff. Let $\{U_i\}_{i \in I}$ be a family of open sets. We need to prove $\bigcap\limits_{i \in I} U_i$ is open. If

$$\bigcap_{i \in I} U_i = \emptyset,$$

then as $\emptyset \in \tau$, we are done. Let

$$\bigcap_{i \in I} U_i \neq \emptyset.$$

We prove $\bigcap\limits_{i \in I} U_i$ is open by proving each point in $\bigcap\limits_{i \in I} U_i$ is an interior point. Let

$$x \in \bigcap_{i \in I} U_i.$$

Then, a least open neighborhood of x exists by assumption. Let it be U_x. As $x \in \bigcap\limits_{i \in I} U_i$, for each $i \in I$, we get $x \in U_i$. As each $U_i \in \tau$, and as U_x is a least open neighborhood of x, so for each $i \in I$, we get $U_x \subseteq U_i$. So,

$$U_x \subseteq \bigcap_{i \in I} U_i.$$

Therefore,

$$x \in U_x \subseteq \bigcap_{i \in I} U_i,$$

that is, x is an interior point. As x is arbitrary, we get

$$\bigcap_{i \in I} U_i \in \tau.$$

Hence, $\mathcal{X}$ is an Alexandroff space. $\square$

In the next example, we see the above criterion in action.

Example 4.2.4. Let $U_m = \{1, \cdots, m \mid m \in \mathbb{N}\}$, and $\mathcal{B} = \{U_m \mid m \in \mathbb{N}\}$ as shown in Figure 4.1.

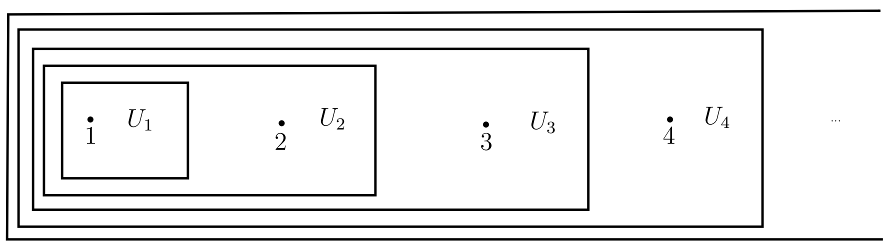

Fig. 4.1: The family $\mathcal{B}$.

We claim that $\mathcal{B}$ forms a basis for $\mathbb{N}$, and the topology generated by $\mathcal{B}$ on $\mathbb{N}$ is Alexandroff.

For $n \in \mathbb{N}$, we have $U_n \in \mathcal{B}$, and $n \in U_n$. So, each element of $\mathbb{N}$ is contained in some element of $\mathcal{B}$.

Let $n \in \mathbb{N}$ such that for some $U_k, U_l \in \mathcal{B}$, we have $n \in U_k \cap U_l$. By the definition of U_k and U_l, this means $n \leq k$, and $n \leq l$. Hence, $n \leq \min\{k, l\}$. By the definition of U_is, we get $n \in U_{\min\{k,l\}} \subseteq U_k \cap U_l$. So $\mathcal{B}$ forms a basis for $\mathbb{N}$.

Let τ be the topology generated by $\mathcal{B}$. Our claim is $\langle \mathbb{N}, \tau \rangle$ forms an Alexandroff space. We prove it by proving that each element in $\mathbb{N}$ has a least open neighborhood. For any $n \in \mathbb{N}$, as $U_n \in \mathcal{B}$, so $U_n \in \tau$, and hence, U_n is a neighborhood of n. We claim that U_n is the least open neighborhood of n. Let $V \in \tau$ contain n. As $\mathcal{B}$ is a basis, there is a basic open set U_m containing n, such that $n \in U_m \subseteq V$. As $n \in U_m$, we get $n \leq m$, which implies $U_n \subseteq U_m$.

So, $U_n \subseteq U_m \subseteq V$, which implies $U_n \subseteq V$. As V is arbitrary, we get that U_n is the least open neighborhood of n. As n is arbitrary, we get that $\langle \mathbb{N}, \tau \rangle$ is an Alexandroff space.

$\dashv$

Alexandroff spaces have been introduced because each reflexive transitive model can be thought of as a topo-model with an Alexandroff topology, such that the logical information at each of the state is preserved. The next section shows how to do this.

4.3 The Topo-completeness of S4

The aim of the section is to prove the completeness of **S4** with respect to **Top**, using Alexandroff spaces. For that, we first need a few definitions.

Definition 4.3.1. Let $\mathcal{F} = (X, R)$ be an **S4**-frame. A subset A of X is called an *upset* of $\mathcal{F}$ if for each $x, y \in X$, if $x \in A$ and xRy, then we have $y \in A$. Dually, A is called a *downset* if for each $x, y \in X$, if $x \in A$ and yRx, then we have $y \in A$. $\dashv$

Informally, upsets are forward closed in terms of R, and downsets are backward closed in terms of R.

Example 4.3.2. Consider the frame as shown in the Figure 4.2, where $X = \{1, \ldots, 6\}$, and R is depicted by arrows. The frame (X, R) is an **S4**-frame as R is reflexive and transitive.

Here, $\{1, 2, 3\}$ is a downset, and its complement $\{4, 5, 6\}$ is an upset. Also, $\{1, 3\}$ is a downset, and its complement $\{2, 4, 5, 6\}$ is an upset. $\dashv$

Using the upsets and downsets on an **S4**-frame (X, R), we can define a topology on the set X in the following way.

Proposition 4.3.3. *Let (X, R) be an **S4**-frame. Let τ_R be defined as*

$$\tau_R = \{A \subseteq X \mid A \textit{ is an upset}\}.$$

Then $\langle X, \tau_R \rangle$ forms a topological space.

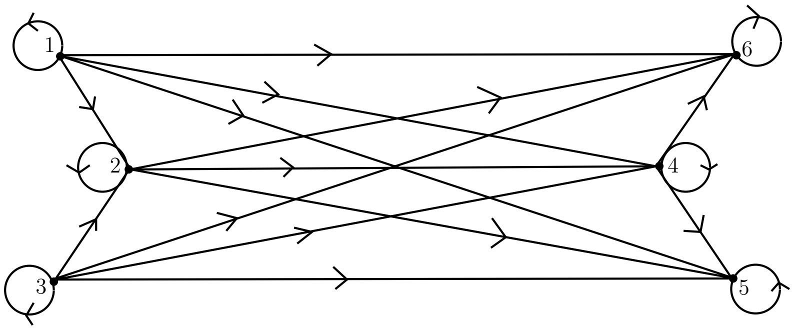

Fig. 4.2: The **S4**-frame.

Proof. We first prove $\emptyset, X \in \tau_R$. As $\emptyset$ is empty, vacuously for each $x \in \emptyset$, if xRy, then $y \in \emptyset$. So, $\emptyset \in \tau_R$. Also, as $R \subseteq X \times X$, so for each $x \in X$, if xRy, then $y \in X$. So, $X \in \tau_R$.

Now we prove τ_R is closed under intersection. Let $U, V \in \tau_R$. We claim $U \cap V \in \tau_R$, that is, $U \cap V$ is an upset. Let $x \in U \cap V$, and let xRy. Then as $U, V \in \tau_R$, so by definition of τ_R we get, U and V are upsets. So, we get $y \in U$, and $y \in V$ which implies $y \in U \cap V$. Thus, we get $U \cap V \in \tau_R$. So, τ_R is closed under intersection.

Now we prove τ_R is closed under arbitrary unions. Let $\{U_i\}_{i \in I} \subseteq \tau_R$, let

$$x \in \bigcup_{i \in I} U_i,$$

and let xRy. As $x \in \bigcup_{i \in I} U_i$, so there exists an $i_o \in I$ such that $x \in U_{i_o}$. As xRy and $U_{i_o} \in \tau_R$, we get $y \in U_{i_o}$, which implies

$$y \in \bigcup_{i \in I} U_i.$$

So,

$$\bigcup_{i \in I} U_i \in \tau_R.$$

Thus, we get that τ_R is closed under arbitrary unions. Hence, $\langle X, \tau_R \rangle$ forms a topological space. $\square$

Next, we see how the closed sets look like. Let $A \subseteq X$ be a closed set. Then A^c is an open set, and hence, an upset. Let $x \in A$, and yRx. If $y \notin A$, then $y \in A^c$, and as A^c is an upset, and yRx holds, we get $x \in A^c$ which contradicts the assumption that $x \in A$. So $y \in A$, and thus, we get that A is a downset. We have shown that complements of upsets are downsets. So, the closed sets in τ_R are exactly the downsets.

The next result says that the upset topology on **S4**-frames is Alexandroff.

Proposition 4.3.4. *$\langle X, \tau_R \rangle$ forms an Alexandroff space.*

Proof. We prove that τ_R is closed under arbitrary intersections. Let $\{U_i\}_{i \in I} \subseteq \tau_R$. So, each U_i is an upset by definition. Let

$$x \in \bigcap_{j \in I} U_j,$$

and xRy. Let $i \in I$. As $x \in \bigcap_{j \in I} U_j$, so $x \in U_i$. As xRy, and as U_i is an upset, so we get $y \in U_i$. So for each $i \in I$, we have $y \in U_i$, which implies

$$y \in \bigcap_{j \in I} U_j.$$

Hence, we get $\bigcap_{j \in I} U_j$ is an upset. So τ_R is closed under arbitrary intersections, and hence $\langle X, \tau_R \rangle$ is Alexandroff. □

The next remark describes how the smallest open neighborhood of a point look like in this topology.

Remark 4.3.5. As $\langle X, \tau_R \rangle$ is Alexandroff, each element has a least open neighborhood. Consider the set

$$R(x) = \{y \in X \mid xRy\}.$$

The set $R(x)$ consists of all the right neighbors of the element x with respect to the relation R.

1. We claim that $R(x)$ is an upset and hence, is open. Let $y \in R(x)$, and for some $z \in X$, let yRz. By assumption, $y \in R(x)$ holds, so by the definition of $R(x)$, we have xRy. As (X, R) is an **S4**-frame, so R is transitive. As xRy holds, as yRz holds by assumption and as R is transitive, so we get xRz. So by definition of $R(x)$, we get $z \in R(x)$. As y, z are arbitrary, we get that $R(x)$ is an upset and hence, is open.

2. We claim $R(x)$ is the least open neighborhood of x. Let $U \in \tau_R$ contain x. We prove $R(x) \subseteq U$. Let $y \in R(x)$. Then, by definition of $R(x)$, we have xRy. As $x \in U$, as U is an upset, and as xRy, we get $y \in U$. As y is arbitrary, we get $R(x) \subseteq U$. So, $R(x)$ is the least open neighborhood of x.

$\dashv$

The next lemma asserts that this topology preserves the logical information if the valuation is taken to be the same.

Lemma 4.3.6. *Let $M_1 = (X, R, v)$ be a model based on the **S4**-frame (X, R). Let τ_R be the topology generated by the upsets. Let M_2 be the model $\langle X, \tau_R, v\rangle$. Then for all modal formulas ϕ, and for each $x \in X$, we have $M_1, x \vDash \phi$ iff $M_2, x \vDash \phi$.*

Proof. We prove this by induction on the set of all modal formulas. Let

$$A = \{\phi \text{ is a modal formula} \mid \text{for each } x \in X, \text{ we have } M_1, x \vDash \phi \text{ iff } M_2, x \vDash \phi\}.$$

We prove by induction that A is the set of all modal formulas.

Let p be a propositional variable, and $x \in X$. Then $M_1, x \vDash p$ iff $x \in v(p)$. As the valuation for both the models is the same, we have $x \in v(p)$ iff $M_2, x \vDash p$. Together, we get $M_1, x \vDash p$ iff $M_2, x \vDash p$. So $p \in A$.

Let us assume $\phi, \psi \in A$. This is our induction hypothesis.

Then $M_1, x \vDash \neg\phi$ iff $M_1, x \nvDash \phi$. As $\phi \in A$ by our induction hypothesis, we get $M_1, x \nvDash \phi$ iff $M_2, x \nvDash \phi$. Also, $M_2, x \nvDash \phi$ iff $M_2, x \vDash \neg\phi$. Together, we get $M_1, x \vDash \neg\phi$ iff $M_2, x \vDash \neg\phi$. As x is arbitrary, we get $\neg\phi \in A$.

Also, $M_1, x \vDash \phi \wedge \psi$ iff both $M_1, x \vDash \phi$ and $M_1, x \vDash \psi$. By the induction hypothesis, both $M_1, x \vDash \phi$ and $M_2, x \vDash \psi$ hold iff both $M_2, x \vDash \phi$ and $M_2, x \vDash \psi$ hold, which is equivalent to $M_2, x \vDash \phi \wedge \psi$. Thus, we get $\phi \wedge \psi \in A$.

Next, we prove $\Diamond\phi \in A$. Let $x \in X$, and let $M_1, x \vDash \Diamond\phi$. So, there exists a $y \in X$ such that xRy holds and $M_1, y \vDash \phi$. As $\phi \in A$, we have $M_2, y \vDash \phi$. Let $U \in \tau_R$ contain x. As xRy holds, so $y \in R(x)$. By Proposition 4.3.4, τ_R is Alexandroff because R is reflexive and transitive. As $R(x)$ is the smallest open neighborhood of x by Remark 4.3.5, we have $R(x) \subseteq U$. So $y \in U$. Thus, for each $U \in \tau_R$ containing x, there exists a $y \in U$ such that $M_2, y \vDash \phi$. Hence, $M_2, x \vDash \Diamond\phi$. So, $M_1, x \vDash \Diamond\phi$ implies $M_2, x \vDash \Diamond\phi$.

Now, we assume $M_2, x \vDash \Diamond\phi$. Then, for each $U \in \tau_R$ containing x, there exists $y \in U$ such that $M_2, y \vDash \phi$. As $R(x) \in \tau_R$, and $x \in R(x)$, particularly, we get that there exists a $y \in R(x)$ such that $M_2, y \vDash \phi$. The statement $y \in R(x)$ is equivalent to xRy, and by the induction hypothesis, the statement $M_2, y \vDash \phi$ is equivalent to $M_1, y \vDash \phi$. Together, we get that there exists a $y \in X$ such that xRy, and $M_1, y \vDash \phi$. Hence, $M_1, x \vDash \Diamond\phi$. So, $M_2, x \vDash \Diamond\phi$ implies $M_1, x \vDash \Diamond\phi$. Therefore, $\Diamond\phi \in A$. Hence by induction, A is the set of all modal formulas. □

The above lemma states that truth of formulas remains conserved for the two interpretations, if the models are related to each other by the construction methods that we have introduced in this section. The next corollary is the final result of the section, and establishes the fact that the theorems of **S4** are exactly the formulas which are valid on **Top**. It should be noted that in the following proof we use the fact that **S4** is complete with respect to the class of all reflexive transitive frames.

Corollary 4.3.7. ***S4*** *is complete with respect to* ***Top****.*

Proof. We use the method explained in Remark 4.0.4. Let $\phi \notin \mathbf{S4}$. As **S4** is complete with respect to the class of all reflexive-transitive frames (Proposition 2.3.50), so we have a model $M_1 = \langle X, R, \tau \rangle$, and $x \in X$, such that (X, R) is an **S4**-frame, and $M_1, x \nvDash \phi$. Consider the model $M_2 = \langle X, \tau_R, v \rangle$. Then by Lemma 4.3.6, we have $M_2, x \nvDash \phi$. Thus, we have a topo-model on which ϕ is falsifiable.

Hence, **S4** is complete with respect to **Top**. □

The soundness and completeness result together imply that the theorems of **S4** are exactly the set of valid formulas on **Top**.

Each formula which is valid in **Top** corresponds to a property of a topological space. For example, consider the formula (T) which is given by $\Box p \to p$. As (T) is valid in **Top**, for any topo-model $\mathcal{M} = \langle X, \tau, v \rangle$, and for any $x \in X$, we have $x \vDash \Box p$ implies $x \vDash p$, that is, $x \in v(\Box p)$ implies $x \in v(p)$. As $v(\Box p) = \mathrm{Int}(v(p))$, we get $x \in \mathrm{Int}(v(p))$ implies $x \in v(p)$. Thus, we get $\mathrm{Int}(v(p)) \subseteq v(p)$. As v is an arbitrary valuation, so $v(p)$ is an arbitrary subset of X. Hence for all $A \subseteq X$, we have $\mathrm{Int}(A) \subseteq A$. Thus, the formula (T) corresponds to the property that the interior of a set is always contained in the set itself. Similarly, every other formula, which is valid in **Top**, corresponds to a property.

The soundness and completeness result has a two-way application.

- Since **S4** is sound with respect to **Top**, so every theorem of **S4** is valid in every topological space. Therefore, using **S4** we get to know the properties of a topological space. Thus, we can use **S4** to study **Top**.

- Since **S4** is complete with respect to **Top**, so every formula valid in **Top** is a theorem of **S4**. Therefore, if a property of a topological space corresponds to a modal formula, then by completeness, that modal formula is a theorem of **S4**. Thus, we can use **Top** to study **S4**.

The next chapter contains a more constructive proof of the same result.

5. THE CANONICAL TOPO-MODEL PROOF

The last proof of the completeness of **S4** with respect to **Top** relied on the fact that **S4** is complete with respect to the class of all reflexive transitive frames. Another completeness proof exists, and it is independent of the relational semantics results for **S4**. The proof is done by building one and only one topo-model, which can falsify every formula which is not in **S4**. This chapter discusses the proof in detail (taken from [10]).

5.1 The Canonical topo-model for S4

In this section, we construct a special topo-model which we will use to prove the completeness of **S4** with respect to the class of all topological spaces. The construction is similar to the construction of the canonical model in the case of relational semantics (can be found in [9]).

Recall that for a normal logic Λ, a set of formulas Γ is said to be Λ-consistent if for no finite set $\{\phi_1, \ldots, \phi_n\} \subseteq \Gamma$, we have $\vdash_\Lambda (\phi_1 \wedge \ldots \wedge \phi_n) \to \bot$, and a Λ-consistent set of formulas Γ is called maximally consistent if there is no consistent set of formulas properly containing Γ. A formula ϕ is said to be Λ-consistent if the set $\{\phi\}$ is Λ-consistent.

The next remark gives equivalent conditions for Λ-consistency of a set Γ.

Remark 5.1.1 (Equivalent conditions for consistency). It should be noted that $(p \to \bot) \to \neg p$ is a propositional tautology. So by uniform substitution, for arbitrary formulas $\phi_1, \ldots, \phi_n$, we have

$$\vdash_\Lambda ((\phi_1 \wedge \ldots \wedge \phi_n) \to \bot) \to \neg(\phi_1 \wedge \ldots \wedge \phi_n).$$

If Γ is not Λ-consistent, then we have formulas $\phi_1, \ldots, \phi_n \in \Gamma$, such that $\vdash_\Lambda (\phi_1 \wedge \ldots \wedge \phi_n) \to \bot$. Then by modus ponens, we get $\vdash_\Lambda \neg(\phi_1 \wedge \ldots \wedge \phi_n)$. Also, as $\neg p \to p \to \bot$ is a propositional tautology, so by a similar reasoning we get the converse, that is, if there are formulas $\phi_1, \ldots, \phi_n \in \Gamma$, such that $\vdash_\Lambda \neg(\phi_1 \wedge \ldots \wedge \phi_n)$, then we have $\vdash_\Lambda (\phi_1 \wedge \ldots \wedge \phi_n) \to \bot$. So, in this case Γ is not Λ-consistent.

Thus, Γ is Λ-consistent iff for no finite set $\{\phi_1, \ldots, \phi_n\} \subseteq \Gamma$, we have $\vdash_\Lambda \neg(\phi_1 \wedge \ldots \wedge \phi_n)$

Similarly, as

$$\neg(p_1 \wedge \ldots \wedge p_n) \to \neg p_1 \vee \ldots \vee \neg p_n,$$

and

$$\neg p_1 \vee \ldots \vee \neg p_n \to \neg(p_1 \wedge \ldots \wedge p_n)$$

are propositional tautologies, so we get that Γ is not Λ-consistent iff there are $\phi_1, \ldots, \phi_n \in \Gamma$ such that $\vdash_\Lambda \neg\phi_1 \vee \ldots \vee \neg\phi_n$. $\dashv$

The next lemma and corollary elucidate a property of maximally consistent sets (or MCSs) which we'll be using frequently in our proofs later.

Lemma 5.1.2. *If Γ is a maximally consistent set of formulas for the normal logic Λ, then for each formula ϕ, not both ϕ and $\neg\phi$ are in Γ.*

Proof. Let Γ be a maximally consistent set. For a formula ϕ, assume both ϕ and $\neg\phi$ are in Γ. We have the following proof that $\neg(\phi \wedge \neg\phi) \in \Lambda$.

1. $\vdash_\Lambda \neg(p \wedge \neg p)$ (Propositional tautology)
2. $\vdash_\Lambda \neg(\phi \wedge \neg\phi)$ (Uniform substitution: 1)

This implies that Γ is not consistent, which contradicts our assumption. Thus, not both ϕ and $\neg\phi$ are in Γ. $\square$

Lemma 5.1.2 and Lemma 2.3.42 together yield the following corollary.

Corollary 5.1.3. *If Γ is a maximally consistent set of formulas for the normal logic Λ, then for each formula ϕ, exactly one of ϕ or $\neg\phi$ is in Γ.*

We now proceed to construct the canonical topo-model for **S4**. From now on, we fix our normal logic as **S4**. Thus, consistent means **S4**-consistent, unless stated otherwise.

Definition 5.1.4. The *canonical topological space* is the pair $\mathcal{X} = \langle X^{\mathcal{L}}, \tau^{\mathcal{L}} \rangle$ where:

- $X^{\mathcal{L}}$ is the set of all maximally consistent sets.
- For each formula ϕ, we define

$$\widehat{\phi} = \{x \in X^{\mathcal{L}} \mid \phi \in x\},$$

$$B^{\mathcal{L}} = \{\widehat{\Box\phi} \mid \phi \text{ is a formula}\},$$

  and $\tau^{\mathcal{L}}$ to be the topology generated by taking $B^{\mathcal{L}}$ as the basis.

$\dashv$

The next lemma shows that $B^{\mathcal{L}}$ indeed forms a basis.

Lemma 5.1.5. $B^{\mathcal{L}}$ *forms a basis.*

Proof. We need to prove two things.

1. For each $x \in X^{\mathcal{L}}$, there is a corresponding formula ϕ_x such that $x \in \widehat{\Box\phi_x}$.
2. If $x \in \widehat{\Box\phi} \cap \widehat{\Box\psi}$, then there is a formula χ such that $x \in \widehat{\Box\chi} \subseteq \widehat{\Box\phi} \cap \widehat{\Box\psi}$.

For the first part, it should be noted that $\top \in$ **S4**. This happens because $\top$ is a propositional tautology. As **S4** is closed under necessitation, so we get $\Box\top \in$ **S4**. As for each maximally consistent set x, we have **S4** $\subseteq x$ by Lemma 2.3.42, so $\Box\top \in x$ for each $x \in X^{\mathcal{L}}$, that is, $x \in \widehat{\Box\top}$, for each $x \in X^{\mathcal{L}}$.

For the second part, let $x \in X^{\mathcal{L}}$ be such that $x \in \widehat{\Box\phi} \cap \widehat{\Box\psi}$.

We first claim that $\widehat{\Box\phi} \cap \widehat{\Box\psi} = \widehat{\Box(\phi \wedge \psi)}$. Proving this suffices, as then for any $x \in \widehat{\Box\phi} \cap \widehat{\Box\psi}$, we have $x \in \widehat{\Box(\phi \wedge \psi)} \subseteq \widehat{\Box\phi} \cap \widehat{\Box\psi}$. Proving the above is equivalent to proving that for any $x \in X^{\mathcal{L}}$, $\Box\phi \in x$ and $\Box\psi \in x$ if and only if $\Box(\phi \wedge \psi) \in x$.

To prove it, we will first prove that both $\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$, and $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$ are in **S4**. For $\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$, we have

1. $\vdash p \wedge q \to p$	(Propositional tautology)
2. $\vdash \phi \wedge \psi \to \phi$	(Uniform substitution: 1)
3. $\vdash \Box(\phi \wedge \psi \to \phi)$	(Generalization: 2)
4. $\vdash \Box(p \to q) \to \Box p \to \Box q$	(K)
5. $\vdash \Box(\phi \wedge \psi \to \phi) \to \Box(\phi \wedge \psi) \to \Box\phi$	(Uniform substitution: 4)
6. $\vdash \Box(\phi \wedge \psi) \to \Box\phi$	(Modus Ponens: 3,5)
7. $\vdash p \wedge q \to q$	(Propositional tautology)
8. $\vdash \phi \wedge \psi \to \psi$	(Uniform substitution: 7)
9. $\vdash \Box(\phi \wedge \psi \to \psi)$	(Generalization: 8)
10. $\vdash \Box(\phi \wedge \psi \to \psi) \to \Box(\phi \wedge \psi) \to \Box\psi$	(Uniform substitution: 4)
11. $\vdash \Box(\phi \wedge \psi) \to \Box\psi$	(Modus Ponens: 9,10)
12. $\vdash (p \to q) \to (p \to r) \to (p \to (q \wedge r))$	(Propositional tautology)
13. $\vdash (\Box(\phi \wedge \psi) \to \Box\phi) \to (\Box(\phi \wedge \psi) \to \Box\psi)$	
$\to (\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi)$	(Uniform substitution: 12)
14. $\vdash (\Box(\phi \wedge \psi) \to \Box\psi) \to (\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi)$	(Modus Ponens: 6,13)
15. $\vdash \Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$	(Modus Ponens: 11,14)

For $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$, we have

1. $\vdash p \to q \to p \wedge q$	(Propositional tautology)
2. $\vdash \Box(p \to q \to p \wedge q)$	(Generalization: 1)
3. $\vdash \Box(p \to q) \to \Box p \to \Box q$	(K)
4. $\vdash \Box(p \to q \to p \wedge q) \to \Box p \to \Box(q \to p \wedge q)$	(Uniform substitution :3)
5. $\vdash \Box p \to \Box(q \to p \wedge q)$	(Modus Ponens: 2,4)
6. $\vdash \Box(q \to p \wedge q) \to \Box q \to \Box(p \wedge q)$	(Uniform substitution: 3)
7. $\vdash (p \to q) \to (q \to r) \to (p \to r)$	(Propositional tautology)
8. $\vdash (\Box p \to \Box(q \to p \wedge q)) \to (\Box(q \to p \wedge q)$	
$\to (\Box q \to \Box(p \wedge q)) \to (\Box p \to \Box q \to \Box(p \wedge q))$	(Uniform substitution: 7)
9. $\vdash (\Box(q \to p \wedge q) \to (\Box q \to \Box(p \wedge q)))$	
$\to (\Box p \to \Box q \to \Box(p \wedge q))$	(Modus Ponens: 5,8)
10. $\vdash \Box p \to \Box q \to \Box(p \wedge q)$	(Modus Ponens: 6,9
11. $\vdash (p \to q \to r) \to (p \wedge q \to r)$	(Propositional tautology)
12. $\vdash (\Box p \to \Box q \to \Box(p \wedge q))$	
$\to (\Box p \wedge \Box q \to \Box(p \wedge q))$	(Uniform substitution: 11)
13. $\vdash \Box p \wedge \Box q \to \Box(p \wedge q)$	(Modus Ponens: 10,12)
14. $\vdash \Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$	(Uniform substitution: 13)

Hence, both $\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$, and $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$ are in **S4**.

To prove the first claim, let $x \in X^{\mathcal{L}}$ such that $\Box\phi, \Box\psi \in x$. We have the proof of $\Box\phi \to \Box\psi \to \Box\phi \wedge \Box\psi$ as follows:

1. $\vdash p \to q \to (p \wedge q)$	(Propositional Tautology)
2. $\vdash \Box\phi \to \Box\psi \to \Box\phi \wedge \Box\psi$	(Uniform substitution: 1)

As **S4** $\subseteq x$ by Lemma 2.3.42, we get that $\Box\phi \to \Box\psi \to \Box\phi \wedge \Box\psi \in x$. As x is closed under Modus Ponens by Lemma 2.3.42, and as $\Box\phi \in x$ by assumption, so we get that $\Box\psi \to \Box\phi \wedge \Box\psi \in x$. As x is closed under Modus Ponens by Lemma 2.3.42, and as $\Box\psi \in x$ by assumption, so we get that $\Box\phi \wedge \Box\psi \in x$. As **S4** $\subseteq x$ by Lemma 2.3.42, and as we have proved $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi) \in$ **S4**, so we get that $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi) \in x$. As x is closed under Modus Ponens by Lemma 2.3.42, so we get $\Box(\phi \wedge \psi) \in x$. Thus, $\Box\phi, \Box\psi \in x$ implies $\Box(\phi \wedge \psi) \in x$.

To prove the converse, assume $\Box(\phi \wedge \psi) \in x$. As we have proved $\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi \in$ **S4**, and as **S4** $\subseteq x$ by Lemma 2.3.42, so we get that $\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi \in x$. As x is closed under Modus Ponens by Lemma 2.3.42, we get $\Box\phi \wedge \Box\psi \in x$. It should be noted that $p \wedge q \to p$ is a propositional tautology, and hence, is in **S4**. By Uniform Substitution, $\Box\phi \wedge \Box\psi \to \Box\phi \in$ **S4**, and as **S4** $\subseteq x$, we get $\Box\phi \wedge \Box\psi \to \Box\phi \in x$. By Modus Ponens, we get $\Box\phi \in x$. Similarly, as $p \wedge q \to q$ is a propositional tautology, so we get that $\Box\psi \in x$. Together we get that, $\Box\phi \wedge \Box\psi \in x$ implies $\Box\phi \in x$ and $\Box\psi \in x$. Hence, we get that $\widehat{\Box\phi} \cap \widehat{\Box\psi} = \widehat{\Box(\phi \wedge \psi)}$. So, $B^{\mathcal{L}}$ forms a basis. $\Box$

Thus, $\tau^{\mathcal{L}}$ is well-defined topology. Next, we make a topo-model out of the canonical topological space.

Definition 5.1.6. The *canonical topo-model* is the pair $M^{\mathcal{L}} = \langle \mathcal{X}, v^{\mathcal{L}} \rangle$, where

- $\mathcal{X}$ is the canonical topological space, and
- $v^{\mathcal{L}}(p) = \{x \in X^{\mathcal{L}} \mid p \in x\}$.

$\dashv$

Having built the canonical topo-model, in the next section, we see what formulas are true at each of the maximally consistent set (or MCS) in the topo-model.

5.2 Completeness through the Canonical Topo-Model

The valuation has been defined on the canonical topological space in such a way that a propositional variable is true at an MCS if and only if that propositional variable is contained in the MCS. The next lemma asserts that the valuation lifts in a similar manner to all the formulas.

Lemma 5.2.1 (Truth Lemma). *For all modal formulas ϕ, and for all $x \in X^{\mathcal{L}}$, we have $M^{\mathcal{L}}, x \vDash \phi$ iff $x \in \widehat{\phi}$.*

Proof. The following proof is by induction. If p is a propositional variable, then

$$M^{\mathcal{L}}, x \vDash p \text{ iff } x \in v^{\mathcal{L}}(p) \text{ iff } p \in x \text{ iff } x \in \widehat{p}.$$

Assume that for all $x \in X^{\mathcal{L}}$, we have $M^{\mathcal{L}}, x \vDash \phi$ iff $x \in \widehat{\phi}$, and $M^{\mathcal{L}}, x \vDash \psi$ iff $x \in \widehat{\psi}$. This is our induction hypothesis. Then for an $x \in X^{\mathcal{L}}$ we have,

$$\begin{aligned} M^{\mathcal{L}}, x \vDash \neg\phi &\text{ iff } M^{\mathcal{L}}, x \nvDash \phi \\ &\text{ iff } x \notin \widehat{\phi} && \text{(induction hypothesis)} \\ &\text{ iff } \phi \notin x \\ &\text{ iff } \neg\phi \in x && \text{(by Corollary 5.1.3)} \\ &\text{ iff } x \in \widehat{\neg\phi}. \end{aligned}$$

Before proving the case for $\wedge$, we prove that for any $x \in X^{\mathcal{L}}$, and for arbitrary formulas ϕ and ψ, we have $\phi \wedge \psi \in x$ iff $\phi \in x$ and $\psi \in x$. Assume $\phi \wedge \psi \in x$. As $p \wedge q \to p$ is a propositional tautology, so by uniform substitution we have $\phi \wedge \psi \to \phi \in \mathbf{S4}$. As $\mathbf{S4} \subseteq x$ by Lemma 2.3.42, we get $\phi \wedge \psi \to \phi \in x$, and as x is closed under Modus Ponens by Lemma 2.3.42, we get that $\phi \in x$. Also, $p \wedge q \to q$ is a propositional tautology. So by a similar reasoning, we get $\psi \in x$. Hence, $\phi \wedge \psi \in x$ implies $\phi \in x$ & $\psi \in x$.

Conversely, assume that $\phi, \psi \in x$. As $p \to q \to p \wedge q$ is a propositional tautology, by uniform substitution we get $\phi \to \psi \to \phi \wedge \psi \in \mathbf{S4}$, and as $\mathbf{S4} \subseteq x$, we get that $\phi \to \psi \to \phi \wedge \psi \in x$. As x is closed under Modus Ponens, and both ϕ and ψ are in x by assumption, so by applying Modus Ponens twice, we get $\phi \wedge \psi \in x$. Thus, $\phi, \psi \in x$ implies $\phi \wedge \psi \in x$. Together, we get $\phi \wedge \psi \in x$ iff $\phi \in x$ and $\psi \in x$.

Now we prove the case for $\wedge$. For arbitrary $x \in X^{\mathcal{L}}$, we have

$$\begin{aligned} M^{\mathcal{L}}, x \models \phi \wedge \psi & \text{ iff } M^{\mathcal{L}}, x \models \phi \text{ and } M^{\mathcal{L}}, x \models \psi \\ & \text{ iff } x \in \widehat{\phi} \text{ and } x \in \widehat{\psi} \qquad \text{(induction hypothesis)} \\ & \text{ iff } \phi \in x \text{ and } \psi \in x \\ & \text{ iff } \phi \wedge \psi \in x \\ & \text{ iff } x \in \widehat{\phi \wedge \psi}. \end{aligned}$$

Next, we prove the case for $\Box$. We need to prove that for all $x \in X^{\mathcal{L}}$, we have $M^{\mathcal{L}}, x \models \Box\phi$ iff $x \in \widehat{\Box\phi}$. Let $x \in X^{\mathcal{L}}$. First, we assume $x \in \widehat{\Box\phi}$. We need to prove that $M^{\mathcal{L}}, x \models \Box\phi$, that is, we need to prove that there is an open set U containing x such that for each $y \in U$, we have $M^{\mathcal{L}}, y \models \phi$. By the induction hypothesis, it suffices to prove that there is an open set U containing x, such that $U \subseteq \widehat{\phi}$.

Consider the basic open set $\widehat{\Box\phi}$. As $x \in \widehat{\Box\phi}$, so it is an open set around x. For $y \in \widehat{\Box\phi}$, by definition we have $\Box\phi \in y$. As $\Box p \to p$ is the axiom (T), by uniform substitution we get $\Box\phi \to \phi \in \mathbf{S4}$. As $\mathbf{S4} \subseteq y$ by Lemma 2.3.42, we get that $\Box\phi \to \phi \in y$. By Modus Ponens, we get $\phi \in y$, that is, $y \in \widehat{\phi}$. So, $\widehat{\Box\phi} \subseteq \widehat{\phi}$, that is, $\widehat{\Box\phi}$ is the needed U. So, $x \in \widehat{\Box\phi}$ implies $M^{\mathcal{L}}, x \models \Box\phi$.

Now, we assume $M^{\mathcal{L}}, x \models \Box\phi$. We need to prove $x \in \widehat{\Box\phi}$. By definition, $M^{\mathcal{L}}, x \models \Box\phi$ implies that there is an open set U containing x such that for each $y \in U$, we have $M^{\mathcal{L}}, y \models \phi$. As each open set is a union of basic open sets, we get that there is a basic open set $\widehat{\Box\psi}$ containing x such that for all $y \in \widehat{\Box\psi}$, we have $M^{\mathcal{L}}, y \models \phi$. By the induction hypothesis, we get that there is a formula ψ such that $\Box\psi \in x$ and $\widehat{\Box\psi} \subseteq \widehat{\phi}$, that is, for all maximally consistent sets y, we have $\Box\psi \in y$ implies $\phi \in y$.

We first prove that the set $\{\Box\psi, \neg\phi\}$ is inconsistent. Assume on the contrary that it's not. Then by the *Lindenbaum's Lemma* (2.3.43), it can be extended to a maximally consistent set. Calling it Σ, as $\Box\psi \in \Sigma$, and as for an arbitrary maximally consistent set y, if $\Box\psi \in y$ then $\phi \in y$, so we get $\phi \in \Sigma$. Hence, $\phi, \neg\phi \in \Sigma$. As $\neg(p \wedge \neg p)$ is a propositional tautology, by uniform substitution we get $\vdash \neg(\phi \wedge \neg\phi)$. Hence, we get that Σ is not consistent, which is a contradiction. So, $\{\Box\psi, \neg\phi\}$ is inconsistent, and thus we have the following three possibilities:

1. If $\vdash \neg(\Box\psi \wedge \neg\phi)$ holds, then as $\neg(p \wedge \neg q) \to p \to q$ is a propositional tautology, by Uniform substitution and Modus Ponens we get $\vdash \Box\psi \to \phi$.

2. If $\vdash \neg\Box\psi$ holds, then as $\neg p \to p \to q$ is a propositional tautology, by Uniform substitution and Modus Ponens, we get $\vdash \Box\psi \to \phi$.

3. If $\vdash \neg(\neg\phi)$ holds, then as $\neg\neg q \to p \to q$ is a propositional tautology, by Uniform substitution and Modus Ponens, we get $\vdash \Box\psi \to \phi$.

Hence, in all the cases we get $\vdash \Box\psi \to \phi$. Next, using $\vdash \Box\psi \to \phi$, we prove that $\vdash \Box\psi \to \Box\phi$.

1.	$\vdash \Box\psi \to \phi$	
2.	$\vdash \Box(\Box\psi \to \phi)$	(Generalization: 1)
3.	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$	(K)
4.	$\vdash \Box(\Box\psi \to \phi) \to \Box\Box\psi \to \Box\phi$	(Uniform substitution: 3)
6.	$\vdash \Box\Box\psi \to \Box\phi$	(Modus Ponens: 2,4)
7.	$\vdash \Box p \to \Box\Box p$	(4)
8.	$\vdash \Box\psi \to \Box\Box\psi$	(Uniform substitution: 7)
9.	$\vdash (p \to q) \to (q \to r) \to (p \to r)$	(Propositional tautology)
10.	$\vdash (\Box\psi \to \Box\Box\psi) \to (\Box\Box\psi \to \Box\phi) \to (\Box\psi \to \Box\phi)$	(Uniform substitution: 9)
11.	$\vdash (\Box\Box\psi \to \Box\phi) \to (\Box\psi \to \Box\phi)$	(Modus Ponens: 8,10)
12.	$\vdash \Box\psi \to \Box\phi$	(Modus Ponens: 6,11)

As **S4** $\subseteq x$ by Lemma 2.3.42, we get $\Box\psi \to \Box\phi \in x$. As x is closed under Modus Ponens by Lemma 2.3.42, and as $\Box\psi \in x$, so we get $\Box\phi \in x$, that is, $x \in \widehat{\Box\phi}$. Thus, $M^{\mathcal{L}}, x \models \Box\phi$ implies $x \in \widehat{\Box\phi}$. Together we get $M^{\mathcal{L}}, x \models \Box\phi$ iff $x \in \widehat{\Box\phi}$. Thus by induction, for all formulas ϕ and all $x \in X^{\mathcal{L}}$, we have $M^{\mathcal{L}}, x \models \phi$ iff $x \in \widehat{\phi}$. $\Box$

Now we know exactly which formulas are true at an MCS in the canonical topo-model. Thus, to show that a formula is falsifiable on the canonical topo-model it is enough to show that its negation is contained in some MCS. The next corollary uses the above fact to prove the completeness.

Corollary 5.2.2 (Completeness). *S4 is complete with respect to the class of all topological spaces.*

Proof. To prove completeness we use Remark 4.0.4, that if $\phi \notin \mathbf{S4}$, then there is a topo-model M and a point x in the topo-model such that $M, x \nvDash \phi$. Assume $\phi \notin \mathbf{S4}$. We claim that $\{\neg\phi\}$ is consistent. Assume on the contrary that it is not. Then $\vdash \neg\neg\phi$. As $\neg\neg p \to p$ is a propositional tautology, so by Uniform substitution and Modus Ponens, we get that $\vdash \phi$ which contradicts the assumption that $\phi \notin \mathbf{S4}$. So $\{\neg\phi\}$ is consistent.

By the *Lindenbaum's Lemma* (2.3.43), $\{\neg\phi\}$ can be extended to a maximally consistent set. Let it be Σ. Then as $\neg\phi \in \Sigma$, by the Truth Lemma (5.2.1), we have $M^{\mathcal{L}}, \Sigma \vDash \neg\phi$, which implies $M^{\mathcal{L}}, \Sigma \nvDash \phi$. Hence, **S4** is complete with respect to the class of all topological spaces. □

The canonical topo-model that we have described is an infinite topo-model. To see this, consider the following. We have countably many propositional variables. Let us call them $a_1, a_2, a_3, \ldots$ for convenience. It can be shown that the sets

$$A_1 = \{\neg a_1, a_2, a_3, \ldots\},$$

$$A_2 = \{a_1, \neg a_2, a_3, \ldots\},$$

$$A_3 = \{a_1, a_2, \neg a_3, \ldots\},$$

$$\vdots$$

are all consistent. Also, no two of them can be extended to the same MCS. For $m \neq n$, if A_m and A_n can be extended to the same MCS Σ, then as $\neg a_m \in A_m$, we get $\neg a_m \in \Sigma$. But as $a_m \in A_n$, we get $a_m \in \Sigma$. Thus, both $a_m, \neg a_m \in \Sigma$, which contradicts the fact that Σ is consistent. So, each A_i can only be to a unique MCS.

Thus, we have infinitely many states in our canonical topo-model. One question which may arise is whether **S4** is complete with respect to the class of all finite topological spaces. The next section answers this in the affirmative.

5.3 Finite Topo-model Property of S4

We want to prove that **S4** is complete with respect to the class of all finite topological spaces, that is, every formula which is not a theorem of **S4**, is falsifiable on a finite topo-model. For a formula ϕ, if $\phi \notin$ **S4**, then as **S4** is complete with respect to the class of all reflexive transitive frames, so ϕ is falsifiable on a reflexive transitive model. In this section, we will see that from that model we can extract a finite topo-model, such that ϕ is falsifiable on the topo-model too. Thus, then $\neg\phi$ is satisfiable on a finite topo-model, and consequently ϕ is falsifiable on a finite topo-model. This gives us the needed completeness result. The detailed proof follows.

Proposition 5.3.1. *S4 is complete with respect to the class of finite topological spaces.*

Proof. Let $\psi \notin$ **S4**. Then, ψ is falsifiable on a reflexive transitive frame by Proposition 2.3.50. Let ϕ denote the formula $\neg\psi$. So, we have a reflexive transitive model $\mathfrak{M} = (X, R, V)$, and some $x \in X$, such that $\mathfrak{M}, x \vDash \phi$.

Let Σ denote the set of formulas containing only ϕ and all its subformulas. We consider a model $\mathfrak{M}_\Sigma = (X_\Sigma, R^t, V_\Sigma)$ which is a filtration of the model $\mathfrak{M}$ through Σ, and where R^t is defined as the following: $R^t|x_1||x_2|$ iff for all formulas λ, if $\Diamond\lambda \in \Sigma$ and $\mathfrak{M}, x_2 \vDash \lambda \vee \Diamond\lambda$, then $\mathfrak{M}, x_1 \vDash \Diamond\lambda$.

As $\mathfrak{M}$ is a reflexive transitive model, so by Theorem 2.3.33 we get that $\mathfrak{M}_\Sigma$ is a transitive model, and by Remark 2.3.32 we get that $\mathfrak{M}_\Sigma$ is a reflexive model. By Proposition 2.3.29, $\mathfrak{M}_\Sigma$ is a finite model, and by the Filtration Theorem (2.3.30) we get that $\mathfrak{M}_\Sigma, |x| \vDash \phi$. Together, we get that ϕ is true on a finite reflexive transitive model.

Let M be the topo-model formed from $\mathfrak{M}_\Sigma$ by taking the upsets as open sets. Then as $\mathfrak{M}_\Sigma, |x| \vDash \phi$, so by Lemma 4.3.6 we have $M, |x| \vDash \phi$. Hence,

ϕ is true on a finite topo-model, which implies ψ is falsifiable on a finite topological space. Thus, **S4** is complete with respect to the class of all finite topological spaces. $\square$

What this says is that the set of formulas valid on the class of all topological spaces, and the set of formulas valid on the class of all finite topological spaces is the same. Essentially, no modal formula can help us in distinguishing between a finite and an infinite space. Our interpretation is not strong enough to capture the property of finiteness, and we cannot distinguish the two different classes of spaces based on the logic alone.

In the next chapter we will see another soundness and completeness result of **S4** with respect to a different class of spaces. All these results together highlight the limitations of the topological interpretation.

6. THE MCKINSEY-TARSKI THEOREM

In this chapter, we see the McKinsey-Tarski theorem which is another soundness and completeness result for **S4**. It was first proved in 1944 by J.C.C. McKinsey and Alfred Tarski ([1]). Recall that a space is said to be separable if it has a countable dense subset, and a space is said to be dense-in-itself if every point is a limit point. The McKinsey-Tarski Theorem states that **S4** is sound and complete with respect to the class of all separable dense-in-itself metric spaces.

The McKinsey-Tarski theorem is stronger than the soundness and completeness result of **S4** with respect to the class of all topological spaces. To see this, consider the following discussion. For a formula not in **S4**, by the McKinsey-Tarski theorem, it is falsifiable on a dense-in-itself separable metric space. As every dense-in-itself separable metric space is a topological space, so we get that every formula not in **S4** is falsifiable on a topological space. Thus, the McKinsey-Tarski theorem implies that **S4** is complete with respect to the class of all topological spaces.

We will be seeing a recent proof of the McKinsey-Tarski theorem (described in [10]). The crux of the proof is to falsify a formula which is not in **S4**, on a topo-model based on $\mathbb{Q}$ equipped with the euclidean topology. As $\mathbb{Q}$ equipped with the euclidean topology is a separable dense-in-itself metric space, so we get the McKinsey-Tarski theorem. To understand the proof in detail, we need a few definitions and results first.

Definition 6.0.1. A frame (X, R) is said to be *rooted* if there exists an $r \in X$ such that for each $x \in X$, we have rRx. $\dashv$

Thus, a frame is said to be rooted if there exists a 'root'. We will call reflexive transitive frames as **S4**-frames.

Proposition 6.0.2. ***S4*** *is complete with respect to the class of all finite rooted* ***S4****-frames.*

Proof. In the proof of Proposition 5.3.1, we prove that if $\psi \notin \mathbf{S4}$, then there is a finite **S4**-frame (X_Σ, R^t), a valuation V_Σ on X_Σ, and $|x| \in X_\Sigma$ such that for the model $\mathfrak{M}_\Sigma = (X_\Sigma, R^t, V_\Sigma)$, we have $\mathfrak{M}_\Sigma, |x| \nvDash \psi$. Let

$$R^t(|x|) = \{y \in X_\Sigma \mid |x| R^t y\}.$$

Let R denote the restriction of R^t to $R^t(|x|)$, and let V denote the restriction of V_Σ to $R^t(|x|)$. Then $M = (R^t(|x|), R, V)$ is a submodel of $\mathfrak{M}_\Sigma$.

By definition of $R^t(|x|)$, we get that $(R^t(|x|), R)$ is a rooted frame, as $|x|$ forms a root.

We claim M is a generated submodel of $\mathfrak{M}_\Sigma$. Let $y \in R^t(|x|)$, and let yRz hold for some $z \in X_\Sigma$. As R is a restriction of R^t, we have yR^tz. As $y \in R^t(|x|)$, we get that xR^ty. As R^t is transitive, we get xR^tz, that is, $z \in R^t(|x|)$. Hence, M is a generated submodel of $\mathfrak{M}_\Sigma$. As $\mathfrak{M}_\Sigma$ is a reflexive transitive model, by Proposition 2.3.24 we get that M is also a reflexive transitive model. Also, as $\mathfrak{M}_\Sigma$ is a finite model, so M is a finite model. By Proposition 2.3.23, we get that $M, |x| \nvDash \psi$. Hence, ϕ is falsifiable on a finite rooted **S4**-frame, and so, **S4** is complete with respect to the class of all finite rooted **S4**-frames. ☐

We will use this completeness result later to prove the completeness of **S4** with respect to $\mathbb{Q}$.

6.1 Dense linearly ordered sets with no endpoints

In this section we prove an isomorphism result which will help us in transferring the completeness result of **S4** with respect to the class of all finite rooted

S4-frames to completeness with respect to $\mathbb{Q}$ equipped with the euclidean topology.

Definition 6.1.1. A frame $(X, <)$ is said to be a *linear order* if it follows the following properties:

- (Transitivity) for all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$, and
- (Trichotomy) for all $x, y \in X$, exactly one of $x < y$, $y < x$, or $x = y$ hold.

A linear order is said to be *dense* if for all $x, y \in X$, if $x < y$, then there exists a $z \in X$ such that $x < z < y$. A linear order is said to have *no endpoints* if for all $x \in X$, there exist $y, z \in X$ such that $y < x < z$. $\dashv$

Example 6.1.2. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$, where $<$ is the usual 'less than', are dense linearly ordered sets with no endpoints. $(\mathbb{N}, <)$, where $<$ is the usual 'less than', is a linearly ordered set, but is neither dense nor has no endpoints. $\dashv$

Definition 6.1.3. For two linearly ordered sets $(A, <_A)$ and $(B, <_B)$, a function $f : A \to B$ is said to be an *order isomorphism* if

- f is bijective, and
- for each $a_1, a_2 \in A$, we have $a_1 <_A a_2$ iff $f(a_1) <_B f(a_2)$.

Two linearly ordered sets $(A, <_A)$ and $(B, <_B)$ are said to be *order isomorphic* if there exists an order isomorphism from one to another. $\dashv$

Example 6.1.4. Any two finite linearly ordered sets with the same number of elements are order isomorphic. $\dashv$

In this section, when we talk about an isomorphism between linearly ordered sets, we mean order isomorphism. The next result proves that every countable dense linearly ordered set with no endpoints is isomorphic to $\mathbb{Q}$ with the usual 'less than' order. The proof uses what is called the *back and forth method*, and can be found in [11].

Proposition 6.1.5. *Any two countable dense linearly ordered sets with no endpoints are isomorphic.*

Proof. Let $(A, <_A)$ and $(B, <_B)$ be two countable dense linearly ordered sets with no endpoints. As an abuse of notation, we will use $<$ both for $<_A$ and $<_B$, and it must understood from the context what $<$ stands for at that particular instance.

It should be noted that a linearly ordered set with no endpoints can't be finite. This is because if it is indeed finite, then there has to be a least and a greatest element, which contradicts the assumption that the set has no endpoints. Hence, both A and B are countably infinite. Thus, the elements of A and B can be enumerated as

$$A = \{a_1, a_2, \ldots\},$$

and

$$B = \{b_1, b_2, \ldots\}.$$

Recall that a partial function f from a set X to a set Y is just a function from a set Z to the set Y, such that $Z \subseteq X$. We inductively define a sequence of partial functions $\{f_i\}_{i \in \mathbb{N}}$ from A to B such that for each $i \in \mathbb{N}$, we have

- $\mathrm{dom}(f_i)$ is finite,
- f_i is injective,
- $f_i \subseteq f_{i+1}$ and
- f_i preserves order both ways, that is, if $a_1, a_2 \in \mathrm{dom}(f_i)$, then $a_1 < a_2$ iff $f_i(a_1) < f_i(a_2)$.

We define

$$f_1 = \{(a_1, b_1)\}.$$

Then $\mathrm{dom}(f_1)$ is finite, f_1 injective, and vacuously f_1 preserves order both ways. Assume that f_k has been defined such that $\mathrm{dom}(f_k)$ is finite, f_k is injective, and f_k preserves order both ways. We construct f_{k+1} in the following way.

- For each $a \in \mathrm{dom}(f_k)$, define

$$f_{k+1}(a) = f_k(a).$$

- If $a_{k+1} \in \mathrm{dom}(f_k)$, we proceed to the next step. Else, as A is linearly ordered, so the elements of $\mathrm{dom}(f_k) \cup \{a_{k+1}\}$ can be written in a sequence of increasing order. We have the following possibilities:

  1. If the sequence is

  $$a'_1 < a'_2 < \ldots < a'_i < a_{k+1} < a'_{i+1} < \ldots < a'_n.$$

  Then as f_k is order preserving by assumption, and as $f_k \subseteq f_{k+1}$, we get $f_{k+1}(a'_i) < f_{k+1}(a'_{i+1})$. As B is dense, so there exists a $b \in B$ such that $f_{k+1}(a'_i) < b < f_{k+1}(a'_{i+1})$. We define $f_{k+1}(a_{k+1})$ to be b.

  2. If the sequence is

  $$a_{k+1} < a'_1 < a'_2 < \ldots < a'_n,$$

  then as f_k preserves order, and as $f_k \subseteq f_{k+1}$, similar to the previous case, we get $f_{k+1}(a'_1) < \ldots < f_{k+1}(a'_n)$. Then, as B has no least point, we can choose a $b \in B$ such that $b < f_{k+1}(a'_1)$. We define $f_{k+1}(a_{k+1})$ to be b.

  3. If the sequence is

  $$a'_1 < a'_2 < \ldots < a'_n < a_{k+1},$$

  as B has no greatest point, similar to the previous case, there exists a $b \in B$ such that $f_{k+1}(a'_n) < b$. We define $f_{k+1}(a_{k+1})$ to be b.

- If $b_{k+1} \in \mathrm{ran}(f_{k+1})$, then we are done. Else, as B is linearly ordered, so the elements in $\mathrm{ran}(f_{k+1}) \cup \{b_{k+1}\}$ can be put in a sequence of increasing order. Here too, we get three cases.

1. If the sequence is of the form

$$b'_1 < \ldots < b'_i < b_{k+1} < b'_{i+1} < \ldots < b'_m,$$

then as f_k preserves order in both the ways, and as $f_k \subseteq f_{k+1}$, we get $f_{k+1}^{-1}(b'_i) < f_{k+1}^{-1}(b'_{i+1})$. As A is dense, so there exists an $a \in A$ such that $f_{k+1}^{-1}(b'_i) < a < f_{k+1}^{-1}(b'_{i+1})$. We define $f_{k+1}(a)$ to be b_{k+1}.

2. If the sequence is

$$b_{k+1} < b'_1 < \ldots < b'_m,$$

then as f_k preserves the order both the ways, and as $f_k \subseteq f_{k+1}$, similar to the previous case, we get $f_{k+1}^{-1}(b'_1) < \ldots < f_{k+1}^{-1}(b'_m)$. Then, as A has no least element, we get that there exists an $a \in A$, such that $a < f_{k+1}^{-1}(b'_1)$. We define $f_{k+1}(a)$ to be b_{k+1}.

3. If the sequence is

$$b'_1 < b'_2 < \ldots < b'_m < b_{k+1},$$

as A has no greatest point, similar to the previous case, there exists an $a \in A$ such that $f_{k+1}^{-1}(b'_m) < a$. We define $f_{k+1}(a)$ to be b_{k+1}.

In this way, we define f_{k+1}.

As no more than one element has been added to $\mathrm{dom}(f_k)$ to form $\mathrm{dom}(f_{k+1})$, and as $\mathrm{dom}(f_k)$ is finite by assumption, so $\mathrm{dom}(f_{k+1})$ is finite. Also, f_{k+1} is injective as the order condition we put in choosing a and b makes sure that $a \neq f_{k+1}^{-1}(b'_j)$, and $b \neq f_{k+1}(a'_l)$ for any $1 \leq j \leq m$, and $1 \leq l \leq n$. Finally, in constructing f_{k+1} we have made sure that f_{k+1} preserves the order both the ways.

Hence, we have constructed a sequence of partial functions $\{f_i\}_{i \in \mathbb{N}}$ from A to B, such that for each $i \in \mathbb{N}$, $\mathrm{dom}(f_i)$ is finite, f_i is injective and f_i preserves the order both the ways. We define

$$f = \bigcup_{i \in \mathbb{N}} f_i.$$

We claim that f is an order isomorphism from A to B.

- For any $a_k \in A$, we have $a_k \in \text{dom}(f_k) \subseteq \text{dom}(f)$. So, $\text{dom}(f) = A$.
- We now prove that f is indeed a function. Assume on the contrary that it is not. Then, there exists some $a \in A$ such that for some $b, b' \in B$ with $b \neq b'$, we have $(a, b), (a, b') \in f$. As $f = \bigcup_{i \in \mathbb{N}} f_i$, so there exist $j, l \in \mathbb{N}$ such that $(a, b) \in f_j$, and $(a, b') \in f_l$. By the inductive process, we have $f_j \subseteq f_{\max\{j,l\}}$, and $f_l \subseteq f_{\max\{j,l\}}$, which implies $(a, b), (a, b') \in f_{\max\{j,l\}}$, which contradicts the fact that $f_{\max\{j,l\}}$ is a partial function. So, our assumption is wrong, and f is indeed a function.
- We now prove that f is injective. For $k, m \in \mathbb{N}$ with $k \neq m$, let there be $a_k, a_m \in A$ such that for some $b \in B$, we have $(a_k, b), (a_m, b) \in f$. Then again there exist $j, l \in \mathbb{N}$ such that $(a_k, b) \in f_j$, and $(a_m, b) \in f_l$, which implies $(a_k, b), (a_m, b) \in f_{\max\{j,l\}}$, which contradicts the fact that $f_{\max\{j,l\}}$ is injective. Hence, f is injective.
- For any $b_k \in B$, as $b_k \in \text{ran}(f_k) \subseteq \text{ran}(f)$, so $\text{ran}(f) = B$. Hence, f is surjective.
- We now claim that f preserves the order both the ways. Let $a_j, a_l \in A$ such that $a_j < a_l$. Then we get $a_j, a_l \in \text{dom}(f_{\max\{j,l\}})$. As $f_{\max\{j,l\}}$ preserves order both ways, we get $f_{\max\{j,l\}}(a_j) < f_{\max\{j,l\}}(a_l)$. As $f_{\max\{j,l\}} \subseteq f$, we get $f(a_j) < f(a_l)$.

  Similarly, for $b_j, b_l \in B$ such that $b_j < b_l$, as $b_j, b_l \in \text{ran}(f_{\max\{j,l\}})$, and as $f_{\max\{j,l\}}$ preserves order both ways, we have $f^{-1}_{\max\{j,l\}}(b_j) < f^{-1}_{\max\{j,l\}}(b_l)$. As $f_{\max\{j,l\}} \subseteq f$, we get $f^{-1}(b_j) < f^{-1}(b_l)$. Hence, f preserves order both ways.

Thus, f is an order isomorphism and any two countable dense linearly ordered sets with no endpoints are isomorphic. $\square$

As $(\mathbb{Q}, <)$ is a countable linearly ordered dense set with no endpoints, so as a consequence of the previous proposition we get that every countable linearly ordered dense set with no endpoints is isomorphic to $(\mathbb{Q}, <)$.

6.2 Homeomorphism result of ordered sets

In this section, we see that we can put a topology on a countable dense linearly ordered set with no endpoints such that it is homeomorphic to $\mathbb{Q}$ with the euclidean topology.

Definition 6.2.1. For a linearly ordered set $(X, <)$, and for each $a, b \in X$, we define

$$(a, b) = \{x \in X \mid a < x < b\}.$$

$\dashv$

Lemma 6.2.2. *If $(X, <)$ is a linearly ordered set with no endpoints then $\{(a, b) \mid a, b \in X\}$ forms a basis.*

Proof. As $(X, <)$ has no endpoints, so for all $x \in X$, there exist $a, b \in X$ such that $a < x < b$, which implies $x \in (a, b) \subseteq X$.

Let $x \in X$ such that $x \in (a, b) \cap (a', b')$ for some $a, b, a', b' \in X$. So $a < x < b$, and $a' < x < b'$. By trichotomy a and a' are comparable, so $\max\{a, a'\}$ exists. Similarly, $\min\{b, b'\}$ exists. Then we get

$$a < \max\{a, a'\} < x < \min\{b, b'\} < b,$$

and

$$a' < \max\{a, a'\} < x < \min\{b, b'\} < b'.$$

Thus

$$x \in (\max\{a, a'\}, \min\{b, b'\}) \subseteq (a, b) \cap (a', b').$$

So, $\{(a, b) \mid a, b \in X\}$ forms a basis for X. $\square$

Definition 6.2.3. We call the topology generated by the above basis as the *order topology* on X. $\dashv$

Example 6.2.4. For the euclidean topology on $\mathbb{Q}$, the set of open intervals forms a basis, which also forms a basis for the order topology on $(\mathbb{Q}, <)$. Thus, the order topology and the euclidean topology on $\mathbb{Q}$ are the same. $\dashv$

We have already proved that there is just one linearly ordered set with no endpoints upto isomorphism. As the order topology is entirely dependent upon the order, it is no surprise that as topological spaces, every linearly ordered set with no endpoints equipped with the order topology is homeomorphic to $\mathbb{Q}$ with the order topology, which in turn is the same as the euclidean topology on $\mathbb{Q}$. The next propositions proves the same.

Proposition 6.2.5. *Every countable dense linearly ordered set with no endpoints is homeomorphic to $\mathbb{Q}$ with the euclidean topology.*

Proof. We prove that every countable dense linearly ordered set with no endpoints is homeomorphic to $\mathbb{Q}$ with the order topology. As both the order topology and euclidean topology are the same on $\mathbb{Q}$, so the result follows.

Let $(X, <)$ be a countable dense linearly ordered set with no endpoints. By Proposition 6.1.5, there exists an order isomorphism $f : X \to \mathbb{Q}$. We claim that this map f is a homeomorphism.

As f is an order isomorphism, hence it is bijective. We now prove that the inverse image of open sets under f is open. It suffices to prove that the inverse image of a basic open set under f is open. Let $(p, q) \subseteq \mathbb{Q}$ be a basic open set. Then

$$\begin{aligned} x \in f^{-1}(p,q) &\Leftrightarrow f(x) \in (p,q) \\ &\Leftrightarrow p < f(x) < q \\ &\Leftrightarrow f^{-1}(p) < x < f^{-1}(q) \quad (f \text{ preserves order both the ways}) \\ &\Leftrightarrow x \in (f^{-1}(p), f^{-1}(q)). \end{aligned}$$

Thus for each $p, q \in \mathbb{Q}$ such that $p < q$, we have $f^{-1}(p,q) = (f^{-1}(p), f^{-1}(q))$. Therefore, f^{-1} maps basic open sets to basic open sets.

Similarly, for each $a, b \in X$ such that $a < b$, we get

$$(f^{-1})^{-1}(a,b) = f(a,b) = (f(a), f(b)).$$

Thus the inverse image of f^{-1} takes basic open sets to basic open sets. Therefore, f is a homeomorphism, and X equipped with the order topology and $\mathbb{Q}$ equipped with the order topology are homeomorphic. $\square$

6.3 S4 frames as an interior image of $\mathbb{Q}$

In this section, we prove that every finite rooted **S4**-frame can be thought of as the image of an interior map from $\mathbb{Q}$. To prove the result, we first need to construct a set which is homeomorphic to $\mathbb{Q}$.

6.3.1 The Construction

Let $\mathfrak{F} = (W, T)$ be a finite rooted **S4**-frame. We construct a topological space (Σ, τ) such that there is an onto interior map from Σ to W where W is equipped with the Alexandroff topology induced from the relation T.

Construction of Σ

Let

$$\Sigma = \{\text{finite sequences on non-zero integers}\}.$$

Let Λ denote the empty sequence. Then Σ contains Λ. Some other elements of Σ are $1, (-3)43, 2(-1)1, 2$, and $21(-3)9(-8)73(-9)8126$. To define the topology τ on Σ, we first construct a linearly ordered set X. Using the order on X, we put an order on Σ such that it forms a linearly ordered dense set with no endpoints. Then, the topology τ is defined to be the order topology on Σ.

Construction of X

Let

$$L = \mathbb{R} \times (-\infty, 0).$$

So, L is the lower half plane without the x-axis. Let

$$L_0 = \mathbb{R} \times \{0\}.$$

So on the xy-plane, L_0 represents the x-axis. Let

$$\mathcal{C} = \{[a, b] \mid a, b \in \mathbb{R}\}.$$

So, $\mathcal{C}$ is the set of all closed and bounded intervals in $\mathbb{R}$. Consider the following maps:

1. $\alpha : L \to \mathcal{C}$ given by

$$\alpha((x, y)) = [x + y, x - y].$$

   Identifying $\mathbb{R}$ with L_0, what α does is, it maps each point in L to the closed and bounded interval which forms the edge of the unique isosceles right triangle, with the right angled vertex at (x, y).

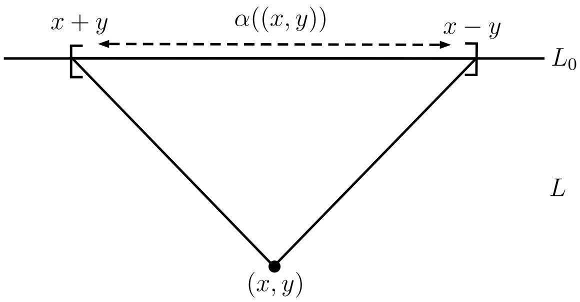

Fig. 6.1: The map α.

2. $\beta : \mathcal{C} \to L$ given by

$$\beta([a, b]) = (\frac{a + b}{2}, \frac{a - b}{2}).$$

Identifying $\mathbb{R}$ with L_0, what β does is, it maps each closed and bounded interval to the point which forms the right angled vertex of the unique right isosceles triangle with the hypotenuse as the interval $[a, b]$.

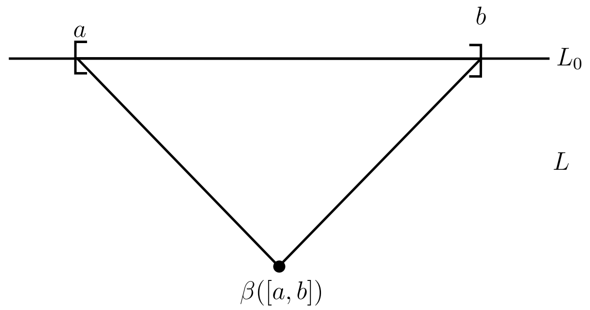

Fig. 6.2: The map β.

Then, $\alpha \circ \beta : \mathcal{C} \to \mathcal{C}$ is given by

$$\begin{aligned}\alpha \circ \beta([a, b]) &= \alpha\left(\frac{a+b}{2}, \frac{a-b}{2}\right) \\ &= \left[\frac{a+b}{2} + \frac{a-b}{2}, \frac{a+b}{2} - \frac{a-b}{2}\right] \\ &= [a, b].\end{aligned}$$

Thus, $\alpha \circ \beta$ is the identity map on $\mathcal{C}$. Similarly, $\beta \circ \alpha$ is the identity map on L.

So, the two sets $\mathcal{C}$ and L are in one-to-one correspondence with each other. We will use all these maps, and Σ to define the needed set X. For that, we first define a map $h : \Sigma \to L$ in an inductive manner. Define

$$h(\Lambda) = (0, -1).$$

To define h on the other elements of Σ, we need a few notations. Let $\sigma \in \Sigma$, and let z be a non-zero integer. Let $\sigma.z$ denote the finite sequence formed by concatenating the sequence σ with z. For example, 123.5 denotes the string 1235, and Λ.2 denotes the string 2.

Let us assume that $h(\sigma)$ is defined, and let $h(\sigma) = (x, y) = p$. Using $h(\sigma)$, we will define $h(\sigma.z)$ for each non-zero integer z. We introduce some more notations which we will be needing to define $h(\sigma.z)$. For each $n \in \mathbb{N}$, let

$$I^p_{-n} = \left[x + \frac{y}{2^{n-1}}, x + \frac{y}{2^n}\right],$$

and

$$I^p_n = \left[x - \frac{y}{2^n}, x - \frac{y}{2^{n-1}}\right].$$

Thus,

$$I^p_{-1} = \left[x + y, x + \frac{y}{2}\right],$$

$$I^p_1 = \left[x - \frac{y}{2}, x - y\right],$$

$$I^p_{-2} = \left[x + \frac{y}{2}, x + \frac{y}{4}\right],$$

and

$$I^p_2 = \left[x - \frac{y}{4}, x - \frac{y}{2}\right].$$

We define

$$h(\sigma.z) = \beta(I^p_z).$$

In this manner, we define h inductively on Σ. As each $\sigma \in \Sigma$ can be constructed in an inductive manner from Λ by concatenating a non-zero integer z finitely many times, and in a unique manner, hence h is a well-defined function. Consider the projection map $\pi_1 : L \to \mathbb{R}$ given by

$$\pi_1((x, y)) = x.$$

We define

$$X = \pi_1 \circ h(\Sigma).$$

Identifying $\mathbb{R}$ with L, we can visualize the construction as shown in Figure 6.3.

As $X \subseteq \mathbb{R}$, a natural order relation $<$ is induced on X from $\mathbb{R}$. The properties of trichotomy and transitivity hold for elements in X, by virtue of them holding in $\mathbb{R}$. Hence, $(X, <)$ also forms a linearly ordered set.

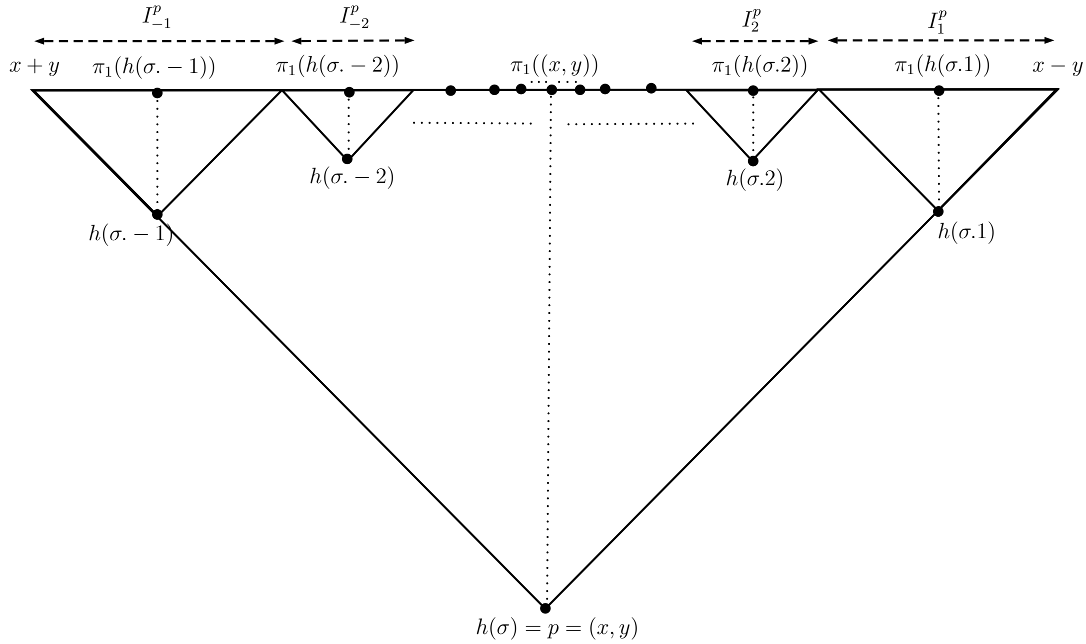

Fig. 6.3: The map h.

We define a relation $<$ on Σ as follows. For $\sigma, \lambda \in \Sigma$,

$$\sigma < \lambda \text{ iff } \pi_1(h(\sigma)) < \pi_1(h(\lambda)).$$

Thus, the order on Σ is the pullback of the order defined on X under the map $\pi_1 \circ h$. We claim that $(\Sigma, <)$ forms a countably infinite linearly ordered dense set with no endpoints.

Σ is countably infinite

Σ can be partitioned into the sets $S_0, S_1, \ldots$, where for a non-negative integer m, S_m contains all finite sequences of length m. For example,

$$S_0 = \{\Lambda\},$$

$$S_1 = \{-1, -2, \ldots, 1, 2, \ldots\},$$

$$S_2 = \{-1(-1), -1(-2), \ldots, (-1)1, (-1)2, \ldots, -2(-1), -2(-2), \ldots\}$$

$$\vdots$$

and so on. Then

$$\Sigma = \bigcup_{m \in \mathbb{N} \cup \{0\}} S_m.$$

S_0 is a singleton, and for $m \in \mathbb{N}$, S_m can be identified with $(\mathbb{Z} - \{0\})^m$. As $\mathbb{Z} - \{0\}$ is countably infinite, hence $(\mathbb{Z} - \{0\})^m$ is countably infinite for each $m \in \mathbb{N}$. Thus, each S_m is countable, and hence, Σ is a countably infinite.

$(\Sigma, <)$ is a linearly ordered set

To establish this, we will use the fact that X is a subset of $\mathbb{R}$ and $(\mathbb{R}, <)$ forms a linearly ordered set. For each $x, y, z \in X$, the properties of trichotomy and transitivity hold as they hold in $(\mathbb{R}, <)$. Thus, $(X, <)$ forms a linearly ordered set under the induced order $<$.

For transitivity in $(\Sigma, <)$, let $\sigma, \tau, \lambda \in \Sigma$ such that $\sigma < \tau$, and $\tau < \lambda$. By the definition of $<$ on Σ, we have $\pi_1(h(\sigma)) < \pi_1(h(\tau))$, and $\pi_1(h(\tau)) < \pi_1(h(\lambda))$. As $\pi_1(h(\sigma)), \pi_1(h(\tau)), \pi_1(h(\lambda)) \in X$, and as $<$ is a transitive relation on X, we get that $\pi_1(h(\sigma)) < \pi_1(h(\lambda))$, which implies $\sigma < \lambda$. Hence, $<$ is transitive on Σ.

For trichotomy, let $\sigma, \lambda \in \Sigma$. Then $\pi_1(h(\sigma)), \pi_1(h(\lambda)) \in X$. As $(X, <)$ is a linearly ordered set, by trichotomy exactly one of the following happens:

1. $\pi_1(h(\sigma)) < \pi_1(h(\lambda))$,
2. $\pi_1(h(\lambda)) < \pi_1(h(\sigma))$, or
3. $\pi_1(h(\sigma)) = \pi_1(h(\lambda))$.

By the definition of $<$ on Σ, for the first case we get $\sigma < \lambda$, and for the second case we get $\lambda < \sigma$. Next, we prove that the map $\pi_1 \circ h$ is injective, and hence the third case implies that $\sigma = \lambda$.

$\pi_1 \circ h$ is an injective map

Before proving the injectivity, we will prove a series of lemmas which can be found in [12]. Recall that by $\mathbb{Z}^*$, we mean $\mathbb{Z} - \{0\}$.

Lemma 6.3.1. *For any $\sigma \in \Sigma$, and $z_1, z_2 \in \mathbb{Z}^*$ such that $z_1 \neq z_2$, the set $I_{z_1}^{h(\sigma)} \cap I_{z_2}^{h(\sigma)}$ has at most one element.*

Proof. Let $\sigma \in \Sigma$, and $z \in \mathbb{Z}^*$. Let $h(\sigma) = (x, y)$. Consider the case when $z > 0$. Then

$$I_z^{h(\sigma)} = \left[x - \frac{y}{2^z}, x - \frac{y}{2^{z-1}}\right].$$

Then for all $z' \in \mathbb{Z}^*$ such that $z' < 0$, we have

$$I_{z'}^{h(\sigma)} = \left[x + \frac{y}{2^{-z-1}}, x + \frac{y}{2^{-z}}\right].$$

As for each $p \in I_z^{h(\sigma)}$, we have $p > x$, and as for each $q \in I_{z'}^{h(\sigma)}$, we have $q < x$, so we get

$$I_z^{h(\sigma)} \cap I_{z'}^{h(\sigma)} = \emptyset.$$

If $z - 1 \in \mathbb{Z}^*$, then we have

$$I_{z-1}^{h(\sigma)} = \left[x - \frac{y}{2^{z-1}}, x - \frac{y}{2^{z-2}}\right].$$

Thus, we get

$$I_z^{h(\sigma)} \cap I_{z-1}^{h(\sigma)} = \left\{x - \frac{y}{2^{z-1}}\right\}.$$

Similarly,

$$I_z^{h(\sigma)} \cap I_{z+1}^{h(\sigma)} = \left\{x - \frac{y}{2^{z}}\right\}.$$

For any $k > 1$, we have

$$I_{z+k}^{h(\sigma)} = \left[x - \frac{y}{2^{z+k}}, x - \frac{y}{2^{z+k-1}}\right].$$

As $k > 1$, we have $k - 1 > 0$, and hence,

$$I_{z+k}^{h(\sigma)} \cap I_z^{h(\sigma)} = \emptyset.$$

Thus, for $z > 0$, and for any $z' \in \mathbb{Z}^*$ such that $z \neq z'$, we get that $I_z^{h(\sigma)} \cap I_{z'}^{h(\sigma)}$ has at most one element. Similarly, we get the result for $z < 0$. □

Lemma 6.3.2. *Let $\sigma, \tau \in \Sigma$. We define a relation R on Σ given by*

$$\sigma R \tau \text{ iff } \sigma \text{ is a proper initial substring of } \tau.$$

Then the following statements are equivalent:

(1) $\sigma R \tau$.

(2) $\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma)$.

(3) $\alpha \circ h(\tau) \subseteq \alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\}$.

(4) $Int(\alpha \circ h(\tau)) \subsetneq Int(\alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\})$.

Proof. We first prove that (1) and (2) are equivalent. Let $\sigma R \tau$. So $\tau = \sigma.z_1.\cdots.z_k$ for some $z_1, \cdots, z_k \in \mathbb{Z}^*$ and $k > 0$.

We claim that $\alpha \circ h(\sigma.z_1) \subsetneq \alpha \circ h(\sigma)$. We have

$$\alpha \circ h(\sigma.z_1) = \alpha(\beta(I_{z_1}^{h(\sigma)})),$$

and as $\alpha = \beta^{-1}$, so we get

$$\alpha \circ h(\sigma.z_1) = I_{z_1}^{h(\sigma)}.$$

Let $h(\sigma) = (x, y)$. Then $\alpha \circ h(\sigma) = [x + y, x - y]$, and we get

$$I_{z_1}^{h(\sigma)} \subsetneq \alpha \circ h(\sigma),$$

for any $z \in \mathbb{Z}^*$. Hence,

$$\alpha \circ h(\sigma.z_1) \subsetneq \alpha \circ h(\sigma).$$

Continuing in this manner, we get

$$\alpha \circ h(\sigma.z_1.\cdots.z_m) \subsetneq \alpha \circ h(\sigma.z_1.\cdots.z_{m-1})$$

for each $m \leq k$. Together, we get $\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma)$. Thus, (1) $\Rightarrow$ (2).

Now, we prove (2) $\Rightarrow$ (1). To prove this, we prove the contra-positive, that the negation of (1) $\Rightarrow$ the negation of (2). Let $(\sigma, \tau) \notin R$. We have the following cases:

- $\tau R \sigma$.
- $(\tau, \sigma) \notin R$.

If $\tau R \sigma$, then we have $\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma)$ because (1) $\Rightarrow$ (2). So, it is **not** the case that $\alpha \circ h(\sigma) \subsetneq \alpha \circ h(\tau)$.

To handle the next case, we need a few notations. For any $\lambda \in \Sigma$, let $l(\lambda)$ denote the length of the finite sequence. For example, $l(\Lambda) = 0$, $l(123) = 3$, and $l(3(-6)2(-4)56(-5)23(-7)26) = 12$. For any $\lambda \in \Sigma$, and for any positive integer $k \leq l(\lambda)$, let λ_k denote the k-th entry of λ. For example $(123)_2 = 2$, and $(3(-6)2(-4)56(-5)23(-7)26)_{10} = -7$.
Consider the case $(\tau, \sigma) \notin R$. Then, there exists a longest $\rho \in \Sigma$ such that $\rho R \sigma$ and $\rho R \tau$. Also, then $\rho \neq \tau$ and $\rho \neq \sigma$, else $\tau R \sigma$ and $\sigma R \tau$, respectively which contradicts our assumption. (Note that $\rho = \Lambda$ is also possible.)

Then as $\rho \neq \sigma$, we have that $\rho.\sigma_{l(\rho)+1}$ is an initial substring (not necessarily proper) of σ. If it is a proper initial substring, from (1) $\Rightarrow$ (2), we get

$$\alpha \circ h(\sigma) \subsetneq \alpha \circ h(\rho.\sigma_{l(\rho)+1}).$$

If it is an initial substring but not proper, then $\sigma = \rho.\sigma_{l(\rho)+1}$, which implies

$$\alpha \circ h(\sigma) = \alpha \circ h(\rho.\sigma_{l(\rho)+1}).$$

In both the cases, we get

$$\alpha \circ h(\sigma) \subseteq \alpha \circ h(\rho.\sigma_{l(\rho)+1}).$$

By a similar reasoning, we get

$$\alpha \circ h(\tau) \subseteq \alpha \circ h(\rho.\tau_{l(\rho)+1}).$$

Thus,

$$\alpha \circ h(\sigma) \cap \alpha \circ h(\tau) \subseteq \alpha \circ h(\rho.\sigma_{l(\rho)+1}) \cap \alpha \circ h(\rho.\tau_{l(\rho)+1}).$$

By Lemma 6.3.1, $\alpha \circ h(\rho.\sigma_{l(\rho)+1}) \cap \alpha \circ h(\rho.\tau_{l(\rho)+1})$ contains at most one point, and as $\alpha \circ h(\tau)$ is a closed and bounded interval, so it contains more than one point. Thus, it is **not** the case that $\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma)$. Thus, (2) $\Rightarrow$ (1).

Now, we prove that (1) $\Rightarrow$ (3). Let us assume $\sigma R \tau$. Then, the finite sequence $\sigma.\tau_{l(\sigma)+1}$ is an initial substring of τ (not necessarily proper). If it is an initial substring but not proper, we have

$$\alpha \circ h(\sigma.\tau_{l(\sigma)+1}) = \alpha \circ h(\tau),$$

and if it is a proper initial substring, from (1) $\Rightarrow$ (2), we have

$$\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma.\tau_{l(\sigma)+1}).$$

In any case,

$$\alpha \circ h(\tau) \subseteq \alpha \circ h(\sigma.\tau_{l(\sigma)+1}).$$

Let, $h(\sigma) = (x, y)$. Then, $\pi_1 \circ h(\sigma) = x$. By the definition of $I_z^{h(\sigma)}$, for any $z \in \mathbb{Z}^*$, we have $x \notin I_z^{h(\sigma)}$. Thus,

$$\pi_1 \circ h(\sigma) \notin \alpha \circ h(\sigma.\tau_{l(\sigma)+1}),$$

which implies

$$\alpha \circ h(\sigma.\tau_{l(\sigma)+1}) - \{\pi_1 \circ h(\sigma)\} = \alpha \circ h(\sigma.\tau_{l(\sigma)+1}).$$

As $\alpha \circ h(\tau) \subseteq \alpha \circ h(\sigma.\tau_{l(\sigma)+1})$, we get

$$\alpha \circ h(\tau) - \{\pi_1 \circ h(\sigma)\} = \alpha \circ h(\tau).$$

By assumption $\sigma R \tau$, and hence, from (1) $\Rightarrow$ (2), we get

$$\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma).$$

Thus,

$$\alpha \circ h(\tau) - \{\pi_1 \circ h(\sigma)\} \subsetneq \alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\},$$

which implies

$$\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\}.$$

Thus, (1) $\Rightarrow$ (3).

Now we prove (3) $\Rightarrow$ (4). Let us assume (3) holds, that is,

$$\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\}.$$

Let $h(\sigma) = (x, y)$. Then $\alpha \circ h(\sigma) = [x + y, x - y]$, and $\pi_1 \circ h(\sigma) = x$. Thus

$$\alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\} = [x + y, x) \cup (x, x - y].$$

Let $h(\tau) = (u, v)$. Then $\alpha \circ h(\tau) = [u + v, u - v]$. Thus by the assumption (3), we have

$$[u + v, u - v] \subsetneq [x + y, x) \cup (x, x - y].$$

Thus $[u + v, u - v] \subsetneq [x + y, x)$, or $[u + v, u - v] \subsetneq (x, x - y]$. Therefore $(u + v, u - v) \subsetneq (x + y, x)$, or $(u + v, u - v) \subsetneq (x, x - y)$. Hence,

$$(u + v, u - v) \subsetneq (x + y, x) \cup (x, x - y).$$

So,

$$\mathrm{Int}(\alpha \circ h(\tau)) \subsetneq \mathrm{Int}(\alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\}).$$

Thus, (3) $\Rightarrow$ (4).

Now, we prove (4) $\Rightarrow$ (2). Let us assume (4) holds, that is,

$$\mathrm{Int}(\alpha \circ h(\tau)) \subsetneq \mathrm{Int}(\alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\}).$$

We first assume $\tau = \sigma$. Let $h(\sigma) = (x, y)$. Then $\alpha \circ h(\sigma) = [x, y]$. So by assumption, we get

$$(x + y, x - y) \subsetneq (x + y, x - y) - \{x\},$$

which is a contradiction. So $\tau \neq \sigma$. As $\alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\} \subsetneq \alpha \circ h(\sigma)$, so

$$\mathrm{Int}(\alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\}) \subseteq \mathrm{Int}(\alpha \circ h(\sigma)).$$

Using (3), we get

$$\mathrm{Int}(\alpha \circ h(\tau)) \subsetneq \mathrm{Int}(\alpha \circ h(\sigma)).$$

Let $h(\tau) = (u, v)$. Then $\alpha \circ h(\tau) = [u+v, u-v]$. As $\mathrm{Int}(\alpha \circ h(\tau)) \subsetneq \mathrm{Int}(\alpha \circ h(\sigma))$, we have

$$(u + v, u - v) \subsetneq (x + y, x - y),$$

and so

$$[u + v, u - v] \subsetneq [x + y, x - y].$$

Thus,

$$\alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma).$$

Hence, (4) $\Rightarrow$ (2), and all the statements are equivalent. □

Now we prove the injectivity of the function $\pi_1 \circ h$, which will finally establish the trichotomy of the relation $<$ on Σ.

Proposition 6.3.3. *The function $\pi_1 \circ h : \Sigma \to \mathbb{R}$ is injective.*

Proof. Let $\sigma, \tau \in \Sigma$ be distinct. We have the following possibilities.

1. $\sigma R \tau$.

2. $\tau R \sigma$.

3. Neither $\sigma R \tau$ nor $\tau R \sigma$.

In the first case by Lemma 6.3.2, we have

$$\pi_1 \circ h(\tau) \in \alpha \circ h(\tau) \subsetneq \alpha \circ h(\sigma) - \{\pi_1 \circ h(\sigma)\}.$$

So $\pi_1 \circ h(\tau) \neq \pi_1 \circ h(\sigma)$. Similarly, for the second case we get $\pi_1 \circ h(\tau) \neq \pi_1 \circ h(\sigma)$. For the last case, let ρ be the longest string such that both $\rho R\tau$ and $\rho R\sigma$ hold. Note that $\rho \neq \sigma$ and $\rho \neq \tau$, else the assumption that neither $\sigma R\tau$ nor $\tau R\sigma$ gets violated. Then

$$\pi_1 \circ h(\sigma) \in \mathrm{Int}(\alpha \circ h(\sigma)),$$

since $\pi_1 \circ h(\sigma)$ is the midpoint of $\alpha \circ h(\sigma)$. Similarly, for τ we get

$$\pi_1 \circ h(\tau) \in \mathrm{Int}(\alpha \circ h(\tau)).$$

By definition of ρ, we have $\rho.\sigma_{l(\rho)+1} \neq \rho.\sigma_{l(\rho)+1}$.

As $\rho.\sigma_{l(\rho)+1}$ is an initial substring (not necessarily proper) of σ, we get

$$\pi_1 \circ h(\sigma) \in \mathrm{Int}(\alpha \circ h(\sigma)) \subseteq \mathrm{Int}(\alpha \circ h(\rho.\sigma_{l(\rho)+1})).$$

Similarly,

$$\pi_1 \circ h(\tau) \in \mathrm{Int}(\alpha \circ h(\tau)) \subseteq \mathrm{Int}(\alpha \circ h(\rho.\tau_{l(\rho)+1})).$$

Now, both $\alpha \circ h(\rho.\tau_{l(\rho)+1})$ and $\alpha \circ h(\rho.\sigma_{l(\rho)+1})$ are closed and bounded intervals, and if they intersect, they intersect only at one of each of their endpoints. $\mathrm{Int}(\alpha \circ h(\rho.\tau_{l(\rho)+1}))$ and $\mathrm{Int}(\alpha \circ h(\rho.\sigma_{l(\rho)+1}))$ are open bounded intervals, and hence, they can't intersect. So,

$$\mathrm{Int}(\alpha \circ h(\rho.\sigma_{l(\rho)+1})) \cap \mathrm{Int}(\alpha \circ h(\rho.\tau_{l(\rho)+1})) = \emptyset.$$

Therefore $\pi_1 \circ h(\sigma) \neq \pi_1 \circ h(\tau)$. Hence, in all the cases, we get $\pi_1 \circ h(\sigma) \neq \pi_1 \circ h(\tau)$, which implies that $\pi_1 \circ h$ is injective. □

$(\Sigma, <)$ is dense

Let $\sigma, \lambda \in X$ such that $\sigma < \lambda$. This implies $\pi_1(h(\sigma)) < \pi_1(h(\lambda))$. We claim there exists $z \in \mathbb{Z}$ such that $z > 0$ and $\pi_1(h(\sigma)) < \pi_1(h(\sigma.z)) < \pi_1(h(\lambda))$. Let $z \in \mathbb{Z}$ be arbitrary such that $z > 0$. If $h(\sigma) = (x, y)$, then

$$\pi_1(h(\sigma.z)) = \pi_1(\beta(I_z^{(x,y)})) = x - \frac{3y}{2^{z+1}}.$$

Thus

$$\pi_1(h(\sigma.z)) - \pi_1(h(\sigma)) = -\frac{3y}{2^{z+1}},$$

which implies

$$\pi_1(h(\sigma.z)) = \pi_1(h(\sigma)) - \frac{3y}{2^{z+1}}.$$

So we can choose a large enough z_0, such that

$$\pi_1(h(\sigma)) - \frac{3y}{2^{z_0+1}} < \pi_1(h(\lambda)).$$

(Note that $y < 0$.) So, there exists $z_0 > 0$ such that $\pi_1(h(\sigma)) < \pi_1(h(\sigma.z_0)) < \pi_1(h(\lambda))$, which implies $\sigma < \sigma.z_0 < \lambda$. Hence, $(\Sigma, <)$ is dense.

$(\Sigma, <)$ has no endpoints

Let $\sigma \in \Sigma$. We claim $\sigma.-1 < \sigma < \sigma.1$. Let $h(\sigma) = (x, y)$. We have

$$\pi_1(h(\sigma.1)) = \pi_1(\beta(I_1^{(x,y)})) = \pi_1(\beta(\left[x - \frac{y}{2}, x - y\right])) = x - \frac{3y}{4},$$

and

$$\pi_1(h(\sigma.-1)) = \pi_1(\beta(I_{-1}^{(x,y)})) = \pi_1(\beta(\left[x + y, x + \frac{y}{2}\right])) = x + \frac{3y}{4}.$$

As $y < 0$, we have

$$x + \frac{3y}{4} < x < x - \frac{3y}{4}.$$

Thus, $\pi_1(h(\sigma.-1)) < \pi_1(h(\sigma)) < \pi_1(h(\sigma))$, which implies $\sigma.-1 < \sigma < \sigma.1$. Hence, $(\Sigma, <)$ has no endpoints.

All of this together says that $\Sigma, <$ is a linearly ordered dense set with no endpoints. By Proposition 6.1.5 and Proposition 6.2.5, Σ equipped with the order topology is homeomorphic to $\mathbb{Q}$ equipped with the euclidean topology.

6.3.2 Interior Map from Σ to the frame

Let $\mathfrak{F} = (W, T)$ be a finite rooted **S4**-frame. We recursively define a function $f : \Sigma \to W$ such that f becomes an interior map. As $\mathfrak{F}$ is rooted, so there exists an $r \in W$ such that r is a root, that is, for each $w \in W$, we have rTw. We define

$$f(\Lambda) = r.$$

The recursive step

Let $\sigma \in \Sigma$, and let $f(\sigma) = w$ for some $w \in W$. Let

$$T(w) = \{v \in W \mid Twv\}.$$

Then as T is a reflexive relation, we have $w \in T(w)$, and hence, $T(w) \neq \emptyset$. Consider a function $g_w : \mathbb{N} \to T(w)$ such that for each $v \in T(w)$, the set $g_w^{-1}(v)$ is an infinite set. It should be noted that such a function always exists. To see this, consider the following. As W is finite by assumption, and as $T(w) \subseteq W$, we get that $T(w)$ is finite. Let

$$T(w) = \{v_1, v_2, \ldots, v_n\}.$$

Then, one of the candidates for g_w is

$$(v_1, v_2, \ldots, v_n, v_1, v_2, \ldots, v_n, v_1, \ldots).$$

We define

$$f(\sigma.n) = f(\sigma. - n) = g_w(n).$$

Thus for each $z \in \mathbb{Z}^*$, we have $f(\sigma.z) \in T(w)$. In this way f is defined.

f is onto

We claim that f is onto. For $w \in W$, as $\mathfrak{F}$ is a rooted, there exists a root r such that rTw. Then by definition of g_r, there exists an $n \in \mathbb{N}$ such that $g_r(n) = w$, and by the definition of f, we have $f(\Lambda.n) = w$. As w is arbitrary, we get that f is onto.

Next, we prove a series of lemmas (can be found in [12]) which will help us in proving that f is an interior map.

Lemma 6.3.4. *The function $\pi_1 \circ h : \Sigma \to X$ is a homeomorphism where both the spaces are equipped with the order topology.*

Proof. We have already proved that $\pi_1 \circ h$ is injective (Proposition 6.3.3). As X is defined to be the range of $\pi_1 \circ h$, we get that $\pi_1 \circ h$ is a bijective map from Σ to X. Also, as the order defined on Σ is the pullback of the order defined on X, so $\pi_1 \circ h$ is an order isomorphism. Using the steps in the proof of Proposition 6.2.5, we get that $\pi_1 \circ h$ is a homeomorphism. □

Definition 6.3.5. We define a relation S on Σ as follows:

$$\sigma S \tau \text{ iff } \sigma \text{ is an initial substring of } \tau.$$

⊣

It can be checked that S is a partial order, that is, it is reflexive, anti-symmetric, and transitive.

The next lemma states that the subspace topology on X, and the order topology on X are one and the same. This will help us later when we prove that $S(\sigma)$ is open in $\langle \Sigma, \mu \rangle$, where μ is the order topology on Σ.

Lemma 6.3.6. *Let τ_1 denote the subspace topology on X, and let τ_2 denote the order topology on X. Then $\tau_1 = \tau_2$.*

Proof. Consider a basic open interval

$$(\pi_1 \circ h(\sigma), \pi_1 \circ h(\sigma'))_{\pi_1 \circ h(\Sigma)} \in \tau_2.$$

Then

$$(\pi_1 \circ h(\sigma), \pi_1 \circ h(\sigma'))_{\pi_1 \circ h(\Sigma)} = (\pi_1 \circ h(\sigma), \pi_1 \circ h(\sigma'))_{\mathbb{R}} \cap \pi_1 \circ h(\Sigma) \in \tau_1.$$

So $\tau_2 \subseteq \tau_1$. Now, we prove the other containment. Let $U \in \tau_1$ be a basic open set of the form

$$U = (x, y)_{\mathbb{R}} \cap \pi_1 \circ h(\Sigma).$$

Let $z \in U$. Then for some $\sigma \in \Sigma$, we have $z = \pi_1 \circ h(\sigma)$. Then by using the proof used to show that $(\Sigma, <)$ is a dense order, we know we can choose a large enough $n \in \mathbb{N}$, such that

$$x < \pi_1 \circ h(\sigma. - n) < \pi_1 \circ h(\sigma) < \pi_1 \circ h(\sigma.n) < y.$$

Then

$$z \in (\pi_1 \circ h(\sigma. - n), \pi_1 \circ h(\sigma.n))_{\pi_1 \circ h(\Sigma))} \subseteq U.$$

As

$$(\pi_1 \circ h(\sigma. - n), \pi_1 \circ h(\sigma.n))_{\pi_1 \circ h(\Sigma)} \in \tau_2,$$

so $\tau_1 \subseteq \tau_2$. Hence, $\tau_1 = \tau_2$. □

Thus, the two topologies on X are the same.

Lemma 6.3.7. *Let $\sigma \in \Sigma$. Then*

1. $\pi_1 \circ h(S(\sigma)) = \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma) = Int_{\mathbb{R}}(\alpha \circ h(\sigma)) \cap \pi_1 \circ h(\Sigma)$.
2. *$S(\sigma)$ is open in $\langle \Sigma, \mu \rangle$.*

Proof. 1. We first prove that

$$\pi_1 \circ h(S(\sigma)) \subseteq \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

Let $\tau \in S(\sigma)$ be arbitrary. We prove the above containment by proving that

$$\pi_1 \circ h(\tau) \in \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

As $\tau \in S(\sigma)$, so either $\tau = \sigma$ or $\tau \in R(\sigma)$, where 'R' is the proper initial substring relation on Σ. If $\tau = \sigma$, we get

$$\pi_1 \circ h(\tau) = \pi_1 \circ h(\sigma) \in \alpha \circ h(\sigma).$$

Thus, we have

$$\pi_1 \circ h(\tau) \in \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

Consider the case $\tau \in R(\sigma)$. Using Lemma 6.3.2, we get

$$\pi_1 \circ h(\tau) \in \alpha \circ h(\tau) \subseteq \alpha \circ h(\sigma).$$

Thus, in both the cases we get

$$\pi_1 \circ h(\tau) \in \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

So,

$$\pi_1 \circ h(S(\sigma)) \subseteq \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

We now prove the other containment. Let

$$x \in \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

We prove that $x \in \pi_1 \circ h(S(\sigma))$. As $x \in \pi_1 \circ h(\Sigma)$ by assumption, so we get that there exists some $\tau \in \Sigma$ such that $\pi_1 \circ h(\tau) = x$, and $x \in \alpha \circ h(\sigma)$. We claim $\tau \in S(\sigma)$. Assume not. Then $\tau \neq \sigma$. We have the following cases:

- $\tau R \sigma$ holds. Then by the Lemma 6.3.2, we have

  $$\alpha \circ h(\sigma) \subseteq \alpha \circ h(\tau) - \{\pi_1 \circ h(\tau)\},$$

  which implies $\pi_1 \circ h(\tau) \notin \alpha \circ h(\sigma)$. But, this contradicts the assumption that $x \in \alpha \circ h(\tau)$. So, this case is not possible.

- Neither $\tau R \sigma$ nor $\sigma R \tau$ holds. Let ρ be the longest common initial segment of τ and σ. Then by assumption, we have $\rho \neq \sigma$ and $\rho \neq \tau$. By Lemma 6.3.2, we have

  $$\pi_1 \circ h(\sigma) \in \alpha \circ h(\sigma) \subseteq \alpha \circ h(\rho.\sigma_{l(\rho)+1}),$$

and

$$\pi_1 \circ h(\tau) \in \alpha \circ h(\tau) \subseteq \alpha \circ h(\rho.\tau_{l(\rho)+1}).$$

By the definition of ρ, we have $\rho.\tau_{l(\rho)+1} \neq \sigma.\tau_{l(\rho)+1}$. So, by Lemma 6.3.1, the intervals $\alpha \circ h(\rho.\tau_{l(\rho)+1})$ and $\alpha \circ h(\sigma.\tau_{l(\rho)+1})$ intersect at at most one point, which if exists, is the one of the endpoints of the closed intervals. As $\pi_1 \circ h(\tau)$ is not an endpoint of the closed interval $\alpha \circ h(\rho.\tau_{l(\rho)+1})$, we get that

$$\pi_1 \circ h(\tau) \notin \alpha \circ h(\rho.\sigma_{l(\rho)+1}).$$

As

$$\alpha \circ h(\sigma) \subseteq \alpha \circ h(\rho.\sigma_{l(\rho)+1}),$$

we get that $\pi_1 \circ h(\tau) \notin \alpha \circ h(\sigma)$ which is a contradiction.

Thus, the only possibility is $\sigma R \tau$. So, either $\sigma = \tau$ or $\sigma R \tau$. Hence, $\tau \in S(\sigma)$, which implies $\pi_1 \circ h(\tau) \in \pi_1 \circ h(\Sigma)$. Thus, $x \in \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma)$ implies $x \in \pi_1 \circ h(S(\sigma))$. So, we get

$$\alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma) \subseteq \pi_1 \circ h(S(\sigma)).$$

Together we get the first equality, which is

$$\alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma) = \pi_1 \circ h(S(\sigma)).$$

For the other one, as $\mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) \subseteq \alpha \circ h(\sigma)$, we get

$$\mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) \cap \pi_1 \circ h(\Sigma) \subseteq \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

We prove the other containment. Let

$$x \in \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

Then by using the first equality, we get that there exists $\tau \in S(\sigma)$ such that $x = \pi_1 \circ h(\tau)$. We have the following two cases:

- If $\tau = \sigma$, then

$$\pi_1 \circ h(\tau) = \pi_1 \circ h(\sigma) \in \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma))$$

(this happens because $\pi_1 \circ h(\sigma)$ is the midpoint of the closed interval $\alpha \circ h(\sigma)$). Thus,

$$\pi_1 \circ h(\tau) \in \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) \cap \pi_1 \circ h(\Sigma).$$

As x is arbitrary, we get

$$\alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma) \subseteq \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) \cap \pi_1 \circ h(\Sigma).$$

- If $\sigma R \tau$ then by Lemma 6.3.2, we have

$$\pi_1 \circ h(\tau) \in \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\tau)) \subsetneq \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) - \{\pi_1 \circ h(\sigma)\} \subseteq \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)).$$

Thus,

$$\pi_1 \circ h(\tau) \in \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) \cap \pi_1 \circ h(\Sigma).$$

Similar to the previous case, we get

$$\alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma) \subseteq \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) \cap \pi_1 \circ h(\Sigma).$$

Thus, the second equality holds too.

2. By the first part, we have

$$\pi_1 \circ h(S(\sigma)) = \mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) \cap \pi_1 \circ h(\Sigma).$$

By Lemma 6.3.6, the set $\mathrm{Int}_{\mathbb{R}}(\alpha \circ h(\sigma)) \cap \pi_1 \circ h(\Sigma)$ is open in X under the order topology too. Thus, $\pi_1 \circ h(S(\sigma))$ is open in X under the order topology. By Lemma 6.3.4, $\pi_1 \circ h$ is a homeomorphism. So, $S(\sigma)$ is open in $\langle \Sigma, \mu \rangle$.

$\square$

Lemma 6.3.8. *For each $\sigma \in \Sigma$, the set*

$$\{(\sigma. - n, \sigma.n) : n \in \mathbb{N}\}$$

is a countable local basis.

Proof. Let $\sigma \in \Sigma$, and (τ, τ') be a basic open set in (Σ, μ) such that $\sigma \in (\tau, \tau')$. Then $\tau < \sigma < \tau'$. By the reasoning used to prove that $(\Sigma, <)$ is dense, we get that there are $n, n' \in \mathbb{N}$ such that

$$\tau < \sigma. - n < \sigma < \sigma.n' < \tau'.$$

Let $n_0 = \max\{n, n'\}$. Then by the construction of Σ, we have

$$\pi_1 \circ h(\sigma. - n) \leq \pi_1 \circ h(\sigma.n_0),$$

and

$$\pi_1 \circ h(\sigma.n') \geq \pi_1 \circ h(\sigma.n_0).$$

Thus, we have

$$\tau < \sigma. - n_0 < \sigma < \sigma.n_0 < \tau'.$$

Also for any $\lambda \in (\sigma. - n_0, \sigma.n_0)$, we have

$$\tau < \sigma. - n_0 < \lambda < \sigma.n_0 < \tau',$$

and thus we get $\sigma \in (\sigma. - n_0, \sigma.n_0) \subseteq (\tau, \tau')$. Thus $\{(\sigma. - n, \sigma.n) \mid n \in \mathbb{N}\}$ forms a local basis for σ.

□

We will use these lemmas to prove that f is an interior map. Let τ denote the Alexandroff topology on the frame (X, T).

f is continuous

We claim that for each basic open set $T(w)$ in $\langle X, \tau \rangle$, we have

$$f^{-1}(T(w)) = \bigcup_{wTf(\sigma)} S(\sigma).$$

Proving this suffices because by Lemma 6.3.7 we have that each $S(\sigma)$ is open in $\langle \Sigma, \mu \rangle$. We first prove that

$$f^{-1}(T(w)) \subseteq \bigcup_{wTf(\sigma)} S(\sigma).$$

Let $\lambda \in f^{-1}(T(w))$. So $f(\lambda) \in T(w)$, which implies $wTf(\lambda)$. As $\lambda \in S(\lambda)$, we get

$$\lambda \in \bigcup_{wTf(\sigma)} S(\sigma).$$

Thus,

$$f^{-1}(T(w)) \subseteq \bigcup_{wTf(\sigma)} S(\sigma).$$

For the converse, let $\lambda \in \Sigma$ such that $wTf(\lambda)$. We claim $S(\lambda) \subseteq f^{-1}(T(w))$. Let $\delta \in S(\lambda)$ be arbitrary. Then $\delta = \lambda.z_1.z_2.\ldots.z_k$ for $k \in \mathbb{N} \cup \{0\}$, and $z_i \in \mathbb{Z}^*$. We prove $\delta \in f^{-1}(T(w))$ by induction on k.

If $k = 0$, then $\delta = \lambda$, and as $f(\lambda) \in T(w)$ by assumption, so $\delta \in f^{-1}(T(w))$. Let us assume $f(\lambda.z_1 \ldots z_{k-1}) \in T(w)$. Then

$$f(\lambda.z_1.\ldots.z_k) = g_{f(\lambda.z_1.\ldots.z_{k-1})}(|z_k|) \in T(f(\lambda.z_1.\ldots.z_{k-1})).$$

As $f(\lambda.z_1.\ldots.z_{k-1}) \in T(w)$ by assumption, and as T is transitive, we have $T(f(\lambda.z_1.\ldots.z_{k-1})) \subseteq T(w))$. Hence, $f(\lambda.z_1.\ldots.z_k) \in T(w)$.

So by induction, we have $S(\lambda) \subseteq f^{-1}(T(w))$. Thus,

$$f^{-1}(T(w)) = \bigcup_{wTf(\sigma)} S(\sigma),$$

and f is continuous.

f is open

We now prove that f is an open map, that is, f maps open sets to open sets. But before that, we prove a needed lemma.

Lemma 6.3.9. *For $\sigma \in \Sigma$, and $n \in \mathbb{N}$, we have $(\sigma.-n, \sigma.n) \subseteq S(\sigma)$.*

Proof. Let $\sigma \in \Sigma$, $n \in \mathbb{N}$, and let $\lambda \in (\sigma.-n, \sigma.n)$. Then $\sigma.-n < \lambda < \sigma.n$, which implies

$$\pi_1 \circ h(\sigma.-n) < \pi_1 \circ h(\lambda) < \pi_1 \circ h(\sigma.n).$$

As

$$\pi_1 \circ h(\sigma.-n), \pi_1 \circ h(\sigma.n) \in \alpha \circ h(\sigma),$$

and as $\alpha \circ h(\sigma)$ is a closed and bounded interval, we get that

$$\pi_1 \circ h(\lambda) \in \alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma).$$

By Lemma 6.3.7, we have

$$\alpha \circ h(\sigma) \cap \pi_1 \circ h(\Sigma) = \pi_1 \circ h(S(\sigma)).$$

Thus, we get

$$\pi_1 \circ h(\lambda) \in \pi_1 \circ h(S(\sigma)),$$

which implies that there exists some $\lambda' \in S(\sigma)$ such that $\pi_1 \circ h(\lambda') = \pi_1 \circ h(\lambda)$. By Proposition 6.3.3, $\pi_1 \circ h$ is an injective map. Hence, $\lambda' = \lambda$, which implies $\lambda \in S(\sigma)$. As λ is arbitrary, we get that $(\sigma.-n, \sigma.n) \subseteq S(\sigma)$. □

Now we prove that f is an open map. We claim that for each $\sigma \in \Sigma$, and $n \in \mathbb{N}$, we have $f(\sigma.-n, \sigma.n) = T(f(\sigma))$. Proving this suffices as for any open set U in $\langle \Sigma, \mu \rangle$, and $\sigma \in U$, by Lemma 6.3.8 there exists $n_\sigma \in \mathbb{N}$ such that $(\sigma.-n_\sigma, \sigma.n_\sigma) \subseteq U$. Thus

$$U = \bigcup_{\sigma \in U} (\sigma.-n_\sigma, \sigma.n_\sigma),$$

and hence

$$f(U) = \bigcup_{\sigma \in U} f(\sigma. - n_\sigma, \sigma.n_\sigma).$$

So if $f(\sigma. - n_\sigma, \sigma.n_\sigma)$ is open, then $f(U)$ is open. As U is arbitrary, proving what we claim proves that f is an open map. By definition of f, for each $\lambda \in \Sigma$, and $z \in \mathbb{Z}^*$, we have $f(\lambda)Tf(\lambda.z)$, and as T is reflexive, we have $f(\lambda)Tf(\lambda)$. Thus, $f(S(\sigma)) \subseteq T(f(\sigma))$. So from Lemma 6.3.9, we get

$$f(\sigma. - n, \sigma.n) \subseteq f(S(\sigma)) \subseteq T(f(\sigma)).$$

Thus, one inclusion is proven. Now, we prove the other inclusion. Let $w \in T(f(\sigma))$. Then $f(\sigma)Tw$. By definition of f, $g^{-1}_{f(\sigma)}(w)$ is an infinite set, and hence there exists an $m \in \mathbb{N}$ such that $m > n$ and $f(\sigma.m) = w$. As $m > n$, we have $\sigma.m \in (\sigma. - n, \sigma.n)$, and hence $w \in f(\sigma. - n, \sigma.n)$. As w is arbitrary, we get $T(f(\sigma)) \subseteq f(\sigma. - n, \sigma.n)$. Thus $f(\sigma. - n, \sigma.n) = T(f(\sigma))$, and f is an open map.

As f is both a continuous map and an open map, we get that f is an interior map from $\langle \Sigma, \mu \rangle$ to the finite rooted **S4**-frame with the Alexandroff topology. By Proposition 6.2.5, $\mathbb{Q}$ with the euclidean topology is homeomorphic to $\langle \Sigma, \mu \rangle$. So we have a homeomorphism from $\langle \Sigma, \mu \rangle$ to $\mathbb{Q}$. Thus, composing the latter map with the former map, we get the following result.

Proposition 6.3.10. *Each finite rooted **S4**-frame with the Alexandroff topology is an interior image of $\mathbb{Q}$ with the euclidean topology.*

6.4 Constructing the Topo-bisimulation

By Proposition 6.0.2, we know that **S4** is complete with respect to the class of all finite rooted **S4**-frames. We will use this completeness to prove that **S4** is complete with respect to the topological space $\mathbb{Q}$ with the euclidean topology. As $\mathbb{Q}$ with the euclidean topology forms a dense-in-itself separable metric space, we get the McKinsey-Tarski theorem.

Theorem 6.4.1 (McKinsey-Tarski). ***S4** is sound and complete with respect to the class of all dense-in-itself separable metric spaces.*

Proof. The soundness part is straight forward. This is because we have already proved that **S4** is sound with respect to the class of arbitrary topological spaces. Hence, **S4** is sound with respect to the class of all dense-in-itself separable metric spaces.

Now, we prove the completeness. Let $\phi \notin \mathbf{S4}$. As **S4** is complete with respect to finite rooted **S4**-frames by Proposition 6.0.2, so there exists a model $M = (W, R, V)$ such that (W, R) is a finite rooted **S4**-frame, and $w \in W$ such that $M, w \nvDash \phi$.

Let τ represent the Alexandroff topology on the frame (W, R). Then by Lemma 4.3.6, we get that for the topo-model $\mathcal{M} = \langle W, \tau, V \rangle$, we have $\mathcal{M}, w \nvDash \phi$. Let ϵ represent the euclidean topology on $\mathbb{Q}$. By Proposition 6.3.10, there is an onto interior map f from $\langle \mathbb{Q}, \epsilon \rangle$ to $\langle W, \tau \rangle$.

Using f, we now construct a topo-model based on $\mathbb{Q}$, and a total topo-bisimulation between the topo-model and $\mathcal{M}$. We first define a valuation v on $\mathbb{Q}$. For each propositional variable p, we define

$$v(p) = f^{-1}(V(p)).$$

Next, we define a relation $Z \subseteq \mathbb{Q} \times W$ as

$$Z = \{(q, f(q) \mid q \in \mathbb{Q}\}.$$

We claim that Z forms a total topo-bisimulation between the topo-models $\langle \mathbb{Q}, \epsilon, v \rangle$ and $\langle W, \tau, V \rangle$. It should be noted that in the definition of Z, q varies in $\mathbb{Q}$. Also, as f is onto, for each $w \in W$, $f^{-1}(w)$ is non-empty. Thus, Z is total. To prove that Z is a topo-bisimulation, we will use Remark 3.2.2.

For the atomic clause, we need to prove that for each propositional variable p, we have $Z(v(p)) \subseteq V(p)$, and $Z^{-1}(V(p)) \subseteq v(p)$. By definition of Z, we have $Z(v(p)) = f(v(p))$. As $v(p)$ is defined to be $f^{-1}(V(p))$, we get that

$$Z(v(p)) = f(f^{-1}(V(p))) = V(p).$$

Similarly,

$$Z^{-1}(V(p)) = f^{-1}(V(p)) = v(p).$$

Hence, the atomic clause is followed.

For the forth condition we need to prove that for each open set $U \in \epsilon$, $Z(U)$ is open, and for the back condition we need to prove that for each open set $V \in \tau$, $Z^{-1}(V)$ is open. As for $U \in \epsilon$, $Z(U) = f(U)$, and for $V \in \tau$, $Z^{-1}(V) = f^{-1}(V)$, and as f is an interior map, so the back and forth conditions are satisfied. Hence, Z is a total topo-bisimulation.

Let $\mathcal{M}_{\mathbb{Q}}$ denote the topo-model $\langle \mathbb{Q}, \epsilon, v \rangle$. and let q_0 be an element of $f^{-1}(w)$. By Theorem 3.2.4, as $\mathcal{M}, w \not\models \phi$, so we get $\mathcal{M}_{\mathbb{Q}}, q_0 \not\models \phi$.
Therefore every formula not in **S4** is falsifiable on a topo-model based on a dense-in-itself separable metric space, and hence **S4** is complete with respect to the class of all dense-in-itself separable metric spaces. $\square$

An upshot of the McKinsey-Tarski Theorem is that our topological interpretation of the basic modal language is too weak to capture the notions of a space being separable, dense-in-itself, or metrizable, and to fully capture these properties in the modal formulas we need more expressive interpretations.

7. CONCLUSION

The thesis aimed to describe the proofs of soundness and completeness of **S4** with respect to the class of all topological spaces, and the McKinsey-Tarski theorem, which states that **S4** is sound and complete with respect to the class of all dense-in-itself separable metric spaces. The results yield an approach to understand topological spaces using logic, and vice-versa. The two results also highlight a limitation of the topological interpretation of the basic modal language described in the thesis, that the interpretation is too weak to encapsulate the property of a topological space being separable, metrizable or dense-in-itself.

To tackle this problem, a number of alternate interpretations have been proposed. McKinsey and Tarski proposed the *derived set semantics* in [1]. The derived set semantics is more expressive than the interpretation we have seen in the thesis. To increase the expressive power it is common to add *universal modality*. In 1999, Shehtman showed that connectedness is expressible by adding a universal modality [13].

Also, analogues of the McKinsey-Tarski theorem have been obtained. The topology of $\mathbb{Q}$ can be thought of as the interval topology of a dense linear order. It has been proved that **S4** is complete with respect to arbitrary (nonempty) *locally compact* dense-in-itself *generalized order* spaces [14]. Some other directions worth exploring are *topological products of modal logics* and their connection to *fusions and products of modal logics*, topological semantics of *predicate modal logics*, and topological semantics of *provability logics*.

APPENDIX

I Topological Preliminaries

We briefly survey the basic topological concepts that are used. They can be found in any textbook on general topology (see [15]).

Definition I.1. A *topological space* is a pair $\mathcal{X} = \langle X, \tau \rangle$, where X is a nonempty set, and τ is a collection of subsets of X satisfying the following three conditions:

1. $\emptyset, X \in \tau$.
2. If $U, V \in \tau$, then $U \cap V \in \tau$.
3. If $\{U_i\}_{i \in I} \in \tau$, then $\displaystyle\bigcup_{i \in I} U_i \in \tau$.

$\dashv$

The elements of τ are called *open sets*. The complement of open sets are called *closed sets*. For any x in X, an open set containing x is called an *open neighborhood* of x.

Definition I.2. Let $\mathcal{X} = \langle X, \tau \rangle$ be a topological space. A family $\mathcal{B} \subseteq \tau$ is called a *basis* for the topology $\mathcal{X}$ if and only if

1. for each $x \in X$, there exists a $U \in \mathcal{B}$ such that $x \in U$, and
2. for each $U, V \in \mathcal{B}$, if $x \in U \cap V$, then there exists a $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.

The elements of $\mathcal{B}$ are called *basic open sets* for the topology. $\dashv$

An equivalent definition of a basis for the topology is the following. A family $\mathcal{B} \subseteq \tau$ is called a basis for the topology if every open set can be represented as the union of elements of a subfamily of $\mathcal{B}$.

Definition I.3. Let X be a set, and $\mathcal{B} \subseteq \mathcal{P}(X)$ such that elements of $\mathcal{B}$ satisfy the conditions numbered 1 and 2 given in Definition I.2. Then $\tau = \{U \subseteq X \mid U \text{ is a union of elements of } \mathcal{B}\}$ forms a topology on X. The topology τ is said to be the *topology generated by* $\mathcal{B}$ on X. $\dashv$

For a topological space $\langle X, \tau \rangle$, any basis for the topology generates the same topology. Also, if a basis generates a topology, then it forms a basis for the topology. We also recall the following points:

- For $A \subseteq X$, a point $x \in X$ is called an *interior point* of A if there is an open neighborhood U of x such that $U \subseteq A$. Let $\mathrm{Int}(A)$ denote the set of interior points of A. It is known that $\mathrm{Int}(A)$ is the biggest open set contained in A, called the *interior* of A.
- A point $x \in X$ is called a *limit point* of $A \subseteq X$, if for each open neighborhood U of x, the set $A \cap (U - \{x\})$ is non-empty. The set of limit points of A is called the *derivative* of A, and is denoted by $d(A)$.
- Let $\mathrm{Cl}(A) = A \cup d(A)$. Then, $x \in \mathrm{Cl}(A)$ if and only if $U \cup A$ is nonempty for each open neighborhood U of x, and $\mathrm{Cl}(A)$ is the smallest closed set containing A, called the *closure* of A.

Let Int and Cl denote the interior and the closure operators on $\mathcal{P}(X)$, respectively, where $\mathcal{P}(X)$ denotes the power set of X. Then the following are satisfied for each $A, B \subseteq X$

$$\begin{aligned}
\mathrm{Int}(X) &= X \\
\mathrm{Int}(A \cap B) &= \mathrm{Int}(A) \cap \mathrm{Int}(B) \\
\mathrm{Int}(A) &\subseteq A \\
\mathrm{Int}(A) &\subseteq \mathrm{Int}(\mathrm{Int}(A)) \\
\mathrm{Cl}(\emptyset) &= \emptyset \\
\mathrm{Cl}(A \cup B) &= \mathrm{Cl}(A) \cup \mathrm{Cl}(B) \\
A &\subseteq \mathrm{Cl}(A) \\
\mathrm{Cl}(\mathrm{Cl}(A)) &\subseteq \mathrm{Cl}(A) \\
\mathrm{Int}(A) &= X - \mathrm{Cl}(X - A)
\end{aligned}$$

Definition I.4. Let $\mathcal{X}$ be topological space, and A be a subset of X. A is called *dense-in-itself* if $A \subseteq d(A)$. $\dashv$

Definition I.5. Let $\mathcal{X}$ be a topological space.

1. $\mathcal{X}$ is called *discrete* if every subset of X is open.
2. $\mathcal{X}$ is called *trivial* if $\emptyset$ and X are the only open subsets of X.
3. $\mathcal{X}$ is called *dense-in-itself* if $X \subseteq d(X)$.
4. $\mathcal{X}$ is called *separable* if there exists a countable dense subset of X.

$\dashv$

Definition I.6. Let $\mathcal{X} = \langle X, \tau \rangle$ be a topological space. For $Y \subseteq X$, let $\tau' = \{B \subseteq Y \mid B = B' \cap Y \text{ for some } B' \in \tau\}$. Then $\langle Y, \tau' \rangle$ forms a topological space and τ' is called the *subspace topology* on Y. $\dashv$

Remark I.7. Let $\langle X, \tau \rangle$ be a topological space and $Y \subseteq X$. Let $Z \subseteq Y$. We say Z is open (closed) in Y if Z is an open (closed) subset of Y equipped with the subspace topology. $\dashv$

Definition I.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, and $f : X \to Y$ be a function.

1. f is called *continuous* if U open in Y implies $f^{-1}(U)$ is open in X.
2. f is called *open* if U open in X implies $f(U)$ open in Y.
3. f is called *interior* if it is both continuous and open.
4. f is called a *homeomorphism* if f is bijective and both f and f^{-1} are continuous.

We call $\mathcal{Y}$ a *continuous image* of $\mathcal{X}$ if there is a continuous map from X to Y. Open and interior images of $\mathcal{X}$ are defined analogously. $\dashv$

Definition I.9. Let $\{\mathcal{X}_i\}_{i \in I}$ be a family of topological spaces.

- Let $\tau = \{U \subseteq \prod_{i \in I} X_i \mid U$ is a union of the sets of the form $\prod_{i \in I} U_i$ where each U_i is open in X_i and $U_i \neq X_i$ for finitely many $i \in I\}$. Then τ forms a topology on $\prod_{i \in I} X_i$ and is called the *product topology.*

- Let $\tau' = \{U \subseteq \prod_{i \in I} X_i \mid U$ is a union of the sets of the form $\prod_{i \in I} U_i$ where each U_i is open in $X_i\}$. Then τ' forms a topology on $\prod_{i \in I} X_i$ and is called the *box topology*.

$\dashv$

Remark I.10. In the case of a finite family, the box topology is the same as the product topology. $\dashv$

Definition I.11. Let $\langle \prod_{i \in I} X_i, \tau \rangle$ be a topological space. For each $i \in I$, we have a map $\pi_i : \prod_{j \in I} X_j \to X_i$ given by $\pi_i((x_j)_{j \in I}) = x_i$, called the *projection map* corresponding to the index i. $\dashv$

Remark I.12. For a family $\{\mathcal{X}_i\}_{i \in I}$, if $\prod_{i \in I} X_i$ is equipped with the box or the product topology, then each projection map is both continuous and open. $\dashv$

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