ACTION OF THE MAPPING CLASS GROUP ON THE PANTS COMPLEX

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CERTIFICATE

This is to certify that Saumya Jain, BS-MS (Dual Degree) student in Department of Mathematics, has completed bonafide work on the thesis entitled 'Action of The Mapping Class Group on the Pants Complex' under my supervision and guidance.

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'When you get these jobs that you have been so brilliantly trained for, just remember that your real job is that if you are free, you need to free somebody else. If you have some power, then your job is to empower somebody else.'

- Toni Morrison.

Saumya Jain

ABSTRACT

Understanding algebraic invariants is an age-old method to better understand topological spaces. One such algebraic invariant for surfaces S is its (extended) mapping class group, denoted by $\mathrm{Mod}(S)$. It is defined as the group of homeomorphisms of S, upto isotopy. There are sevaral known combinatorial models for studying $\mathrm{Mod}(S)$. Two such models are the curve complex $C(S)$ and the pants complex $C_P(S)$. The automorphism groups of both these combinatorial objects encode complete information about the group structure of $\mathrm{Mod}(S)$ for most surfaces.

In the following thesis, we will introduce the mapping class groups, the curve complex and the pants complex of a surface, and study their properties. We will study the result that for almost all closed, oriented, finite-type surfaces S, $\mathrm{Aut}\, C_P(S) \cong \mathrm{Mod}(S)$. This is based on the work of Margalit [12].

CONTENTS

1. PRELIMINARIES

In this chapter, we develop the basic theory of curves and mapping class groups. It is based on [2, Chapter 1-3].

1.1 Curves on surfaces

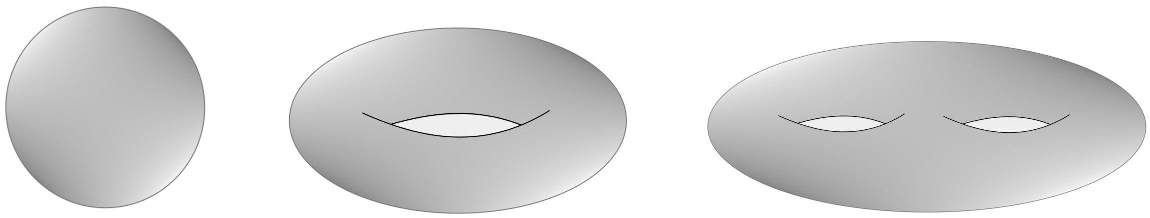
The aim of this section is to give an overview of curves on hyperbolic surfaces. In particular, we consider isotopy classes of simple closed curves through their geodesic representatives. We will study the geometric intersection number between isotopy classes of simple closed curves. We will also see some results about collections of such curves and study cut-surfaces.

Definition 1.1. A *surface* S is a 2-dimensional manifold, that is, a second countable, Hausdorff space such that for every $x \in S$, there is a neighbourhood $U \subset S$ that is homeomorphic to $\mathbb{R}^2$.

We shall consider compact, connected, orientable surfaces. A *closed* surface is one that is compact and without boundary. We give the following classification theorem about such surfaces without proof.

Theorem 1.2 (Classification of Surfaces)**.** *Any closed, connected, orientable surface is homeomorphic to the connected sum of a 2-dimensional sphere with $g \geq 0$ tori. Any compact, connected, orientable surface is obtained from a closed surface by removing $b \geq 0$ open disks with disjoint closures. The set of compact surfaces (upto homeomorphisms) is in bijective correspondence with the set $\{(g, b) : g, b \geq 0\}$.*

Henceforth, we will use $S = S_{g,b}$ to denote a compact, connected, orientable surface of genus $g \geq 0$ with $b \geq 0$ boundary components.



(a) $g = 0$. (b) $g = 1$. (c) $g = 2$.

Fig. 1.1: Closed surfaces S_g for $g = 0, 1, 2$.

It is well-known that any compact surface can be triangulated [13]. (That is, it is homeomorphic to a surface "built" from triangles.) Since $S_{g,b}$ is a compact surface, its triangulation leads to the following topological invariant.

Definition 1.3. The *Euler characteristic* $\chi(S)$ of a surface $S = S_{g,b}$ is defined as $\chi(S) = 2 - 2g - b$.

Example 1.4. Let $S = S_{g,b}$.

1. If $\chi(S) > 0$, that is $2 > 2g + b \geq 0$, then $g = 0$ and $b = 0$ or 1. That is, S is a sphere or a closed disk.

2. If $\chi(S) = 0$, that is $2 = 2g + b$, then either $g = 1$ or $b = 2$. That is, S is either the torus or an annulus.

Besides these finitely many exceptions, the remaining surfaces satisfy $\chi(S) < 0$. The following theorem is about such surfaces (see [2, Chapter-1]).

Theorem 1.5. *Any surface S with $\chi(S) < 0$ admits a hyperbolic metric.*

Now, we define curves on surfaces and cut-surfaces.

Definition 1.6. A *closed curve* on a surface S is the image of a continuous map from S^1 to S. If this map is injective, we say that the closed curve is *simple*.

Definition 1.7. A closed curve is said to be *essential* if it cannot be homotoped to a point and *peripheral* if it cannot be homotoped to a boundary component. A *non-trivial* closed curve is one that is essential and non-peripheral.

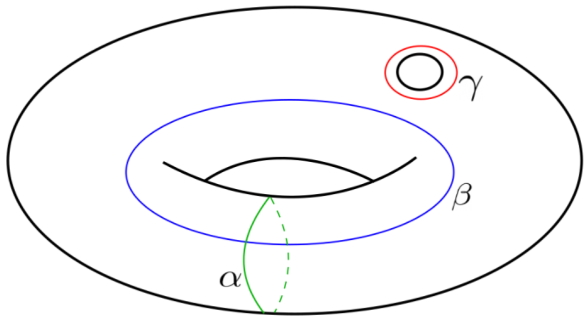


Fig. 1.2: The curves α and β are non-trivial while the curve γ is trivial on $S_{1,1}$.

We can cut a given surface along a closed curve to get one or more subsurfaces. These new surfaces are called *cut-surfaces*, defined as follows.

Definition 1.8. Given a simple closed curve α in a surface S, the surface obtained by cutting S along α is a compact surface S_α equipped with a homeomorphism h between two of its boundary components, so that:

1. $S_\alpha/(x \sim h(x)) \approx S$, and

2. the image of these distinguished boundary components under this quotient map is α,

is called a *cut-surface*.

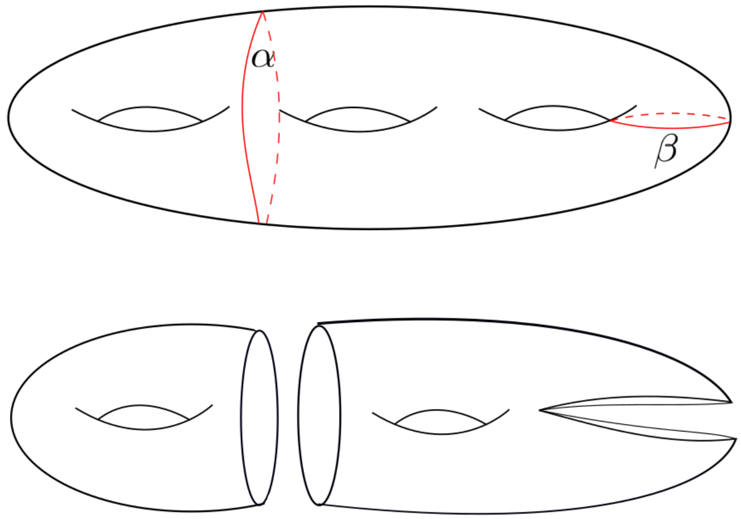


Fig. 1.3: A collection of simple closed curves and the associated cut-surfaces.

When cut along a simple closed curve, the cut-surface may have more than one connected components. If it does, then the simple closed curve is called a *separating curve*; otherwise it is called a *non-separating curve*. In the Figure 1.3, α is separating while β is non-separating. A consequence is the following.

Theorem 1.9. *There is an orientation-preserving homeomorphism of a surface S taking one simple closed curve to another if and only if the corresponding cut surfaces (which may be disconnected) are homeomorphic.*

Henceforth, for surfaces with $\chi(S) < 0$, we will assume S endowed with a hyperbolic metric.

Theorem 1.10. *Let S be a surface with $\chi(S) < 0$. If α is a closed curve in S that is not peripheral, then α is homotopic to a unique geodesic closed curve γ. Moreover, if α is simple, so is γ.*

Definition 1.11. An *isotopy* between two simple closed curves α_1, α_2 is a homotopy α_t between them so that each α_t is a simple closed curve for every $t \in [0, 1]$.

Proposition 1.12. *Let α and β be two non-trivial simple closed curves on a surface S. Then α is isotopic to β if and only if α is homotopic to β.*

Thus, geodesics are the natural representatives of isotopy classes of non-peripheral simple closed curves. Let α and β be a pair of transverse, oriented, simple closed curves in S. Let their free homotopy classes be a and b, respectively.

Definition 1.13. The *geometric intersection number* $i(a, b)$ of a and b is the minimal number of intersection points between a representative curve in the class a and a representative curve in the class b, that is:

$$i(a, b) = \min\{|\alpha \cap \beta| : \alpha \in a, \beta \in b\}.$$

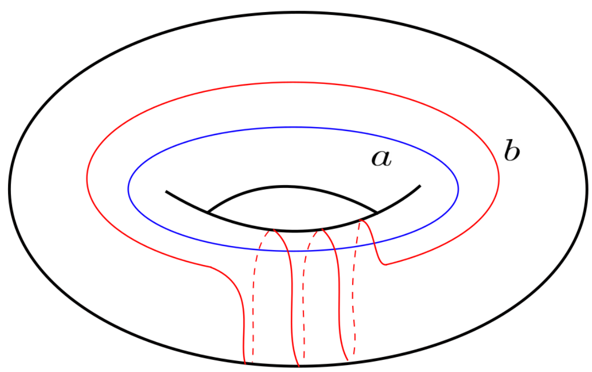


Fig. 1.4: Two curves a and b with $i(a, b) = 3$.

Definition 1.14. We say two curves α and β of free homotopy classes a and b are in *minimal position* if they intersect exactly $i(a, b)$ many times.

Two natural questions then arise:

1. How do we know two curves are in minimal position?
2. Given two intersecting curves, how do we find curves homotopic to them that are in minimal position?

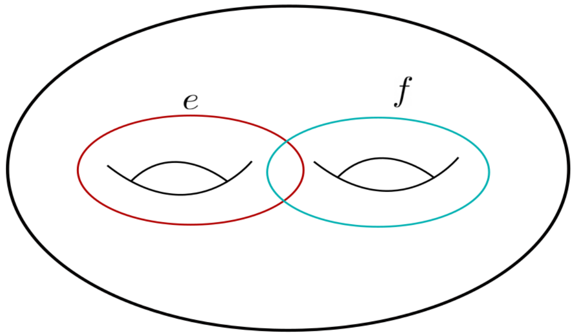


Fig. 1.5: Two curves not in minimal position.

Definition 1.15. We say that two transverse simple closed curves α and β in a surface S form a *bigon* if there is an embedded disk in S whose boundary is the union of an arc of α and an arc of β intersecting in exactly two points.

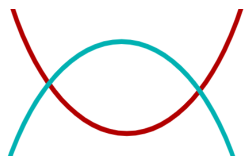


Fig. 1.6: A bigon formed by arcs of two curves.

Proposition 1.16. *Two transverse simple closed curves in a surface S are in minimal position if and only if they do not form a bigon.*

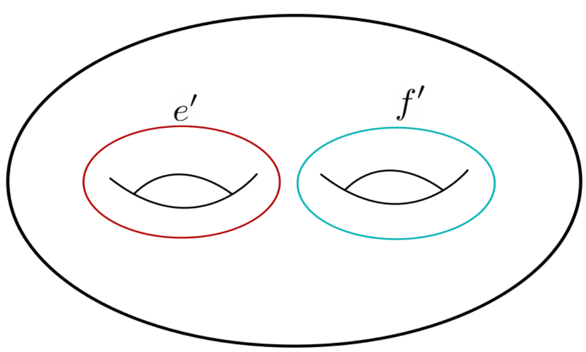


Fig. 1.7: Two curves in minimal position.

Since geodesics do not form bigons, the following holds true.

Proposition 1.17. *Distinct simple closed geodesics in a surface S with $\chi(S) < 0$ are in minimal position.*

Thus, to find geometric intersection numbers, we consider the geodesic representative of the isotopy classes of curves.

Proposition 1.18. *For a surface S with $\chi(S) < 0$, the maximum number of disjoint simple closed curves upto isotopy is $3g + b - 3$.*

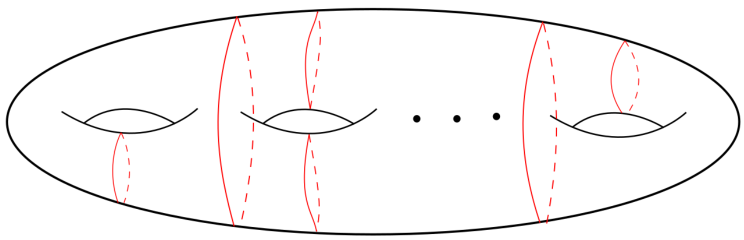


Fig. 1.8: $3g + b - 3$ disjoint non-isotopic simple closed curves on S_g.

The above proposition follows from considering the geodesic representatives as shown in Figure 1.8. By cutting along these $3g+b-3$ curves, we obtain a collection of $2g+b-2$ cut-surfaces homeomorphic to $S_{0,3}$. There is no non-trivial simple closed curve lying on $S_{0,3}$. Thus, no more disjoint curves can be added to the collection of curves (geodesics) shown in Figure 1.8. This fact will be used extensively in the later chapters.

1.2 Mapping class groups

We now move on to defining the mapping class group of surfaces, give examples of mapping classes, and explicitly compute it for some surfaces.

Definition 1.19. The *mapping class group* of S, denoted by $\mathrm{Mod}^+(S)$, is defined as the group of isotopy classes of orientation-preserving self-homeomorphisms of S which preserve the boundary components of S setwise. The elements of this group are called *mapping classes*.

Example 1.20. The following examples can be found discussed in detail in the book [2].

1. For a closed disk $D^2 = S_{0,1}$, $\mathrm{Mod}^+(D^2)$ is trivial.

2. For an annulus $A = S_{0,2}$, $\mathrm{Mod}^+(S)$ is trivial.

3. For $S = S_{1,0}$, $\mathrm{Mod}^+(S) \cong \mathrm{SL}_2(\mathbb{Z})$.

4. For $S = S_{0,3}$, $\mathrm{Mod}^+(S) \cong \Sigma_3$, the symmetric group on 3 letters.

Now we define an explicit mapping class called the *Dehn twist*. Define the *twist map* $T : A \to A$ with $A = S^1 \times [0,1]$ by $T(\omega, \rho) = (\omega + 2\pi\rho, \rho)$.

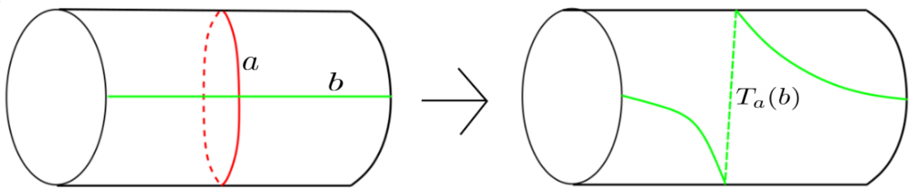


Fig. 1.9: The Dehn twist of b about a curve a.

We define a Dehn twist (see Figure 1.9) as follows.

Definition 1.21. Let N be a closed annular neighbourhood of a simple closed curve α in a surface S. Choose an orientation-preserving homeomorphism $\psi : A \to N$. Define a *Dehn twist* about α, $T_\alpha : S \to S$ as

$$T_\alpha(x) = \begin{cases} \psi \circ T \circ \psi^{-1}(x), & \text{if } x \in N \\ x, & \text{otherwise,} \end{cases} \tag{1.1}$$

where T is the twist map.

Remark 1.22. The map T_α is independent of the choice of the annular neighbourhood N of α. Moreover, if two curves belong to the same isotopy class, then the respective Dehn twists about them are isotopic. Henceforth, we will make no distinction between the Dehn twist about curves that are isotopic. Thus, Dehn twists (about an isotopy class of simple closed curves) are well-defined mapping classes.

Now, we define a bigger group than $\mathrm{Mod}^+(S)$ that also contains mapping classes represented by orientation-reversing homeomorphisms.

Definition 1.23. The *extended mapping class group* of S, denoted by $\mathrm{Mod}(S)$, is defined as the group of isotopy classes of self-homeomorphisms of S.

We have the following split short exact sequence

$$1 \to \mathrm{Mod}^+(S) \to \mathrm{Mod}(S) \xrightarrow{d} \mathbb{Z}_2 \to 1.$$

$\mathrm{Mod}^+(S)$ is the kernel of the map d and hence, a normal subgroup of $\mathrm{Mod}(S)$ of index 2. Following are some examples of $\mathrm{Mod}(S)$.

1. For $S = S_{0,0}$, we have $\mathrm{Mod}(S) = \mathbb{Z}_2$.

2. For $S = S_{1,0}$, $\mathrm{Mod}(S) = \mathrm{GL}_2(\mathbb{Z})$. This is the same for $S = S_{1,1}$.

3. For $S = S_{0,3}$, $\mathrm{Mod}(S) = \Sigma_3 \rtimes \mathbb{Z}_2$.

2. THE CURVE COMPLEX $C(S)$

In 1980, William J. Harvey introduced [3] the idea of the curve complex of a surface while studying Teichmüller spaces of Reimann surfaces. The curve complex is a combinatorial model that has been used to study algebraic and geometric properties of the extended mapping class group. The curve complex has many important properties that make it a useful tool for studying the topology of surfaces. For example, the dimension of the curve complex is related to the complexity of the surface [3] and its connectivity was a key idea in the proof of finite generation of the mapping class group [1] [9]. In this chapter, we will develop the basics of simplicial complexes to understand the curve complex and prove that it is connected. We will also define the Farey graph, which will be useful in proving the main theorems of Chapter 3.

2.1 Simplicial complex

In this section, we will define simplicial complexes. Simplicial complexes are fundamental objects in mathematics that play a central role in fields such as algebraic topology and combinatorics.

Definition 2.1. An k-*simplex* is the smallest convex set of $k+1$ points in $\mathbb{R}^n$ which do not lie in the same $(k-1)$-dimensional hyperplane.

Example 2.2. The k-simplices are thus generalised triangles.

1. 0-*simplex* is a point.

2. 1-*simplex* is a line segment.

3. 2-*simplex* is a triangle.

4. 3-*simplex* is a tetrahedron.

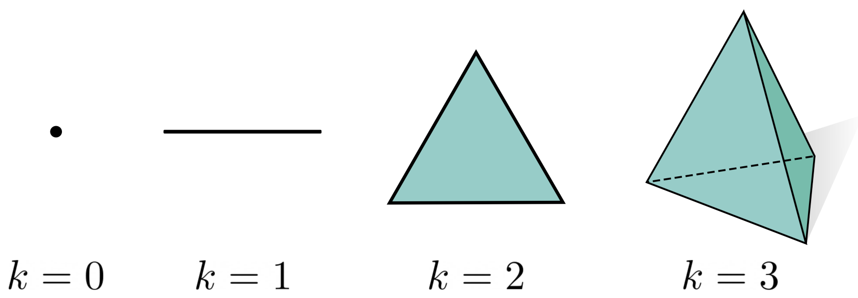


Fig. 2.1: k-simplices for $k = 0, 1, 2, 3$.

Definition 2.3. The convex hull of any nonempty subset of the $k+1$ points that define an k-simplex is called a *face* of the simplex.

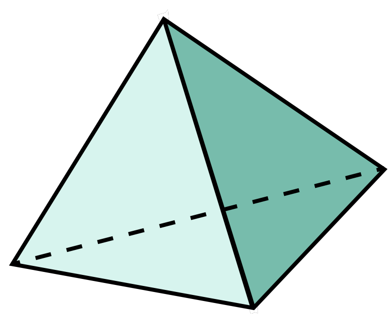


Fig. 2.2: A 2-face of a 3-simplex.

Definition 2.4. A *simplicial complex* K is a set of simplices satisfying the following conditions:

1. If K contains a simplex σ, then K also contains every face of σ.

2. The non-empty intersection of any two simplices σ_1 and σ_2 in K is a face of both σ_1 and σ_2.

A simplicial complex is, thus, a collection of generalised n-triangles (simplices) that fit together in a way that is consistent with their faces.

Definition 2.5. The *dimension of a simplicial complex* is the highest dimension of any of its simplices.

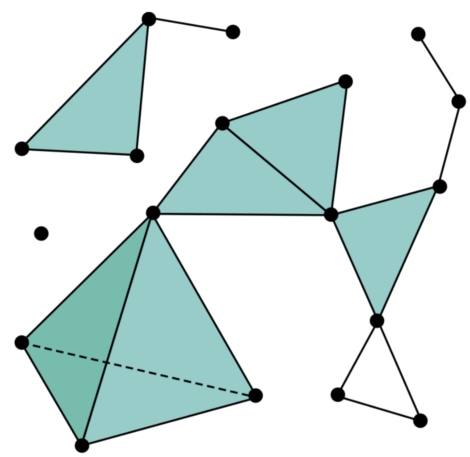


Fig. 2.3: A simplicial complex of dimension 3.

2.2 Curve complex of a surface

In this section, we study the curve complex of a surface S and see it as a combinatorial model for $\mathrm{Mod}(S)$.

Definition 2.6. A *curve complex* is an abstract simplicial complex associated to a surface S, defined as follows.

1. The vertices are the isotopy classes of non-trivial simple closed curves in S (to be used interchangeably from here on).

2. There is an edge between any two vertices representing isotopy classes a and b if $i(a, b)$ is the minimum geometric intersection number, say t, that can be achieved by any two simple closed curves on the surface S.

3. A set of $(n + 1)$ vertices $\{v_0, v_1, \ldots, v_n\}$ span an n-simplex if $i(v_i, v_j) = t$ for all $0 \leq i, j \leq n$, $i \neq j$.

We will denote the curve complex of a surface S by $C(S)$ and its 1-skeleton, consisting of vertices and edges, by $C^1(S)$.

Example 2.7. The following are examples of how some curves look in the curve complex of a given surface.

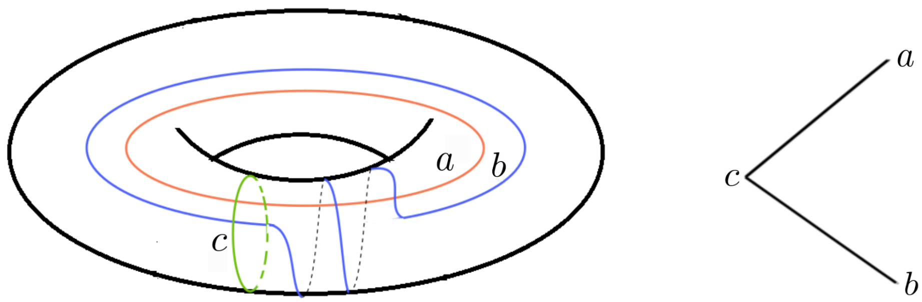


Fig. 2.4: Three curves on $S_{1,0}$ and the corresponding subgraph in $C(S_{1.0})$.

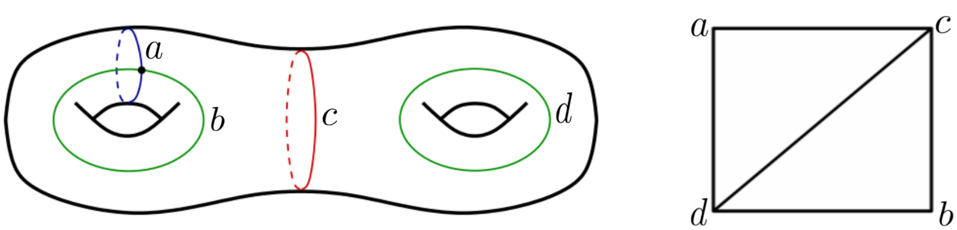


Fig. 2.5: Four curves on $S_{2,0}$ and the corresponding subgraph in $C(S_{2.0})$.

2.2.1 The Farey graph

In this section, we will define the Farey graph and prove that the 1-skeleton of curve complex of a torus is isomorphic to the Farey graph.

Definition 2.8. The *Farey graph* F is the graph with vertices $\mathbb{Q} \cup \{\infty\}$. Two vertices a/b and c/d share an edge if

$$\left|\frac{a}{b} - \frac{c}{d}\right| = \left|\frac{1}{bd}\right|.$$

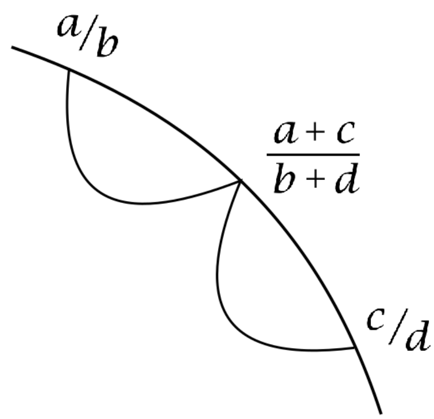


Fig. 2.6: Edges in the Farey graph.

Since $\mathbb{R} \cup \{\infty\}$ is homeomorphic to the circle (by one-point compactification), and $\mathbb{Q}$ is dense in $\mathbb{R}$, we can place the vertices of the Farey graph on a dense subset of the circle and place edges on geodesics according to the Poincaré disc model of $\mathbb{H}^2$ as shown in the Figure 2.7.

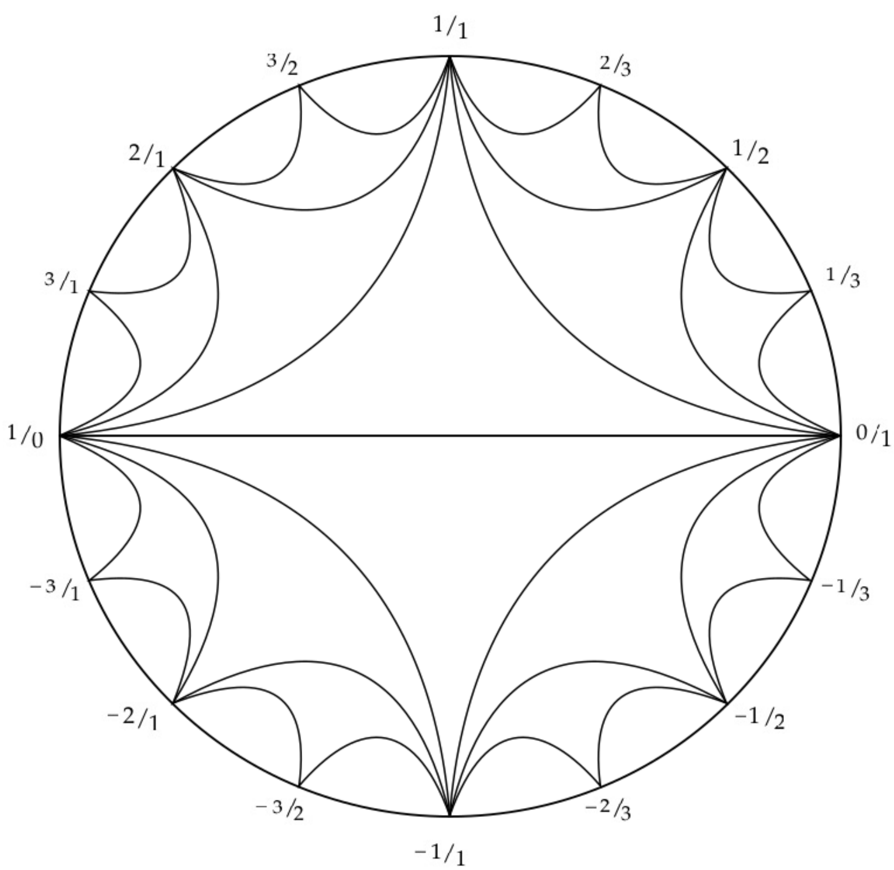


Fig. 2.7: The Farey graph on a Poincaré disc model.

Proposition 2.9. *The* 1*-skeleton of curve complex of a torus is the Farey graph.*

Proof. The fundamental group of the torus, $\pi_1(S_{1,0})$ is $\mathbb{Z}^2$. The primitive elements of $\mathbb{Z}^2$ represent the isotopy classes of non-trivial simple closed curves in $S_{1,0}$. Therefore, the set of vertices of $C(S_{1,0})$ are in a one-to-one correspondence with the set $\{(p,q) \in \mathbb{Z}^2 : \gcd(p,q) = 1\} = \mathbb{Q} \cup \{\infty\}$. There is an edge between two vertices $a = (1,0)$ and $b = (p', q')$ if $i(a,b) = |q'|$. This can be seen by lifting the curves to the universal cover $\mathbb{R}^2$ of $S_{1,0}$.

Given (p,q) such that $\gcd(p,q) = 1$, there exists $a, b \in \mathbb{Z}$ such that $ap + bq = 1$. Thus, there is a $p \times q$ matrix A

$$A = A_{p,q} = \begin{pmatrix} a & b \\ -q & p \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

such that $A(p\ q)^t = (1\ 0)^t$. Thus, A is an orientation-preserving homeomorphism of $\mathbb{R}^2$ that preserves $\mathbb{Z}^2$. Therefore, it induces an orientation-preserving homeomorphism $A' : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. Let A' take (p', q') to $(\tilde{p}', \tilde{q}')$

$$\begin{pmatrix} a & b \\ -q & p \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ -q & p \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} \tilde{p}' \\ \tilde{q}' \end{pmatrix}.$$

Since homeomorphisms preserves the geometric intersection number, we have

$$i((1,0), (\tilde{p}', \tilde{q}')) = i((p,q), (p', q')) = |\tilde{q}'| = |pq' - p'q|. \tag{2.1}$$

Thus, there is an edge between two vertices in $C(S_{2,0})$ if and only if they share an edge in the Farey graph. □

From Proposition 1.18, we know that the number of (isotopy classes of) disjoint non-trivial simple closed curves on $S_{g,b}$ is $3g + b - 3$.

Remark 2.10. The following cases describe the curve complex of a surface on which the minimal intersection number is positive.

1. $C(S_{0,b})$ is an graph for $b = 0, 1$ as there are no nontrivial curves.

2. The curve complex of $S_{0,4}$ and $S_{1,1}$ is the Farey graph.

Remark 2.11. We will use the isotopy classes of nontrivial simple closed curves and the associated vertex in the curve complex interchangeably.

2.2.2 Connectedness of $C(S)$

We now restrict our attention to hyperbolic surfaces S of genus g with b boundary components such that $3g + b \geq 5$ as we have dealt with the other cases in Remark 2.10. Then, $C(S)$ has the following four basic properties.

1. *$C(S)$ is an infinite complex.* This can be seen as follows: given a curve $a \in C(S)$, there exists a b such that a and b intersect once. Taking Dehn twist of a about the curve b gives infinitely many vertices in the curve complex.

2. *$C(S)$ IS LOCALLY INFINITE.* That is, for each vertex in the curve complex, there are infinitely many edges attached to it. Given a vertex a, take another vertex b such that ab forms an edge. Dehn twists of b about a curve disjoint from a and intersecting b once gives infinite number of half-edges emerging from a.

3. *$C(S)$ is full, that is, if $v_0, v_1, .., v_n$ be $(n+1)$-vertices such that $i(v_i, v_j) = 0$, for $0 \leq i, j \leq n+1$, then they form an n-simplex.* It follows from the definition of $C(S)$.

4. *$C(S)$ is finite-dimensional and* $\dim(C(S_{g,b})) = 3g + b - 4$. This follows from the fact that there are at most $3g + b - 3$ many pairwise disjoint curves on $S_{g,b}$.

Remark 2.12. Simplicial maps are defined between two simplicial complexes by taking the set of vertices to the set of vertices, edges to edges, and k-simplices to k-simplices. If these maps are bijections, then the pre-image of every k-simplex is also a k-simplex. The way $C(S)$ is defined (Property 3) implies that a simplicial map on the whole complex can be thought of a simplicial maps at the 1-skeleton level extended linearly to higher-dimensional simplices. Thus, the automorphism group of the curve complex $\mathrm{Aut}(C(S))$ is isomorphic to the automorphism group of its 1-skeleton $\mathrm{Aut}(C^1(S))$.

Theorem 2.13 (Harvey [3])**.** *For $3g + b \geq 5$, the curve complex $C(S_{g,b})$ is connected.*

We note that the statement of the Theorem 2.13 is equivalent to saying that given any $a, b \in C(S)$, there is a sequence of isotopy classes

$$a = c_1, c_2, .., c_k = b$$

such that $i(c_i, c_{i+1}) = 0$, for $1 \leq i \leq k$. The following proof is due to Lickorish [10].

Proof. We prove this result by induction on $i(a, b)$.

If $i(a, b) = 0$, then there is nothing to prove. If $i(a, b) = 1$, then a closed neighborhood of $a \cup b$ is a torus with one boundary component. Take c to be the isotopy class of this boundary component. Since $3g + b \geq 5$, the curve c is non-trivial.

We assume that $i(a, b) \geq 2$. Let α and β be simple closed curves in minimal position representing a and b. Consider two points of their intersection x and y that are consecutive along β. Orient α and β near x and y so that it makes sense to talk about the index of their intersection points, $+1$ or -1, as shown in the Figure 2.8.

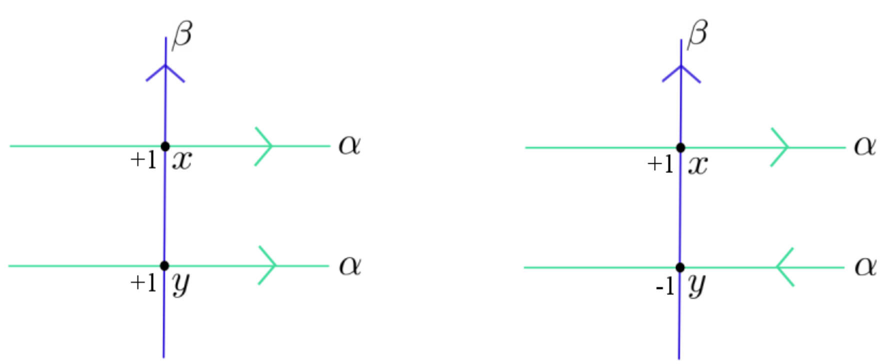


Fig. 2.8: Oriented curves α and β.

Case 1: If indices at both points of intersection are of the same sign, then we choose a γ which is as follows. Let γ be a simple closed curve such that it stays to the right of α except on one part of the arc xy, where it intersects α once to skip one intersection point with β, as shown in Figure 2.9. The curve γ is non-trivial since it intersects α once minimally.

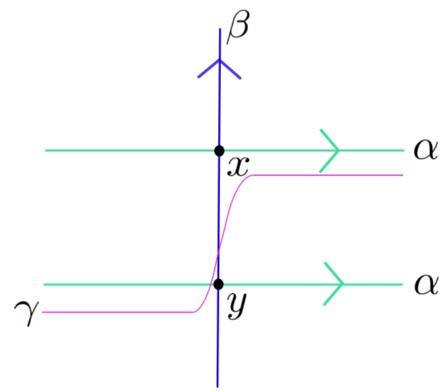


Fig. 2.9: The curve γ in Case 1.

Case 2: If both indices at the points of intersection x and y are not of the same sign, then we choose the simple closed curve γ as follows. Consider the distinct closed curves γ_1 and γ_2 as shown in Figure 2.10. Both γ_1 and γ_2 are essential, otherwise there would be a bigon. If γ_1 and γ_2 are non-peripheral, we are done. If not, choose γ_3 and γ_4 on the other side of α as shown in the Figure 2.10. Again, both γ_3 and γ_4 are essential, otherwise there would be a bigon. Both γ_3 and γ_4 cannot be peripheral, otherwise we get $S = S_{0,4}$ which s not possible since $3g + b \geq 5$. Without loss of generality, say γ_3 is non-peripheral, and let $\gamma = \gamma_3$.

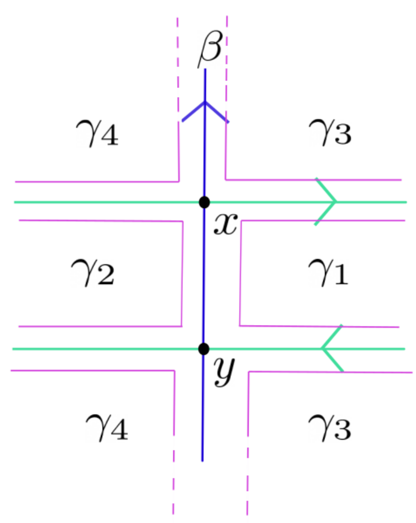


Fig. 2.10: The curves $\gamma_1, \gamma_2, \gamma_3$ and γ_4 in Case 2.

We denote the isotopy class of the γ chosen in the above cases by c. Since $i(a,c) < i(a,b)$ and $i(b,c) < i(a,b)$, by induction hypothesis, there is a path from a to c and another from c to b. Thus, a, b, c is the required sequence. $\square$

Owing to its connectedness, there is a natural way to define the notion of distance on $C(S)$. We give each edge a length of 1, then define distance between any two vertices to be the length of the shortest path between them. This leads us to the following result by Hempel [6]:

Theorem 2.14. *Let $S = S_{g,b}$ with $3g + b \geq 5$ and let $C(S)$ be its curve complex. For vertices x, y representing isotopy classes of non-trivial simple closed curves of $C(S)$ with the geometric intersection number $i(x,y) > 0$, we have*

$$d(x,y) \leq 2 + 2log_2(i(x,y)),$$

where $d(x,y)$ denotes the distance between the vertices x, y in $C(S)$.

2.2.3 Action of $\mathrm{Mod}(S)$ on $C(S)$

The mapping class group acts on the curve complex by acting on the curve associated with each vertex. This action is well-defined because homeomorphisms preserve the geometric intersection number between curves. The following result by Ivanov [7], Korkmaz [8] and Luo [11] describes the curve complex of a surface as a combinatorial model for its extended mapping class group.

Theorem 2.15 (Ivanov [7], Korkmaz [8], Luo [11])**.** *Let $S \neq S_{0,3}$ be an orientable surface with $\chi(S) < 0$ and $\eta : \mathrm{Mod}(S) \to \mathrm{Aut}(C(S))$ be the natural map. Then the following statements hold.*

(i) *η is surjective when $S \neq S_{1,2}$.*

(ii) $\ker(\eta) \cong \mathbf{Z}_2$ *for* $S \in \{S_{1,1}, S_{1,2}, S_{2,0}\}$, $\ker(\eta) \cong \mathbf{Z}_2 \bigoplus \mathbf{Z}_2$ *for* $S = S_{0,4}$, *and* $\ker(\eta)$ *is trivial otherwise.*

(iii) $\mathrm{Im}(\eta) = \mathrm{Aut}^*(C(S)) \lneqq \mathrm{Aut}(C(S))$ *when* $S = S_{1,2}$, *where* $\mathrm{Aut}^*(C(S))$ *is the subgroup of* $\mathrm{Aut}(C(S))$ *which preserves the set of vertices of* $C(S)$.

Thus, $\mathrm{Mod}(S)$ acts on $C(S)$ by permuting its vertices. Theorem 2.15 will play a central role in establishing that the *pants complex*, a complex we will define shortly, shares a similar relationship with $\mathrm{Mod}(S)$.

3. THE PANTS COMPLEX $C_P(S)$

In this chapter, we develop the theory of a 2-dimensional cell complex associated to a surface, the pants complex. Its 1-skeleton was introduced by Hatcher and Thurston to give a proof of the fact that $\mathrm{Mod}(S)$ is finitely presented [5]. Our goal is to see this complex as a combinatorial model for $\mathrm{Mod}(S)$. This chapter is primarily based on the work of Margalit [12].

3.1 Pair of pants

Let $S = S_{g,b}$ be an orientable surface of genus g with b boundary components and $\chi(S) < 0$. Recall that the *extended mapping class group* of S, denoted by $\mathrm{Mod}(S)$, is defined as the group of isotopy classes of self-homeomorphisms of S.

A *pair of pants* is any surface that is homeomorphic to $S_{0,3}$ (see Figure 3.1). In this section, we will understand how to decompose a surface into pairs of pants, and study a simplicial complex called the pants graph associated with this decomposition.

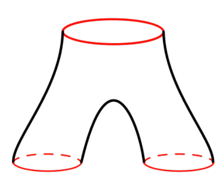


Fig. 3.1: A pair of pants.

Definition 3.1. A *pants decomposition* of S is a maximal collection of distinct isotopy classes of pairwise disjoint nontrivial simple closed curves on S.

Proposition 3.2. *The number of pair of pants in the complement of the pants decomposition is $2g + b - 2$.*

The dimension of the curve complex for $S_{g,b}$ is $3g + b - 4$. This is because a pants decomposition corresponds to a top simplex in the curve complex. We also note that there may be infinitely many ways to cut a surface into pairs of pants. For example, given a pants decomposition $\{\alpha_1, \alpha_2, \ldots\}$ with α_1 lying on the subsurface $S_{1,1}$, consider the Dehn twist $T_c^n(\alpha_1)$ of α_1 about a curve c intersecting α_1 minimally once. Then $\{T^n{}_c(\alpha_1), \alpha_2, \ldots\}$ is a distinct pants decomposition of S for each $n \in \mathbb{N}$.

We aim to construct a graph with vertices representing pants decompositions of the surface. Therefore, we need to define when two vertices are joined by an edge.

Definition 3.3. Given two pants decompositions p and p' of S, we say that they differ by an *elementary move* if:

(i) They differ by one curve, say α_1 and α_1'.

(ii) The geometric interaction number between α_1 and α_1' is minimal on the subsurface that they lie on in the complement of $p \cap p'$.

Let α_1 and α_1' lie on the subsurface S' of S. Then $S = S_{1,1}$ or $S_{0,4}$ depending upon whether α_1 forms the boundary of one pair of pants or two pair of pants, respectively. For $S' = S_{1,1}$, minimal intersection means $i(\alpha_1, \alpha_1') = 1$, and for $S = S_{0,4}$, it means $i(\alpha_1, \alpha_1') = 2$. We will represent an elementary move by $\alpha_1 \to \alpha_1'$. Since the curve complex of S, where $S = S_{1,1}$ or $S_{0,4}$, is the Farey graph, there are infinitely but countably many elementary moves of the form $\alpha_1 \to \star$.

Definition 3.4. The *pants graph* of S, denoted $C_P^1(S)$, is an abstract graph with vertices as pants decompositions and edges when two pants decompositions differ by an elementary move.

For $S = S_{1,0}, S_{1,1}$ and $S_{0,4}$, the definitions of $C_P^1(S)$ and the 1-skeleton of $C(S)$, thus, coincide, which are all abstractly isomorphic to the Farey Graph.

Theorem 3.5. *Let $S = S_{g,b}$ be an orientable surface of genus g with b boundary components and Euler characteristic $\chi(S) < 0$. Then* $\mathrm{Aut}(C_P^1(S)) \cong \mathrm{Aut}(C(S))$.

To construct an isomorphism $\phi : \mathrm{Aut}(C_P^1(S)) \to \mathrm{Aut}(C(S))$, we will prove that the set of abstract marked Farey graphs in $C_P^1(S)$ surjects onto the set of isotopy classes of non-trivial simple closed curves, denoted by $C^{(0)}(S)$. An *abstract marked Farey graph* is an ordered pair (F, X), where X is a vertex in $C_P^1(S)$ and F is a subgraph of $C_P^1(S)$ abstractly isomorphic to the Farey graph.

Remark 3.6. Since a Farey graph is essentially made up of chain-connected triangles, let us first understand what triangles represent in a pants graph. Suppose P,Q and R are the vertices of a triangle in $C_P^1(S)$, and let $P = \{\alpha_1^1, \alpha_2, \ldots, \alpha_n\}$, then $Q = \{\alpha_1^2, \alpha_2, \ldots, \alpha_n\}$ such that $i(\alpha_1^1, \alpha_1^2) > 0$ is minimal. Similarly, we must have $R = \{\alpha_1^3, \alpha_2, \ldots, \alpha_n\}$. Thus, the vertices in a triangle in $C_P^1(S)$ has pants decompositions with $(n-1)$ curves common, and only one moving curve.

Lemma 3.7. *There is a natural surjection f from the set of abstract marked Farey graphs in $C_P^1(S)$ to $C^{(0)}(S)$.*

Proof. The Farey graph is chain-connected, that is, any two triangles in the graph can be connected by a sequence of triangles such that consecutive triangles share an edge. Since the pants decomposition associated to the vertices in a triangle have only one moving curve, chain-connectedness implies that every vertex of the Farey graph has the pants decomposition of the form $\{\star, \alpha_2, \alpha_3, \ldots, \alpha_n\}$. Given a Farey graph, a marked vertex corresponds to a unique moving curve, say α, which corresponds to a vertex v in $C(S)$. Let f be a map that assigns to a marked Farey graph (F, X) in $C_P^1(S)$ this unique vertex v of $C(S)$.

Now, given any vertex v in the curve complex of S associated to a curve β, consider any pants decomposition containing β, say $\{\beta, \beta_2, \ldots, \beta_n\}$, associated to a vertex X' in $C_P^1(s)$. Now consider the subgraph F' formed by all the vertices of the form $\{\star, \beta_2, \ldots, \beta_n\}$. Since $S \setminus \{\beta_2, \ldots, \beta_n\}$ is $n-1$ pairs of pants and a $S_{1,1}$ or $S_{0,4}$, F' is an abstract Farey graph. Thus, for any v in $C^{(0)}(S)$, there is a (F', X') associated to it. $\square$

Notation 3.8. For any abstract marked Farey graph (F, X) in $C^1_P(S)$, we will denote $f((F, X)) = v_{(F,X)}$.

Now, we define the map $\phi : \mathrm{Aut}(C^1_P(S)) \to \mathrm{Aut}(C(S))$ as follows.

Definition 3.9. The map $\phi : \mathrm{Aut}(C^1_P(S)) \to \mathrm{Aut}(C(S))$ is defined as

$$\phi(A)\,(v) = (f \circ A \circ f^{-1})\,(v).$$

Since f is not injective, in the next section we show that ϕ is well-defined.

3.2 Relation between $C_P(S)$ and $C(S)$

In this section, we will prove that the map $\phi : \mathrm{Aut}(C^1_P(S)) \to \mathrm{Aut}(C(S))$ is well-defined. To do so, we will give a classification of small circuits. We will also attach 2-cells to the pants graph based on sequences of elementary moves, resulting in a 2-dimensional cell complex.

3.2.1 Construction of $C_P(S)$

We define two elementary moves and then describe the five basic sequences of these elementary moves which we recognise as the 2-cells attached to $C^1_P(S)$.

Definition 3.10. Let P be a pants decomposition of S containing a curve α such that $S \setminus \alpha$ has $S_{1,1}$ as a connected component. Then there exists a β such that β intersects α in exactly one point transversely and does not intersect any other curve in P. The elementary move associated with $\alpha \to \beta$ is called a *simple move* or an S*-move*.

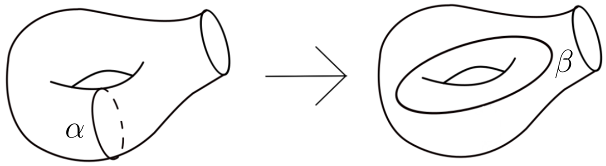


Fig. 3.2: An S-move.

Definition 3.11. Let P be a pants decomposition of S containing a curve α such that $S \setminus \alpha$ has $S_{0,4}$ as a connected component. Then there exists a β such that β intersects α in exactly two points transversely and does not intersect any other curve in P. The elementary move associated with $\alpha \to \beta$ is called an *associative move* or an A-*move*.

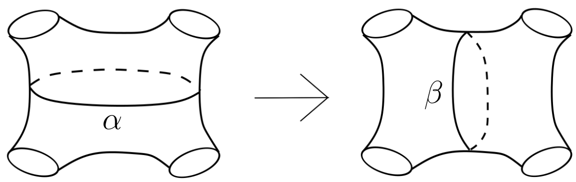


Fig. 3.3: An A-move.

It was shown by Hatcher and Thurston [5] that any two pants decompositions can be joined by a finite sequence of elementary moves. Later, Hatcher [4] proved that any sequence of moves joining two pants decompositions can be obtained from another such sequence by finitely many insertions or deletions of the five basic sequences of moves described below. We will attach the following 2-cells to the pants graph, corresponding to these five sequences of moves.

(i) **Triangular: (3S) or (3A)**

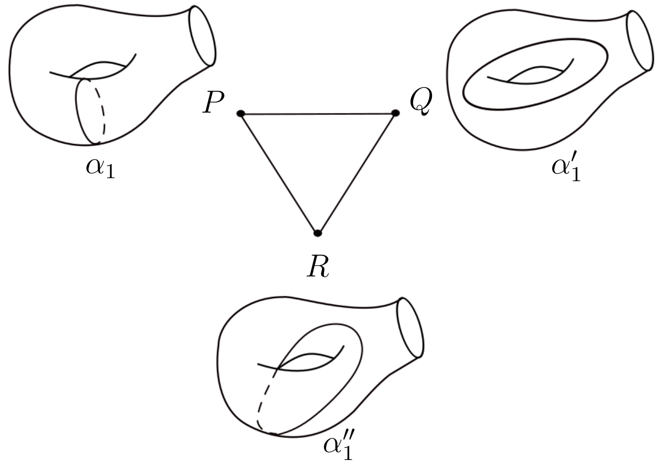


Fig. 3.4: A triangle corresponding to a 3S-move.

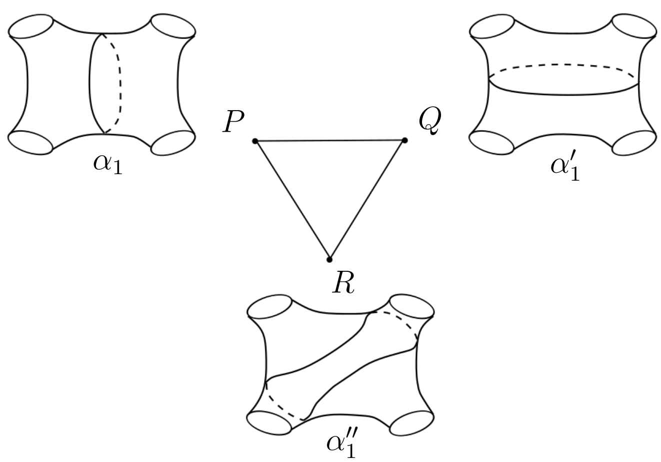


Fig. 3.5: A triangle corresponding to a 3A-move.

(ii) **Square: (C)**

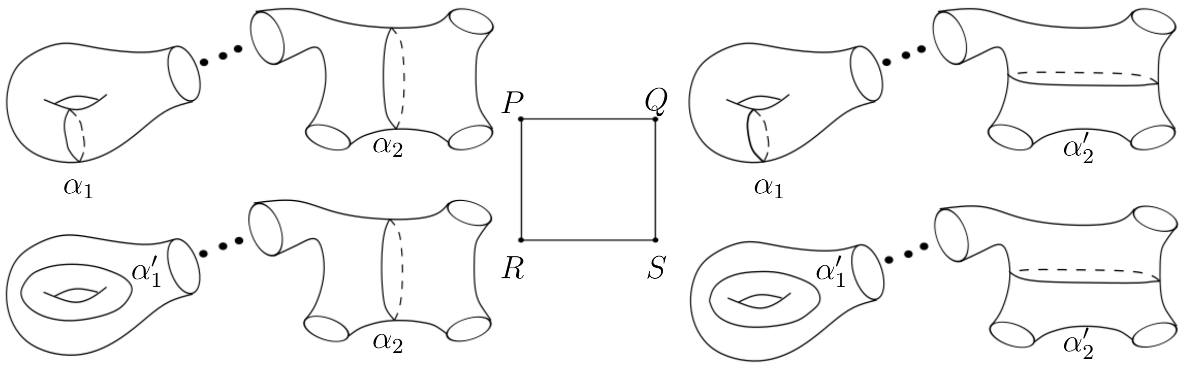


Fig. 3.6: A square corresponding to moves on disjoint subsurfaces.

(iii) **Pentagonal: ($5A$)**

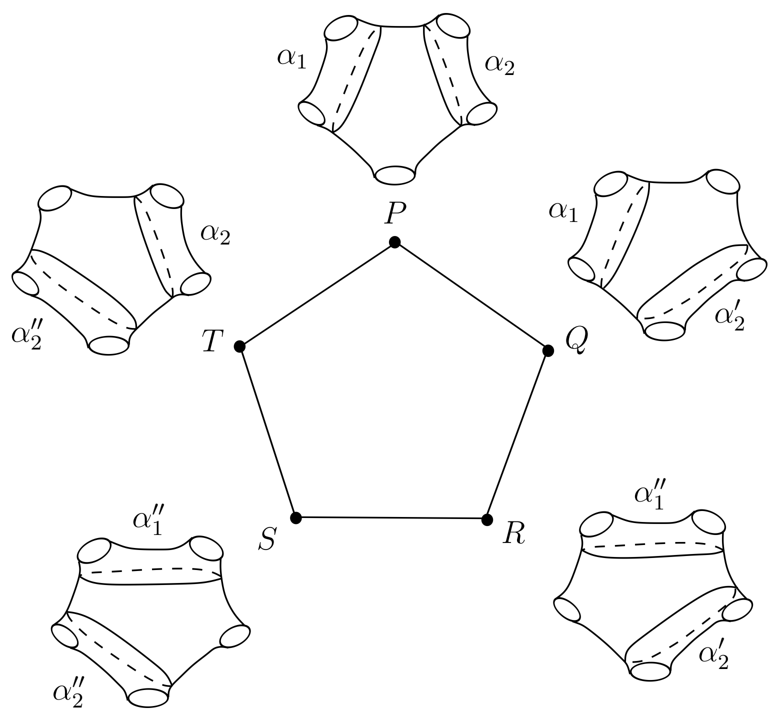


Fig. 3.7: A pentagon corresponding to a $5A$-move.

(iv) **Hexagonal: (6AS)**

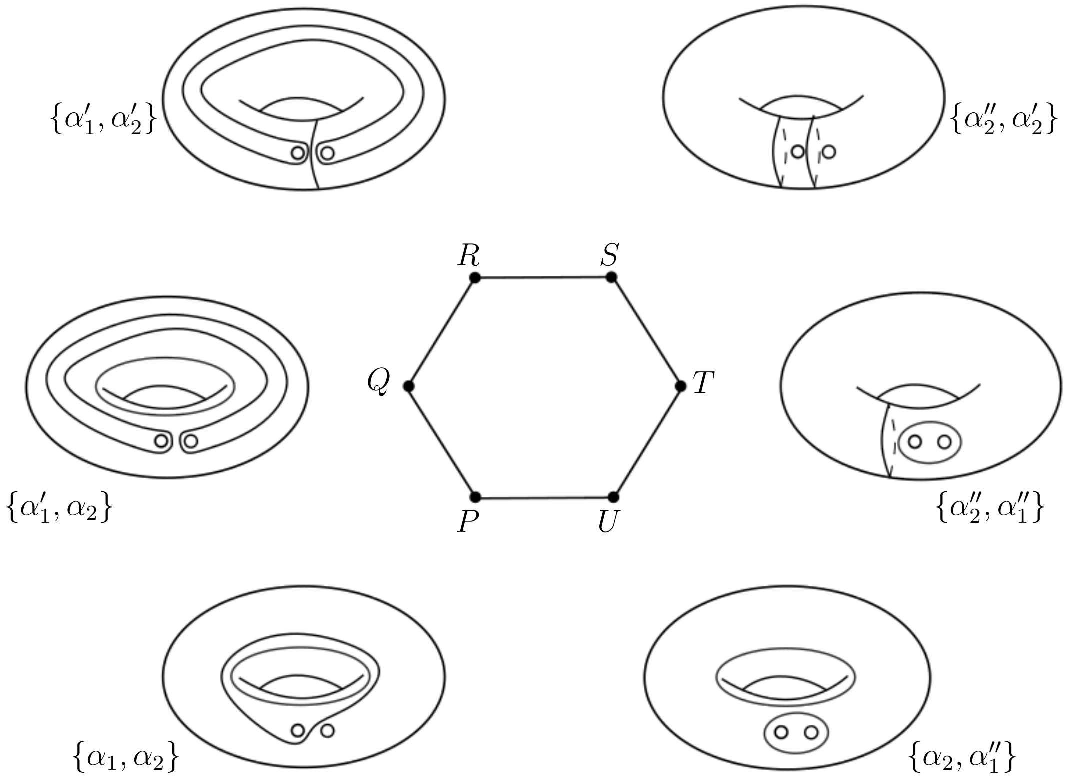


Fig. 3.8: A hexagon corresponding to a 6AS-move.

Definition 3.12. The 2-dimensional cell complex obtained by pasting 2-cells to the pants graph as described above is called the *pants complex*. The pants complex of a surface S is denoted by $C^P(S)$.

We define the action of extended mapping class group on the pants graph via a map θ as follows. For $[g] \in \text{Mod}(S)$ and a map $\theta : \text{Mod}(S) \to \text{Aut}(C_P(S))$, we have $\theta([g]) : C_P^1(S) \to C_P^1(S)$ defined as

$$\theta([g])(\{\alpha_1, \alpha_2, \dots, \alpha_n\}) = \{g(\alpha_1), g(\alpha_2), \dots, g(\alpha_n)\}.$$

The well-definedness of θ follows from the fact that homeomorphisms preserve geometric intersection number between isotopy classes of curves.

Now, we introduce the notion of circuits and alternating sequences.

Definition 3.13. A *circuit* is a subgraph of $C^1_P(S)$ that is homeomorphic to a circle.

We define triangles, squares, pentagons and hexagons to be circuits of the appropriate number of vertices. The following lemma characterizes the triangular 2-cells in the pants complex.

Lemma 3.14. *Every triangle in $C^1_P(S)$ is the boundary of a triangular 2-cell of $C_P(S)$.*

Proof. By Remark 3.6, since the vertices of a triangle PQR correspond to the pants decompositions $\{\alpha_1^1, \alpha_2, \ldots, \alpha_n\}$, $\{\alpha_1^2, \alpha_2, \ldots, \alpha_n\}$ and $\{\alpha_1^3, \alpha_2, \ldots, \alpha_n\}$, respectively, such that α_1^1, α_1^2 and α_1^3 lie on a common subsurface in the complement of the common $(n-1)$ curves. On this common subsurface, they intersect minimally as the pants decompositions associated to P, Q and R differ by an elementary move. Thus, PQR corresponds to a triangular 2-cell. □

3.2.2 Action of $\mathrm{Aut}(C_P(S))$ on $C(S)$

In this section, we prove that the map ϕ is well-defined. To do so, we first state and prove results about small circuits and rely on the fact that the pants graph is connected.

Remark 3.15. We can associate any edge in $C^1_P(S)$ to a unique Farey graph in the following way. Consider an edge PR in $C^1_P(S)$. The elementary move associated with the edge fixes a moving curve, denoted by $\star$. Let the associated abstract Farey graph be the subgraph with the vertices as $\{\star, \alpha_2, \ldots, \alpha_n\}$. Clearly, the edge PR belongs to this Farey graph.

Definition 3.16. A sequence of consecutive vertices $P_1, P_2, \ldots, P_m$ in a circuit is called *alternating* if for $1 < i < m$, the unique Farey graph containing the edge $P_{i-1}P_i$ is not the same as the unique Farey graph containing the edge P_iP_{i+1}. Equivalently, the pants decompositions associated to the vertices P_{i-1}, P_i and P_{i+1} do not have any $(n-1)$ curves in common.

Remark 3.17. A sequence PQR is alternating if and only if given PQ of the form $\star \to \alpha$, QR is not of the form $\alpha \to \star$.

Definition 3.18. A circuit with the property that any three vertices form an alternating sequence is called an *alternating circuit.*

Definition 3.19. A *small circuit* in $C_P^1(S)$ is a circuit with at most six edges. Triangles, squares, pentagons and hexagons are small circuits with $3, 4, 5$ and 6 edges, respectively.

Definition 3.20. A 2-*curve small circuit* is a small circuit such that the pants decompositions corresponding to its vertices have $(n-2)$ curves in common. That is, the pants decomposition of its vertices is of the form $\{\star, \star, \alpha_3, \ldots, \alpha_n\}$.

Lemma 3.21. *Automorphisms of $C_P^1(S)$ preserve alternating sequences.*

Proof. Let $A \in \mathrm{Aut}(C_P^1(S))$. Observe that A preserves circuits and sequences. Consider an alternating sequence $P_1, P_2, \ldots, P_m$. Suppose $A(P_{i-1}P_i)$ and $A(P_iP_{i+1})$ belong to the same Farey graph F for some $1 < i < m$. Consider the restriction of the automorphism $A\big|_{A^{-1}(F)} : A^{-1}(F) \to F$. Then $P_{i-1}P_i$ and P_iP_{i+1} belong to the same Farey graph $A^{-1}(F)$, which is a contradiction. Thus, A must preserve alternating sequences. $\square$

We have the following partial characterization of the small circuits that is preserved by any automorphism of the pants complex.

Lemma 3.22. *Any small circuit which is not a 2-curve small circuit is an alternating hexagon.*

Proof. Let ζ be a small circuit which is not a 2-curve small circuit and let one of its vertices be associated with the pants decomposition $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Without loss of generality, let $\alpha_1 \xrightarrow{r_1} \alpha_1'$, $\alpha_2 \xrightarrow{r_2} \alpha_2'$, and $\alpha_3 \xrightarrow{r_3} \alpha_3'$ be three elementary moves in ζ. To be a circuit, it has edges corresponding to the moves $\star \xrightarrow{r_1'} \alpha_1$, $\star \xrightarrow{r_2'} \alpha_2$ and $\star \xrightarrow{r_3'} \alpha_3$. These are six distinct moves because $i(\alpha_i', \alpha_i) > 0$ and $i(\alpha_i, \alpha_j) = 0$, and therefore, $\alpha_i \neq \alpha_j'$ for any $1 \leq i, j \leq 3$. Thus, there are at least six edges in ζ. Since it is a small circuit, ζ has exactly 6 edges. That is, ζ is a hexagon. Now, we have to prove that ζ is alternating.

We claim that r_i' is given by $\alpha_i' \to \alpha_i$ for $i = 1, 2, 3$. Suppose the claim is not true and r_1' is $\alpha_2' \to \alpha_1$. Then, $i(\alpha_2', \alpha_1) > 0$ and $i(\alpha_2', \alpha_2) > 0$. Out of $\alpha_1, \alpha_1', \alpha_2, \alpha_3, \alpha_3'$, we deduce that α_2' can appear in a pants decomposition only with α_1', α_3 and α_3'. Therefore, the only possibilities for r_1' are:

$$\begin{aligned} & \{\alpha_2', \alpha_1', \alpha_3\} \xrightarrow{r_1'} \{\alpha_1, \alpha_1', \alpha_3\} \\ or \quad & \{\alpha_2', \alpha_3, \alpha_3'\} \xrightarrow{r_1'} \{\alpha_1, \alpha_3, \alpha_3'\} \\ or \quad & \{\alpha_2', \alpha_1', \alpha_3'\} \xrightarrow{r_1'} \{\alpha_1, \alpha_1', \alpha_3'\}. \end{aligned}$$

For $i = 1, 2, 3$, since α_i and α_i' cannot appear in the same pants decomposition, none of the above three cases are possible. Thus, the claim is true.

Now, either ζ is alternating or r_i and r_i' appear as consecutive edges for at least one i. The latter cannot happen since it would imply that number of edges in ζ is less than six. Thus, ζ is an alternating hexagon. □

The following lemma is an immediate consequence of Lemmas 3.21- 3.22.

Lemma 3.23. *If $A \in \mathrm{Aut}(C_P^1(S))$ and ζ is a small circuit which is not an alternating hexagon, then $A(\zeta)$ is a 2-curve small circuit.*

To prove the following proposition, we will use the fact that $C_P^1(S)$ is connected [5]. Given any non-trivial simple closed curve α on a surface S with $\chi(S) < 0$, $S' = S \setminus \alpha$ also has $\chi(S) < 0$. Therefore, $C_P^1(S')$ is connected.

Proposition 3.24. *The map $\phi : \mathrm{Aut}(C_P^1(S)) \to \mathrm{Aut}(C(S))$ as in Definition 3.9 is well-defined.*

Proof. We recall from Definition 3.9 that for $A \in \mathrm{Aut}(C_P^1(S))$ and $v \in C^{(0)}(S)$, we have $\phi : \mathrm{Aut}(C_P^1(S)) \to \mathrm{Aut}(C(S))$ defined as $\phi(A)(v) = (f \circ A \circ f^{-1})(v)$. We show that given $v \in C^{(0)}(S)$, $\phi(A)(v)$ is independent of the choice of the abstract marked Farey graph in $f^{-1}(v)$. Let $v \in C^{(0)}(S)$ be associated to the curve α_1 on S. Let two distinct marked Farey graphs (F_v, X), $(F_v', X') \in f^{-1}(v)$. Let p and p' be the pants decompositions associated to the vertices X and X' in $C_P^1(S)$. Owing to the connectedness of $S \setminus \alpha_1$, we may assume that p and p' differ by an elementary move, say $\alpha_2 \to \alpha_2'$.

Suppose there exists a 2-curve small circuit ζ which is not an alternating hexagon such that four of its vertices, say $WXX'Y$, form an alternating sequence, with $WX \in (F_v, X)$ and $X'Y \in (F_v', X')$ as shown in Figure 3.9.

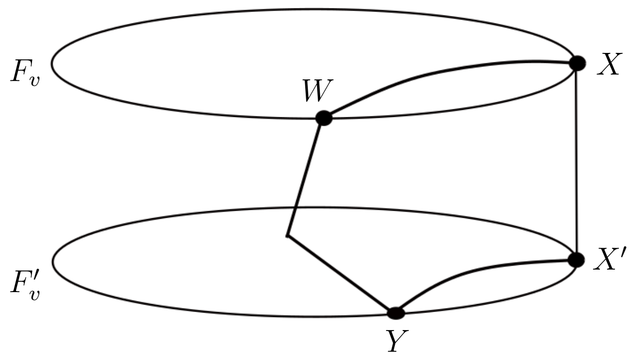


Fig. 3.9: The 2-curve small circuit ζ containing $WXX'Y$.

Suppose $(A(F_v), A(X))$ corresponds to a curve β_1 and $A(X)$ is associated to the pants decomposition $\{\beta_1, \beta_2, \ldots, \beta_n\}$. We show that $(A(F_v'), A(X'))$ corresponds to the same curve β_1. Since $A(W)A(X)$ forms an edge in $(A(F_v), A(X))$, it corresponds to a move $\star \to \beta_1$. Since $A(W)A(X)A(X')$ is an alternating sequence, $A(X)A(X')$ corresponds to a move of the form $\beta_2 \to \star$, say $\beta_2 \to \beta_2'$. Again, since $A(X)A(X')A(Y)$ forms an alternating sequence, $A(X')A(Y)$ cannot have β_2' as the moving curve. But since this sequence is part of a 2-curve small circuit, we get that $A(X')A(Y)$ is associated to the elementary move of the form $\beta_1 \to \star$, that is, the moving curve associated to the abstract marked Farey graph is β_1. Thus, ϕ is well-defined, if such a 2-curve small circuit exists.

Now, we find a 2-curve small circuit that is not an alternating hexagon and has four vertices that form an alternating sequence. Let $S' = S \setminus \{\alpha_3, \alpha_4, \ldots, \alpha_n\}$, then α_1 and α_2 on S' have the following possibilities.

(i) α_1 and α_2 on disconnected components of S'.

(ii) α_1 and α_2 lie on $S_{0,5}$.

(iii) α_1 and α_2 lie on $S_{1,2}$ and one of α_1, α_2 or α_2' is separating.

(iv) α_1 and α_2 lie on $S_{1,2}$ and all of α_1, α_2 or α_2' are non-separating.

The last two cases follow from the fact that if α_1 and α_2 lie on a connected component with genus g' and b' boundary components, then $3g' + b' - 3 = 2$. To find the 2-curve small circuit ζ, we take the boundary of the obvious 2-cells in each case, as follows. In (i) and (ii), let ζ be the boundary of a square 2-cell and a pentagonal 2-cell containing X and X', respectively.

In (iii), observe that a curve is separating on $S_{1,2}$ if and only if it is separating on S. Moreover, exactly one of α_1, α_2 and α_2' can be separating. This is because two separating curves in $S_{1,2}$ can't differ by an elementary move, and $S_{0,3}$ cannot have two separating curves. So, without loss of generality, we have the following two subcases:

(a) α_1 is separating.

(b) α_2 is separating. (The subcase when α_2' is separating is analogous to this case).

In both cases, choose ζ to be the hexagonal 2-cell (see Figure 3.8) . In case (a), let X and U correspond to T and U, and in case (b), they should correspond to S and T, respectively.

Case (iv) can be reduced to Case (iii) in the following manner. We claim that any elementary move on $S_{1,2}$ of the form $\{\alpha_1, \alpha_2\} \to \{\alpha_1, \alpha_2'\}$ with $\alpha_1, \alpha_2, \alpha_2'$ all non-separating, can be realised by a pair of elementary moves of the form $\{\alpha_1, \alpha_2\} \to \{\alpha_1, \alpha_2''\} \to \{\alpha_1, \alpha_2'\}$ where α_2'' is a separating curve.

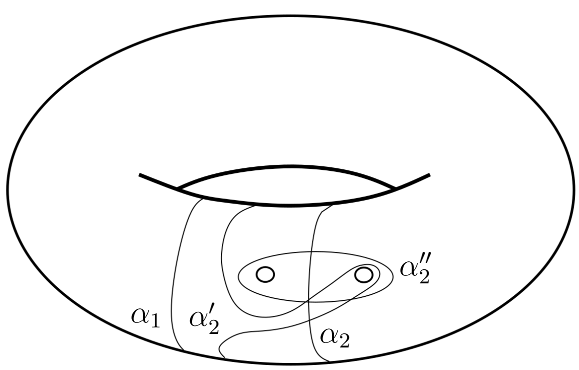


Fig. 3.10: Reduction of Case (iv) to Case (iii).

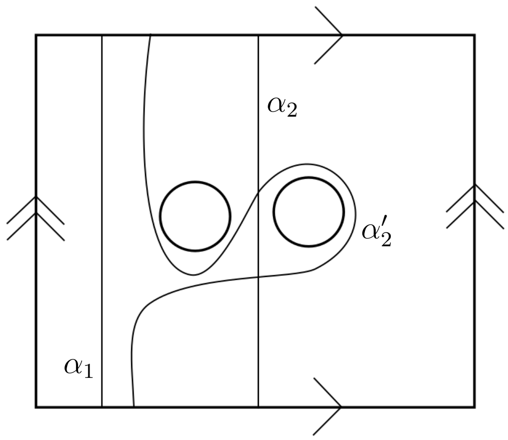


Fig. 3.11: Curves α_1, α_2 and α'_2 in Case (iv).

To complete the proof of well-definedness, we also show that $\phi(A) \in \mathrm{Aut}(C(S))$. By Remark 2.12 , it suffices to show that $\phi(A)$ is in $\mathrm{Aut}(C^1(S))$. Let v and w be vertices in $C(S)$ such that they form an edge in $C(S)$. Let F_v and F_w be the Farey graphs associated with the pants decompositions of the form $\{\star, \beta, \gamma_3, \ldots, \gamma_n\}$ and $\{\alpha, \star, \gamma_3, \ldots, \gamma_n\}$. They intersect at the vertex $X = \{\alpha, \beta, \gamma_3, \ldots, \gamma_n\}$ in $C_P^1(S)$. The marked Farey graphs (F_v, X) and (F_w, X) correspond to the vertices v and w that are joined by an edge. We claim that the vertices in $C(S)$ associated with the marked Farey graphs $(A(F_v), A(X))$ and $(A(F_w), A(X))$, say v' and w', are also joined by an edge. This is true because the curves corresponding to v' and w' occur together in the pants decomposition associated to the vertex $A(X)$. $\square$

3.3 Identifying 2-cells of $C_P(S)$ using its 1-cells

The aim of this section is to recognise the 2-cells of $C_P(S)$ simply by considering the combinatorics of its 1-skeleton, $C_P^1(S)$. The idea is to obtain a result analogous to Remark 2.13. Let $S = S_{g,b}$ be an orientable surface with $\chi(S) < 0$. The number of curves in any pants decomposition of S is $n = 3g + b - 3$. Recall that a circuit in $C_P^1(S)$ is a subgraph homeomorphic to a circle and triangles, squares, pentagons and hexagons are circuits with the appropriate number of vertices.

We have already shown that a triangular 2-cell can be recognised as a triangle in the 1-skeleton of the pants complex. We observe that the square and pentagonal 2-cells have boundaries as alternating squares and alternating pentagons, respectively.

Lemma 3.25. *Every alternating square in $C_P^1(S)$ is the boundary of a square 2-cell in $C_P(S)$.*

Proof. Let $ABCD$ be an alternating square. By Lemma 3.22 , it is a 2-curve small circuit. Let the $(n-2)$ common curves be $\alpha_3, \ldots, \alpha_n$. Using the alternating property of the circuit, $A \to B \to C \to D \to A$ is of the form

$$\{\alpha_1, \alpha_2\} \to \{\alpha_1, \alpha_2'\} \to \{\alpha_1', \alpha_2'\} \to \{\alpha_1', \alpha_2\} \to \{\alpha_1, \alpha_2\}.$$

We show that α_1 and α_2 lie on disconnected subsurfaces. Assume that $S' = S \setminus \{\alpha_3, \ldots, \alpha_n\}$ is a connected subsurface of S containing $\{\alpha_1, \alpha_2\}$. Because $3g' + b' - 3 = 2$ for $S' = S_{g',b'}$, S' must be $S_{0,5}$ or $S_{1,2}$. If $S' = S_{0,5}$, there is only one topological possibility for α_1 and α_2. If $S' = S_{1,2}$, there are two possibilities for α_1 and α_2, case 3 and case 4 in Proposition 3.24. In each of these cases, by observing the subsurface $S' \setminus \alpha_2$, it is clear that there is no α_1' disjoint from α_2 and α_2' that intersects α_1 minimally, which is not possible. Hence, α_1 and α_2 lie on disconnected subsurfaces. $\square$

Lemma 3.26. *Every alternating pentagon in $C_P^1(S)$ is the boundary of a pentagonal 2-cell in $C_P(S)$.*

Proof. **Step (i):** Let $ABCDE$ be an alternating pentagon in $C_P^1(S)$. By Lemma 3.22, $ABCDE$ is a 2-curve small circuit. Let the $(n-2)$ common curves be $\alpha_3, \ldots, \alpha_n$. Using the alternating property of the circuit, $A \to B \to C \to D \to E \to A$ is of the form

$$\{\alpha_1, \alpha_2\} \to \{\alpha_1, \alpha_2'\} \to \{\alpha_1', \alpha_2'\} \to \{\alpha_1', \alpha_2''\} \to \{\alpha_2'', \alpha_2\} \to \{\alpha_1, \alpha_2\}.$$

Since BCD and EAB are alternating, $\alpha_2 \in E$ and $\alpha_1' \notin E$. Also, $\alpha_1' \in D$ and $\alpha_2' \notin D$. **Step (ii):** We observe that $K = \alpha_1\alpha_1'\alpha_2\alpha_2'\alpha_2''\alpha_1$ forms a sequence such that adjacent curves differ by an elementary move (intersect minimally) and non-adjacent curves are disjoint. We also observe that α_1 and α_2 cannot lie on different subsurfaces since α_1' intersects them both non-trivially. Therefore, they must lie on $S_{0,5}$ or $S_{1,2}$.

Step (iii): Suppose $\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_1$ is a sequence of curves such that consecutive curves intersect minimally and others are disjoint. We will show that such a sequence of curves cannot lie on $S_{1,2}$. Since two separating curves on $S_{1,2}$ intersect at least four times (in the neighbourhood of the boundary components), at most one of these curves can be separating. So we divide this case into two subcases:

(a) Exactly one curve in the sequence K is separating.

(b) No curve in the sequence K is separating.

Case (a): In this case, suppose γ_1 is the separating curve, then $S_{1,2} \setminus \gamma_1 = S_{1,1} \cup S_{0.3}$. Since $i(\gamma_1, \gamma_4) = i(\gamma_1, \gamma_3) = 0$, and γ_3, γ_4 intersect minimally on $S_{1,1}$, we have $i(\gamma_3, \gamma_4) = 1$. We will say a curve is of (p, q)-type if it is of (p, q)-type if we ignore the boundary components and just consider them on a torus. Suppose γ_3 is $(1, 0)$-type (this assumption works by the Theorem 1.9), and γ_4 be of $(0, 1)$-type, as in Figure 3.12. Since γ_2 and γ_4 are both non-separating on $S_{1,2}$ with $i(\gamma_2, \gamma_4) = 0$, γ_2 must be of $(0, q)$-type. Similarly, γ_5 must be of $(p, 0)$-type. But this implies that $i(\gamma_2, \gamma_5) > 0$, which is not possible.

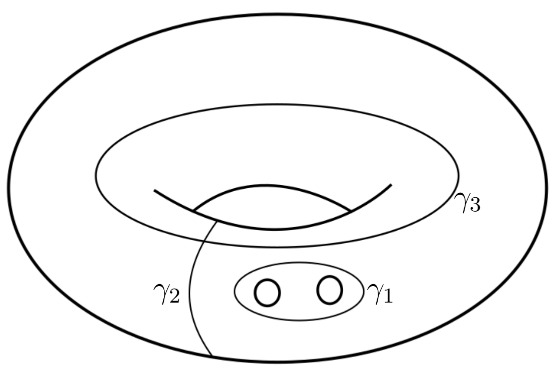


Fig. 3.12: The curve γ_1 is separating in Case (a).

Case (b): Let all the curves in the sequence K be non-separating. The complement of any curve is homeomorphic to $S_{0,4}$, and therefore, adjacent curves in K intersect exactly twice. As explained in the proof of Proposition 3.24, all elementary moves involving three nonseparating curves on $S_{1,2}$ are topologically equivalent. Thus, without loss of generality, we assume that γ_1, γ_3 and γ_4 are the curves as in the Figure 3.13.

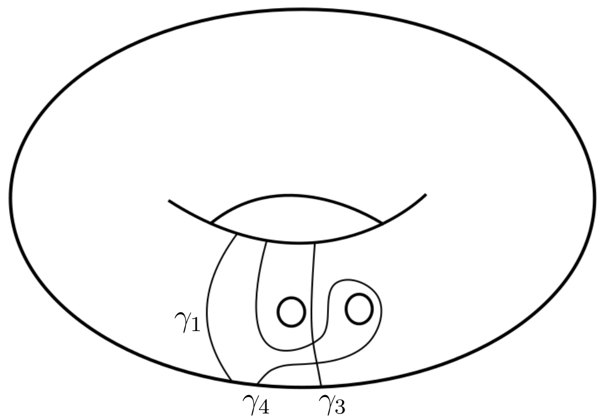


Fig. 3.13: The curve γ_1 is nonseparating in Case (b).

If $S_{1,2}$ is cut along γ_1 and γ_4, we get two pairs of pants such that their boundary components are γ_1, γ_4, and the boundaries of $S_{1,2}$. Since $i(\gamma_1, \gamma_2) = 2$, the two components of γ_2 form an essential arcs of both pairs of pants and intersect the boundaries formed by γ_1. However, any essential arc with end points on γ_1 will intersect γ_3 at least twice in each pair of pants (see Figure 3.14). Thus, $i(\gamma_2, \gamma_3) \geq 4$, which is not possible.

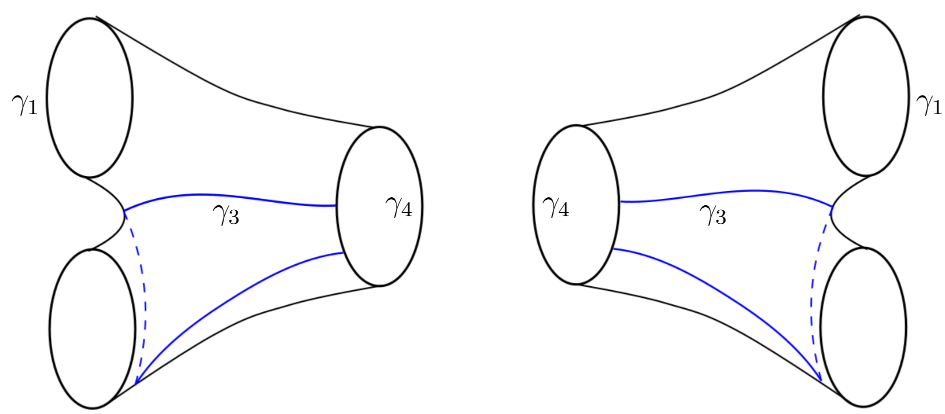


Fig. 3.14: The curve γ_3 on $S \setminus \{\gamma_1, \gamma_4\}$ in Case (b).

Step (iv): Thus, the only possibility is that α_1 and α_2 lie on $S_{0,5}$. Now, we show that the sequence of curves K must be equivalent to the sequence in the definition of pentagonal 2-cells. The complement of any curve of K on $S_{0,5}$ is $S_{0,4}$, hence adjacent curves in K

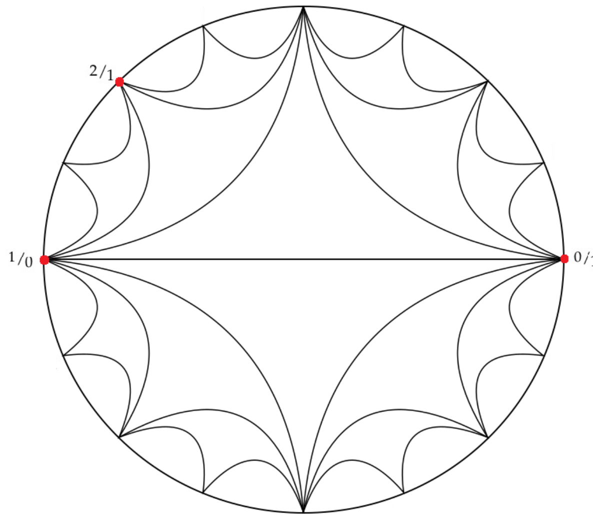


Fig. 3.15: An example of a square triplet.

intersect twice. Therefore, K is the boundary of a pentagonal 2-cell. □

We note that contrary to square and pentagonal 2-cells, the boundary of a hexagonal 2-cell is not an alternating hexagon as the pants decompositions associated to the vertices Q, P, U have $(n-1)$ curves in common. However, we say that the boundary of a hexagonal 2-cell is almost alternating, as we define shortly.

Definition 3.27. A *square triplet* is a set of three vertices in the pants graph $C_P^1(S)$, which lie on a common square in a Farey graph but not on a common triangle. The unique point that has edges connected to two points is called the *central point*. The other two points are called *outer points*.

Example 3.28. The vertices $(1, 0), (0, 1)$ and $(2, 1)$ form a square triplet, as shown in Figure 3.15.

Remark 3.29. All square triplets are equivalent on the pants graph of $S_{1,1}$ and $S_{0,4}$ upto the action of $\mathrm{Mod}(S)$.

Definition 3.30. An *almost-alternating hexagon* in $C_P^1(S)$ is a hexagon with an alternating sequence of 6 vertices, and a sequence of 3 vertices that form a square triplet.

Lemma 3.31. *Every almost-alternating hexagon in $C^1_P(S)$ is the boundary of a hexagonal 2-cell in $C_P(S)$.*

Proof. Let $ABCDEFA$ form an almost alternating hexagon with FAB forming a square triplet. Then $ABCDEF$ must be the alternating sequence.

Step 1: We find the configuration of the curves. Since an almost alternating hexagon is not alternating, it is a 2-curve small circuit. Let $\{\alpha_3, \dots, \alpha_n\}$ be the $(n-2)$ common curves and $A = \{\alpha_1, \alpha_2\}$. Since FAB forms a square triplet, we have $B = \{\alpha_1', \alpha_2\}, F = \{\alpha_1'', \alpha_2\}$, and $i(\alpha_1', \alpha_1'')$ is not minimal. Since $C = \{\alpha_1', \alpha_2'\}$ and BCD is alternating, we have $\alpha_1' \notin D$ but $\alpha_2' \in D$. Similarly, since EFA is alternating, we have $\alpha_1'' \in E$. Thus, the pants decompositions associated to the hexagon $ABCDEFA$ is

$$\{\alpha_1, \alpha_2\} \to \{\alpha_1', \alpha_2\} \to \{\alpha_1', \alpha_2'\} \to \{\alpha_2'', \alpha_2'\} \to \{\alpha_2'', \alpha_1''\} \to \{\alpha_2, \alpha_1''\} \to \{\alpha_1, \alpha_2\}.$$

Step 2: We show that α_1 and α_2 do not lie on disjoint subsurfaces. We observe that there is a chain of curves connecting α_1 to α_2 in $S \setminus \{\alpha_3, \dots \alpha_n\}$, as follows.

$$\alpha_1 \to \alpha_1'' \to \alpha_2' \to \alpha_2.$$

Therefore, α_1 and α_2 must lie on a connected subsurface.

Step 3: Assume that α_1 and α_2 do lie on $S_{0,5}$. FAB forms the square triplet such that $\alpha_2 \in F, A, B$. If we take $S_{0,5} \setminus \alpha_2 = S_{0,4}$, then FAB lies on the pants graph associated to the subsurface $S_{0,4}$. Since squares are characterized by the topological property of the curves in the associated pants decompositions (the geometric intersection numbers), any two squares in $C^1_P(S_{0,4})$ are topologically equivalent. Thus, the pants decompositions associated to F, A and B are as in Figure 3.16. In this figure, let a boundary component be represented by a puncture and a curve be represented by an arc. To recover the curve, take a boundary of a small neighbourhood of the arc. In such a presentation, it is easy to see that α_2' and α_2'' must be represented by arcs which have an endpoint at the puncture a. This implies that $i(\alpha_2', \alpha_2'') > 0$, which is not possible since they both

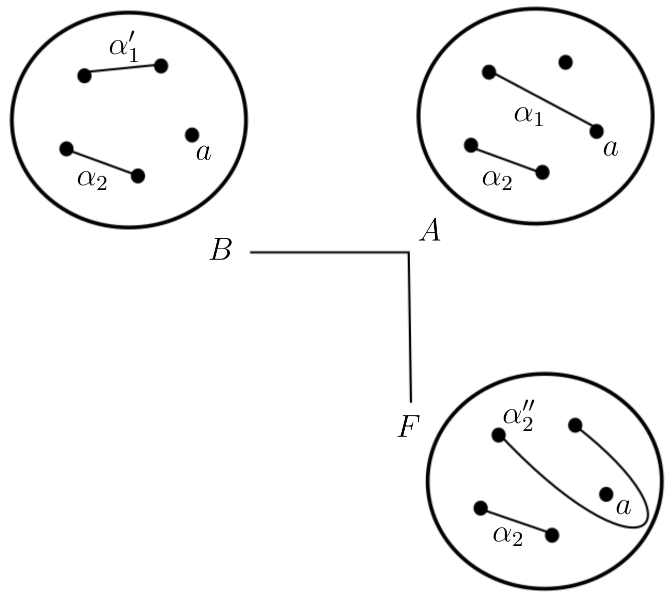


Fig. 3.16: The square triplet FAB as in Step 3.

appear in the pants decomposition associated to the vertex D. Thus, α_1 and α_2 must lie on $S_{1,2}$.

Step 4: Suppose α_2 is separating on $S_{1,2}$, then $S_{1,2} \setminus \alpha_2 = S_{0,3} \cup S_{1,1}$. Then, $\{\alpha_1\}, \{\alpha_1'\}$ and $\{\alpha_1''\}$ are pants decompositions of $S_{1,1}$, whose associated vertices in $C_P^1(S_{1,1})$ form a square triplet. Since square triplets are topologically unique, assume that α_1, α_1' and α_1'' are of $(2,1), (0,1)$ and $(1,0)$-type, respectively. As in the proof of Lemma 3.26, we can assume α_1, α_1' and α_1'' to be of the same type on $S_{1,2}$ as well.

Since there are elementary moves of the form $\alpha_2 \to \alpha_2'$ and $\alpha_2'' \to \alpha_2$, and two separating curves on $S_{1,2}$ must intersect at least four times, we have that α_2' and α_2'' must both be nonseparating. Also, since curves of different types intersect at least ones, and $\{\alpha_2', \alpha_1'\}$ and $\{\alpha_2'', \alpha_1''\}$ are pants decompositions of $S_{1,2}$, we have that α_2' and α_2'' must be of the type $(0,1)$ and $(1,0)$, respectively. But α_2' and α_2'' appear in a pants decomposition together, which is not possible. Therefore, α_2 is nonseparating.

Step 5: Assume that α_1 is separating on $S_{1,2}$. Since all pants decompositions of $S_{1,2}$ containing a separating curve are topologically equivalent, α_1 and α_2 must be as shown in the Figure 3.17. Since, $i(\alpha_1', \alpha_1) = 2$ and $i(\alpha_1', \alpha_2) = 0$, we have α_1' as shown in Figure 3.17. Since FAB forms a square triplet, the choice of α_1'' is topologically unique, and is as shown in the Figure 3.17. Therefore, $i(\alpha_2', \alpha_1') = 0$ and $\alpha_2 \to \alpha_2'$ and $\alpha_2' \to \alpha_1''$ form elementary moves. It follows from the proof of Lemma 3.26 that there can be no such α_2' on $S_{1,2}$. Therefore, α_1 is nonseparating.

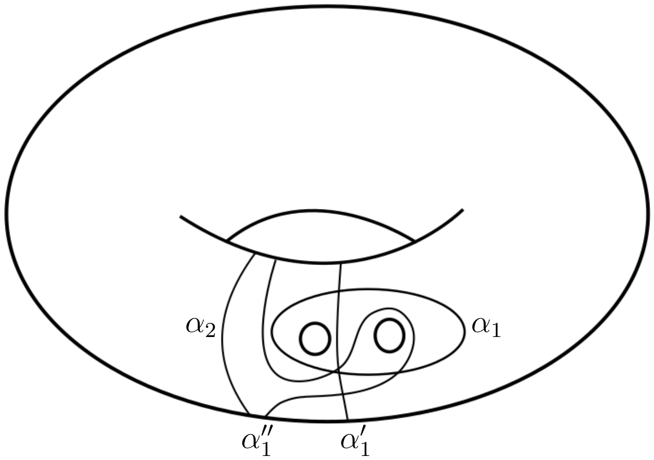


Fig. 3.17: Curves α_1 as in Step 5.

Step 6: Suppose that α_1, α_2 are nonseparating on $S_{1,2}$. Without loss of generality, assume they are as shown in the Figure 3.18. If we assume that α_1' is nonseparating on $S_{1,2}$, then because $\alpha_1 \to \alpha_1'$ in an elementary move, the choice of α_1' is topologically unique. Since FAB forms a square triplet in $C_P^1(S_{0,4}) = S \setminus \alpha_2$, we have $i(\alpha_1, \alpha_1'') = i(\alpha_1, \alpha_1') = 2$ and $i(\alpha_1', \alpha_1'') > 2$. Then the choice of α_1' and α_1'' should be as shown in the Figure 3.18.

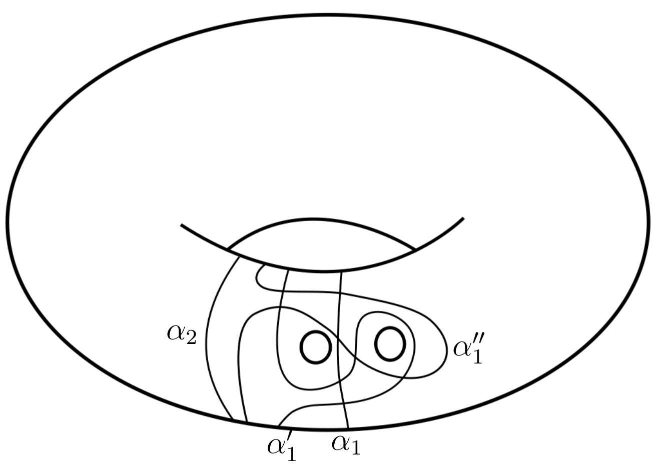


Fig. 3.18: Nonseparating curve α_1' in Step 6.

Since $i(\alpha_2', \alpha_1') = 0$, and $\alpha_2' \to \alpha_1''$ and $\alpha_2 \to \alpha_2'$ appear as elementary moves, it follows as in proof of Proposition 3.24 that no such α_2' can exist. Thus, α_1' must be separating. Similarly, it follows that α_1'' must be separating.

Step 7: We show that the choices of $\alpha_1, \alpha_1', \alpha_1'', \alpha_2, \alpha_2', \alpha_2''$ on $S_{1,2}$ are topologically unique. By Step 4, 5 and 6, it follows that $\alpha_1, \alpha_2, \alpha_1', \alpha_1''$ are as shown in the Figure 3.19.

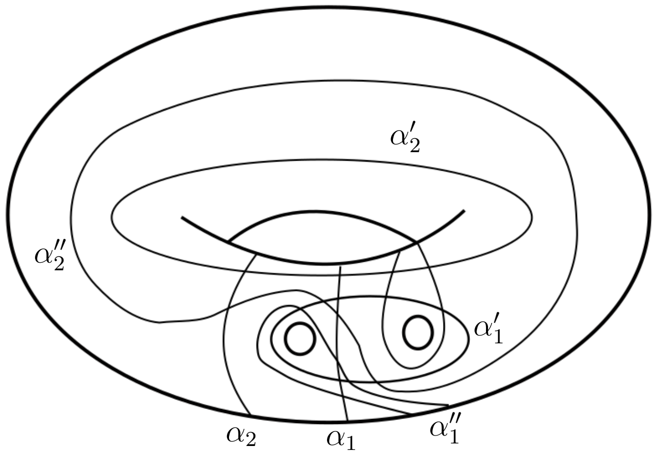


Fig. 3.19: Curves as in Step 7.

The curves α_2' and α_2'' are non-separating since they have intersection number 2 with each separating curve α_1' and α_1''. Since α_2' appears in a pants decomposition with α_1' and α_2'', these curves have intersection number 0 with α_2'. With α_2 and α_1'', it has intersection numbers 1 and 2, respectively. A similar analysis for α_2'' gives us the unique choice of these curves, which is as shown in the Figure 3.19, that corresponds to the circuit associated with a hexagonal 2-cell. □

3.4 Main results

We now state the results that follow as a consequence of the theory developed in the previous sections. The following theorem is a direct consequence of the Lemmas 3.14, 3.25, 3.26 and 3.31.

Theorem 3.32. *If S is an orientable surface with $\chi(S) < 0$, then:*

$$\mathrm{Aut}(C_P(S)) \cong \mathrm{Aut}(C_P^1(S)).$$

Theorem 3.33. *If $S \neq S_{0,3}$ is an orientable surface with $\chi(S) < 0$, and $\theta : \mathrm{Mod}(S) \to \mathrm{Aut}(C_P(S))$ is the natural map, then the following statements hold.*

(i) θ is surjective.

(ii) $\ker(\theta) \cong \mathbf{Z}_2$ for $S \in \{S_{1,1}, S_{1,2}, S_{2,0}\}$, $\ker(\theta) \cong \mathbf{Z}_2 \bigoplus \mathbf{Z}_2$ for $S = S_{0,4}$, and $\ker(\theta)$ is trivial otherwise.

To prove the Theorem 3.33, we will prove a series of lemmas to show that the map ϕ is an isomorphism of groups. We will consider the following diagram which we will show to be commutative.

$$\begin{tikzcd} & \mathrm{Mod}(S) \arrow[dl, "\theta"'] \arrow[dr, "\eta"] & \\ \mathrm{Aut}(C_P(S)) \arrow[r, "i"] & \mathrm{Aut}(C_p^1(S)) \arrow[r, "\phi"] & \mathrm{Aut}(C(S)) \end{tikzcd} \tag{3.1}$$

Here, i is the isomorphism given by the Theorem 3.32 and η is the map in Theorem 2.15.

Lemma 3.34. *The map ϕ is a group homomorphism.*

Proof. For $A, B \in \mathrm{Aut}(C_P^1(S))$ and $v, w \in C^{(0)}(S)$, let $f^{-1}(v)$ contain (F, X) and $f^{-1}(w)$ contain $(B(F), B(X))$. Then, $\phi(AB)(v)$ corresponds to the vertex in $C(S)$ associated with $(AB(F), AB(X))$. We observe that $\phi(A)\phi(B)(v) = \phi(A)(w) = (f \circ A \circ f^{-1})(w)$. Since we have already shown that ϕ is well-defined irrespective of the choice of the marked Farey graph under f^{-1}, we can choose $f^{-1}(w) = (B(F), B(X))$. Thus, ϕ is a homomorphism. □

Lemma 3.35. *The map ϕ is surjective.*

Proof. Let $[g] \in \mathrm{Mod}(S)$, and v be the vertex associated to a curve α_1 on S. Then

$$(\phi \circ i \circ \theta)([g])(v) = \phi(i \circ \theta([g]))(v) = f \circ (i \circ \theta[g]) \circ f^{-1}(v) \tag{3.2}$$

is the vertex corresponding to the marked Farey graph $(i \circ \theta\,[g]\,(F_v),\ i \circ \theta\,[g]\,(X_v))$. If $X_v = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $F_v = \{\star, \alpha_2, \ldots, \alpha_n\}$, then $(i \circ \theta\,[g]\,(F_v),\ i \circ \theta\,[g]\,(X_v))$ corresponds to the marked Farey graph $(\{\star, g(\alpha_2), \ldots, g(\alpha_n)\},\ \{g(\alpha_1), g(\alpha_2), \ldots, g(\alpha_n)\})$.

Thus, $(\phi \circ i \circ \theta)([g])(v) = \eta([g])(v)$, and Diagram 3.1 commutes. Then, surjectivity of the map ϕ follows from the surjectivity of the map η. $\square$

Lemma 3.36. *The map ϕ is injective.*

Proof. For $A \in \mathrm{Aut}(C_P^1(S))$, let $\phi(A)$ is be trivial in $\mathrm{Aut}(C(S))$, that is, $A \in \ker \phi$. Let X be a vertex in $C_P^1(S)$ associated to the pants decomposition $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let v_i be the vertex in the curve complex associated to the curve α_i for $1 \leq i \leq n$. For the Farey graph $F_{v_i} = \{\alpha_1, \ldots, \alpha_{i-1}, \star, \alpha_{i+1}, \ldots, \alpha_n\}$, each (F_{v_i}, X) corresponds to α_i such that the intersection of all F_{v_i}'s is X in $C_P^1(S)$. Since $(A(F_{v_i}), A(X))$ is a marked Farey graphs corresponding to v_i and $\phi(A)$ is identity, the intersection point of all $A(F_{v_i})$'s is $A(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} = X$. That is, A is the identity of $\mathrm{Aut}(C_P^1(S))$. Hence, the kernel of ϕ is trivial, implying that ϕ is injective. $\square$

Proposition 3.37. *ϕ is an isomorphism of groups.*

Finally, we prove Theorem 3.33.

Proof. The commutativity of Diagram 3.1 is shown in the proof of Lemma 3.35. Here, i and ϕ are isomorphisms (from Proposition 3.37 and Theorem 3.32), and η is surjective for $S \neq S_{1,2}$. For $S = S_{1,2}$, the image of η is the subgroup $\mathrm{Aut}^* C(S)$ of $\mathrm{Aut}(C(S))$ which preserves the set of vertices of $C(S)$ associated to nonseparating curves on S. We will show that image of ϕ lies in $\mathrm{Aut}^* C(S)$.

Let v be a vertex of $C(S)$ associated to a nonseparating curve α and $X \in C_P^1(S)$ be the vertex associated to the pants decomposition $\{\alpha, \beta\}$, where β is also nonseparating, with $(F_v, X) \in f^{-1}(v)$. Since α, β lie on $S_{1,2}$, there exists a hexagonal 2-cell containing X, where X is the center of the square triplet in the circuit. By Lemma 3.31, it follows that α is nonseparating. Since almost-alternating and non-alternating sequences are preserved by $\mathrm{Aut}(C_P^1(S))$, the independence of ϕ from the choice of marked Farey graph under f^{-1} proves that image of ϕ lies in $\mathrm{Aut}^* C(S)$. Thus, θ is surjective. The second part follows from the second part of the Theorem 2.15. $\square$

BIBLIOGRAPHY

[1] Max Dehn. *Papers on group theory and topology*. Springer Science & Business Media, 2012.

[2] Benson Farb and Dan Margalit. *A primer on mapping class groups (pms-49)*. Princeton university press, 2011.

[3] Willam J Harvey. "Boundary structure of the modular group". In: *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, NY, 1978)*. Vol. 97. 1981, pp. 245–251.

[4] Allen Hatcher. "Pants decompositions of surfaces". In: *arXiv preprint math/9906084* (1999).

[5] Allen Hatcher and William Thurston. "A presentation for the mapping class group of a closed orientable surface". In: *Topology* 19.3 (1980), pp. 221–237.

[6] John Hempel. "3-manifolds as viewed from the curve complex". In: *Topology* 40.3 (2001), pp. 631–657.

[7] Nikolai V Ivanov. "Automorphisms of complexes of curves and of Teichmuller spaces". In: *International Mathematics Research Notices* 1997.14 (1997), pp. 651–666.

[8] Mustafa Korkmaz. "Automorphisms of complexes of curves on punctured spheres and on punctured tori". In: *Topology and its Applications* 95.2 (1999), pp. 85–111.

[9] William BR Lickorish. "A finite set of generators for the homeotopy group of a 2-manifold". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 60. 4. Cambridge University Press. 1964, pp. 769–778.

[10] William BR Lickorish. “A finite set of generators for the homeotopy group of a 2-manifold”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 60. 4. Cambridge University Press. 1964, pp. 769–778.

[11] Feng Luo. “Automorphisms of the complex of curves”. In: *Topology* 39.2 (2000), pp. 283–298.

[12] Dan Margalit. “Automorphisms of the pants complex”. In: *Duke Mathematical Journal* 121 (Feb. 2004). DOI: 10.1215/S0012-7094-04-12133-5.

[13] Tibor Radó. “Über den begriff der riemannschen fläche”. In: *Acta Litt. Sci. Szeged* 2.101-121 (1925), p. 10.