FIXED POINT THEOREMS AND THEIR APPLICATIONS FOR PROVING THE EXISTENCE OF NASH EQUILIBRIUM

A Thesis

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CERTIFICATE

This is to certify that **Prutha Bhide**, Bs-Ms (Dual Degree) student in Department of **Mathematics**, has completed bonafide work on the thesis entitled **'Fixed Point Theorems and their applications for proving the existence of Nash Equilibrium'** under my supervision and guidance.

April 2022
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Prutha Bhide

ABSTRACT

In this thesis, we study the paper titled "Browder type fixed point theorems and Nash equilibria in generalized games" [10], which derives an existence theorem for Nash equilibrium in generalized games with non-compact strategy sets from any Hausdorff topological vector spaces. We will start by studying the well-known Browder fixed point theorem [2] on Hausdorff topological vector spaces and later study how one of its generalizations is equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz Theorem [5]. These fixed-point theorems will form the basis for the derivation of the main result of [10].

Contents

Chapter 1

Fixed Point Theorems and Existence of Nash Equilibrium

1.1 Introduction

Fixed point theory is vital in many branches of mathematics, particularly in nonlinear analysis, topology, and mathematical game theory. Brouwer's fixed point theorem forms a basis for more general fixed point theorems like Kakutani fixed point theorem [9]. Kakutani fixed point theorem is used to prove the existence of equilibrium points in n-person games. We will start with an introduction to mathematical game theory and the significance of the Nash equilibrium. A Nash equilibrium occurs when we assume that players know the equilibrium strategy of other players and still are not willing to change their equilibrium strategy. Nash equilibrium can be seen as the fixed point of a game's strategy sets. Though Nash equilibrium has its origin in mathematics, it is widely applied in various fields such as economics and social sciences. We will briefly discuss Sperner's Lemma, derive crucial fixed point theorems and end the first chapter by proving the existence of Nash equilibrium.

In the following chapter we study, the paper titled "Browder type fixed point theorems and Nash equilibrium in generalized games" [10]. We present some useful generalizations of the famous Browder fixed point theorem [2] on Hausdorff topological vector spaces. We also prove that one of the generalizations is equivalent to well-known Fan–Knaster–Kuratowski–Mazurkiewicz theorem [5]. Most Nash equilibrium existence theorems for generalized games assume the strategy spaces to be subsets of a Euclidean space or a locally convex Hausdorff topological vector space or a metrizable locally convex vector space. We conclude the second chapter by deriving existence theorems for Nash equilibrium in generalized games, where the strategy spaces are noncompact subsets of a Hausdorff topological vector spaces.

1.2 Game Theory

Game theory is the study of interactive decision-making in conflict situations between different players. In this chapter, we analyze the mathematical framework of game theory and prove some well-known fixed point theorems. We start by discussing some basic concepts of rational decision-making, the notion of normal games, and the much-celebrated Nash equilibrium. This section is based on [7, Chapter 2].

Interactive decision situations are characterized by the following elements:

1. **Players**: A group of players.

2. **Actions**: A set of actions each players can choose from.

3. **Outcomes**: A set of possible results caused by any of the actions described in the action set.

4. **Preferences**: Describes the player preference on the set of possible outcomes.

The preference relation $\succeq$ is defined as a binary relation that describes the player's preferences, from the most desired to least desired outcomes. The notation $x \succeq y$ means "x is preferred over y" or "x is at least as good as y". To define rational preference relation, we need some essential axioms, including:

Axiom 1.2.1. *(The Completeness Axiom) The preference relation $\succeq$ is complete if any two outcomes $x, y \in X$ can be ranked by the preference relation so that either $x \succeq y$ or $y \succeq x$ or both.*

Axiom 1.2.2. *(The Transitivity Axiom) The preference relation $\succeq$ is transitive if for any three outcomes $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.*

Definition 1. A preference relation that is complete and transitive is called a *rational preference relation.*

Example 1.2.1. The $\geq$ relation over real numbers is a rational preference relation.

Definition 2. A *payoff function* (or *utility function*) $u : X \to \mathbb{R}$ represents the preference relation $\succeq$ if for any pair $x, y \in X, u(x) \geq u(y)$ if and only if $x \succeq y$.

Remark 1.2.1. A player facing a decision problem with a payoff function $u(.)$ over actions is rational, if he chooses an action $a \in A$ that maximizes his payoff. That is, $a^* \in A$ is chosen if and only if $u\left(a^*\right) \geq u(a)$ for all $a \in A$.

1.2.1 Strategic Games

A strategic game describes interactive situations among several players. All decisions taken by the players are simultaneous and independent. They are characterized by the players' strategies and their payoff functions. We assume that each player is rational because the player tries to maximize the payoff.

Definition 3. A *normal-form game* is a game that includes three components as follows:

1. A finite set of players, $N = \{1, 2, \ldots, n\}$.

2. Action profile $a = (a_1, \ldots, a_n) \in A = A_1 \times \ldots \times A_n$; where A_i is a action choice for individual i.

3. A set of payoff functions, $\{u_1, u_2, \ldots, u_n\}$, each assigning a payoff value to each combination of chosen actions, that is, a set of functions $u_i : A_1 \times A_2 \times \ldots \times A_n \rightarrow \mathbb{R}$ for each $i \in \mathbb{N}$.

We study some examples to better understand the notion of normal-form games.

Example 1.2.2. (Prisoner's Dilemma) Suppose that two suspects in a small robbery are put into different cells. We know that they are guilty of the robbery, but there is no evidence. Both of them are given a chance to confess. If both of them confess, then each will spend 10 years in jail. If only one confesses, she will act as a witness against the other, receive no punishment, and the other will spend 15 years in jail. On the other hand, if no one confesses, then each of them will spend 1 year in jail.

We refer to the confession as "defect" (D) and to no confession as "not defect" (ND). Then, the prisoner's dilemma game is a strategic game with players $N = 1, 2$, action set $A_i = \{ND, D\}$, and payoff functions

$$u_1(ND, ND) = u_2(ND, ND) = -1$$

$$u_1(D, D) = u_2(D, D) = -10$$

$$u_1(ND, D) = u_2(D, ND) = -15$$

$$u_1(D, ND) = u_2(ND, D) = 0.$$

Figure 1.1 gives a better representation of this game. The prisoner's dilemma

	ND	*D*
ND	$-1, -1$	$-15, \ 0$
D	$0, -15$	$-10, -10$

Figure 1.1: The prisoner's dilemma.

game is a classic example of game theory. The "cooperative" outcome $(-1, -1)$ is quite suitable for both players; it is the most they can get in the game. Although, for each player, confessing leads to a strictly higher payoff than not confessing, regardless of the strategy chosen by the other. Hence, a rational decision-maker should always confess. Thus, if both players behave rationally, they get payoffs (-10,-10), which are much worse than the payoffs in the "cooperative" outcome.

Example 1.2.3. (Matching pennies) This is a game in which two players have to choose a natural number simultaneously and independently. If the sum of the selected numbers is even, then the first player wins; if the sum is odd, the second player wins. All that matters in this game is whether a player chooses an even number (E) or an odd one (O). Thus, the game has players $N = \{1, 2\}$, action set $A_i = \{E, O\}$, and payoff functions

$$\begin{aligned} u_1(E,E) = u_1(O,O) &= u_2(O,E) = u_2(E,O) = 1. \\ u_2(E,E) = u_2(O,O) &= u_1(O,E) = u_1(E,O) = -1. \end{aligned}$$

Figure 1.2 is a more convenient representation of this game.

	E	O
E	$1, -1$	$-1, 1$
O	$-1, 1$	$1, -1$

Figure 1.2: The matching pennies game.

It would not be suitable to play any deterministic strategy in matching pennies. To optimize, we need to choose an even number and odd number alternatively, with a probability of 0.5. Thus, we need the notion of a mixed strategy.

Definition 4. A *strategy* S_i for agent i is a probability distribution over the actions A_i. There are two kinds of strategies, namely:

1. Pure Strategy: A strategy in which only one action is played with positive probability.

2. Mixed Strategy: A strategy in which more than one action is played with positive probability.

Let the set of all strategies for the player i be S_i and $S = S_1 \times \ldots \times S_n$.

1.2.2 Best Response and Nash Equilirbrium

If the player knows the actions of others, then it is easy to pick for the player to choose their own action. A strategy that leads to an outcome with the highest payoff given the strategy of other players is called the best response strategy. For each $s_i \in S_i$, let $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Thus, $s = (s_{-i}, s_i) \in S$ and $s_{-i} \in S_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$.

Definition 5. A response s_i^* is said to be *best response* with respect to opponents' strategies s_{-i}, if $\forall s_i \in S_i$,

$$u_i\left(s_i^*, s_{-i}\right) \geq u_i\left(s_i, s_{-i}\right).$$

We denote best response as BR_i for agent i. That is s_i^* is best response if $s_i^* \in BR\left(s_{-i}\right)$

Definition 6. A collection $s = (s_1, \ldots, s_n)$ of strategies is a *Nash equilibrium*, if $\forall i, s_i \in BR\left(s_{-i}\right)$.

The Nash equilibrium Theorem states that:

Theorem 1.2.1 (Existence of Nash equilibrium)**.** *Any finite-player game with finite strategies for all players has at least a Nash equilibrium.*

Example 1.2.4. Consider the matching pennies game Example 1.2.3. Suppose that the players, besides choosing E or O, have a mixed strategy L such that they choose E with a probability of 0.5 and O with a probability of 0.5. Figure 1.3 give s a more convenient representation of this game. Note that:

	E	O	L
E	1, −1	−1, 1	0, 0
O	−1, 1	1, −1	0, 0
L	0, 0	0, 0	0, 0

Figure 1.3: The matching pennies game.

$$u_1(L, L) = \frac{1}{4}u_1(E, E) + \frac{1}{4}u_1(E, O) + \frac{1}{4}u_1(O, E) + \frac{1}{4}u_1(O, O) = 0.$$

The game has a Nash equilibrium: (L, L). The mixed extension of Example 1.2.3 is a new game in which players can choose not only L but also any other mixed strategy over $\{E, O\}$. But here, the only Nash equilibrium of the game is (L, L).

1.2.3 Generalized Games

We will be dealing with generalized form of normal-form games and proving Nash equilibrium existence theorems for generalized games.

Definition 7. A *generalized game* $(X_i, F_i, u_i)_{i\in I}$ is a game with player set I such that each player $i \in I$ has strategy space X_i ; each player i has a payoff function $u_i : X \longrightarrow \mathbb{R}$ that depends on x_i as well as on x_{-i} of all other players; and for each $i \in I, F_i : X_{-i} \mapsto 2^{X_i}$ and player i's strategy must belong to the set $F_i\left(x_{-i}\right) \subseteq X_i$ that depends on the rival players' strategies.

Definition 8. A vector $x^* = \left(x_i^*\right)_{i \in I} \in X = \prod_{i \in I} X_i$ is called a *generalized Nash equilibrium* of a generalized game $\left(X_i, F_i, u_i\right)_{i \in I}$ if

$$u_i\left(x_i^*, x_{-i}^*\right) \geq u_i\left(x_i, x_{-i}^*\right), \forall x_i \in F_i\left(x_{-i}^*\right). \tag{1.1}$$

1.3 Fixed Point Theorems

Kakutani's and Brouwer's fixed point theorem are two classical results from fixed point theory that we will need to establish Nash's Theorem (Theorem 1.2.1). We will first prove Sperner's lemma and then use it to prove Brouwer's well-known fixed point theorem (Theorem 2.1.1), which states that every continuous map of an n-dimensional ball to itself has a fixed point. The first part of this section is based on *Jacob Fox's notes* [6].

Definition 9. An *n-dimensional simplex* is a convex linear combination of $n+1$ points in a general position.

A simplex is a generalization of the notion of a triangle or tetrahedron in dimensions 2 and 3, respectively, to higher dimensions.

Example 1.3.1. Given vertices $v_1, \ldots, v_{n+1} \in \mathbb{R}^n$, the standard n-simplex is given by the set:

$$S = \left\{ \sum_{i=1}^{n+1} \alpha_i v_i : \alpha_i \geq 0, \sum_{i=1}^{n+1} \alpha_i = 1 \right\}.$$

Figure 1.4 shows a representation of the standard simplices up to the fourth dimension.

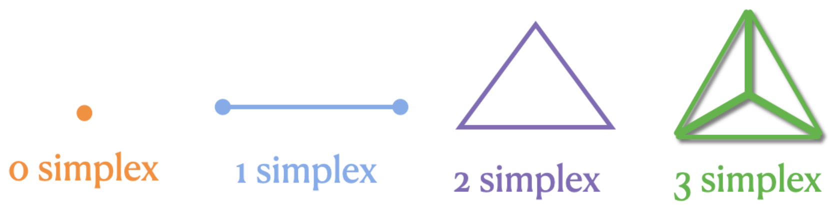

Figure 1.4: n-dimensional simplices.

Definition 10. A *simplicial subdivision* of an n-dimensional simplex S is a partition into smaller n-dimensional simplices (cells) such that any two cells are either disjoint, or they share a full face of a certain dimension.

Definition 11. A *proper coloring* of a simplicial subdivision is an assignment of $n+1$ colors to the vertices of the subdivision so that the vertices of S receive all different colors, and points on each face of S use only the colors of the vertices defining the respective face of S

Definition 12. A *barycentric subdivision* of an n-dimensional simplex consists of $(n+1)!n$ dimensional simplices.

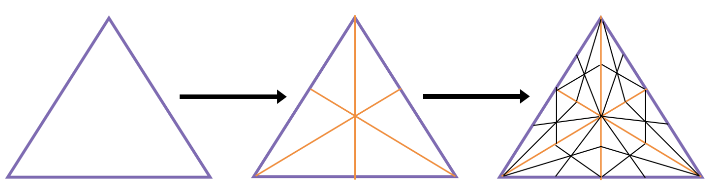

Figure 1.5: Barycentric subdivision of a 2 simplex.

1.3.1 Sperner's Lemma

Lemma 1.3.1. *(Sperner's Lemma) Every properly colored simplicial subdivision contains a cell whose vertices have all different colors.*

Proof. Let us call a cell of the subdivision a colored cell if its vertices get different colors. We are done if we prove that the number of colored cells for any proper coloring is odd.

For $n = 1$ that is the 1-dimensional simplex. We have a line segment, say (a, b). After subdividing it into smaller segments, we color the vertices of the subdivision with 2 different colors. Both a and b will have different colors. Thus, going from a to b, we must switch colors an odd number of times to get a different color for b. Hence, there is an odd number of small segments that receive two different colors. Thus, there are an odd-numbered of colored cells.

For $n = 2$ that is the 2-dimensional simplex. We have a triangle, say A and a properly colored simplicial subdivision with colors 1, 2 and 3. Let B denote the number of cells colored $(1, 2, 2)$ and $(1, 1, 2)$, and C denote number of colored cells. Consider edges in the subdivision whose endpoints receive colors 1 and 2. Let X denote the number of boundary edges colored $(1, 2)$, and Y the number of interior edges colored $(1, 2)$.

Here we can count in two different ways: For each cell of type B, we get two edges colored $(1, 2)$, while for each cell of type C, we get only such edge. Note that this way, we count internal edges of type $(1, 2)$ twice, whereas we count boundary edges only once. We conclude that $2B + C = X + 2Y$.

We know that no two vertices of A will have the same color. Between vertices colored 1 and 2, there must be an odd number of edges colored $(1, 2)$, following the same argument in the 1-dimensional case. Hence, X is odd. This implies that no of colored cells C is odd.

In general, for n-dimensional simplex S, we use induction on n. We have a proper coloring of a simplicial subdivision of S using $n+1$ colors. Let C denote the number of colored cells, using all $n+1$ colors. Let B denote the number of simplicial cells that are colored using $1, 2, ..., n$ so that exactly one of these colors is used twice and the other colors once. Also, we consider $(n-1)$ dimensional faces that use exactly the colors $1, 2, ..., n$. Let X denote the number of such faces on the boundary of S, and Y the number of such faces inside S. Again, we count in two different ways.

Each cell of type C contributes exactly one face colored $1, 2, ..., n$. Each cell of type B contributes exactly two faces colored $1, 2, ..., n$. However, inside faces appear in two cells, while boundary faces appear in one cell. Hence, we get the equation $2B + C = X + 2Y$.

On the boundary, the only $(n-1)$ dimensional faces colored $1, 2, ..., n$ can be on the face $F \subset S$ whose vertices are colored $1, 2, ..., n$. We use the inductive hypothesis for F, which forms a properly colored $(n-1)$ dimensional subdivision. By the hypothesis, F contains an odd number of colored $(n-1)$-dimensional cells, i.e., X is odd. We conclude that C is odd as well. □

1.3.2 Brouwer's Fixed Point Theorem

Theorem 1.3.1. *(Brouwer's Fixed Point Theorem) Let B^n denote an n-dimensional ball. For any continuous map $f : B^n \to B^n$, there is a point $x \in B^n$ such that $f(x) = x$.*

Proof. We will work with a simplex instead of a ball since they are homeomorphic. Let S be a simplex embedded in R^{n+1} so that the vertices of S are $s_1 = (1, 0, \ldots, 0), s_2 = (0, 1, \ldots, 0), \ldots,\ s_{n+1} = (0, 0, \ldots, 1)$. Let $f : S \to S$ be a continuous map and assume that it has no fixed point. We construct a sequence of subdivisions of S that we denote by $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots$. Each $\mathcal{S}_j$ is a subdivision of $\mathcal{S}_{j-1}$, so that the size of each cell in $\mathcal{S}_j$ tends to zero as $j \to \infty$. Now we define a coloring of $\mathcal{S}_j$. For each vertex $x \in \mathcal{S}_j$, we assign a color $c(x) \in [n+1]$ such that $(f(x))_{c(x)} < x_{c(x)}$. To see that this is possible, note that for each point $x \in S, \sum x_i = 1$, and also $\sum f(x)_i = 1$. Unless $f(x) = x$, there are coordinates such that $(f(x))_i < x_i$ and also $(f(x))_{i'} > x_{i'}$.

In case there are multiple coordinates such that $(f(x))_i < x_i$, we pick the smallest i. We check that this is a proper coloring in Sperner's lemma. For vertices of $S, s_i = (0, \ldots, 1, \ldots, 0)$, we have $c(x) = i$ because i is the only coordinate where $(f(x))_i < x_i$ is possible. Similarly, for vertices on a certain face say of S, e.g. $x = co(s_i : i \in A)$, the only coordinates where $(f(x))_i < x_i$ are those where $i \in A$, and hence $c(x) \in A$. Here $co(.)$ denotes the convex hull of (.). Sperner's lemma implies that there is a colored cell with vertices $x^{(j,1)}, \ldots, x^{(j,n+1)} \in \mathcal{S}_j$. In other words, $\left(f\left(x^{(j,i)}\right)\right)_i < x_i^{(j,i)}$ for each $i \in [n+1]$. Since this is true for each $\mathcal{S}_j$, we get a sequence of points $\left\{x^{(j,1)}\right\}$ inside a compact set S which has a convergent subsequence. We can assume that $\left\{x^{(j,1)}\right\}$ itself is convergent.

Since the size of the cells in $\mathcal{S}_j$ tends to zero, the limits $\lim_{j\to\infty} x^{(j,i)}$ are the same in fact for all $i \in [n+1]$. Let's call this common limit point $x^* = \lim_{j\to\infty} x^{(j,i)}$. We assumed that there is no fixed point, therefore $f(x^*) \neq x^*$.

This means that $(f(x^*))_i > x_i^*$ for some coordinate i. But we know that $\left(f\left(x^{(j,i)}\right)\right)_i < x_i^{(j,i)}$ for all j and $\lim_{j\to\infty} x^{(j,i)} = x^*$, which implies $(f(x^*))_i \leq x_i^*$ by continuity. This contradicts the assumption that there is no fixed point. □

1.3.3 Correspondences

We need some concepts and theorems related to correspondences, which will be used in this and the following chapter. Best response is not a function as there can be more than one strategy that belongs to BR_i for player i, it is a multivalued map.

Definition 13. A *correspondence (or a set-valued map or a multivalued map)* $F : X \mapsto 2^Y$ from X to Y, such that both X and Y are nonempty sets, is a map that associates some $x \in X$ to a subset $F(x)$ of Y. The set $F(x)$ is the image of x under F.

Remark 1.3.1. A correspondence $F : X \mapsto 2^Y$ from X to Y is *nonempty-valued, closed-valued, or convex-valued* if, for each $x \in X, F(x)$ is a nonempty, closed, or convex subset of Y, respectively.

Definition 14. A correspondence $F : X \mapsto 2^Y$ from X to Y is:

1. *Upper hemicontinuous* if, for each sequence $\{x^n\} \subset X$ converging to $\bar{x} \in X$ and each open set $Y^* \subset Y$ such that $F(\bar{x}) \subset Y^*$, there is $k_0 \in \mathbb{N}$ such that, for each $n \geq n_0, F(x^n) \subset Y^*$. Equivalently, $F : X \mapsto 2^Y$ is *upper hemicontinuous* if for any open neighbourhood V of $F(a), a \in X, \{x \in X \mid F(x) \subseteq V\}$ is open.

2. *Lower hemicontinuous* if, for each sequence $\{x^n\} \subset X$ converging to $\bar{x} \in X$ and each open set $Y^* \subset Y$ such that $F(\bar{x}) \cap Y^* \neq \varnothing$, there is $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0, F(x^n) \cap Y^* \neq \varnothing$. Equivalently, $F : X \mapsto 2^Y$ is *lower hemicontinuous* if the set $\{x \in X \mid F(x) \cap V \neq \emptyset\}$ is open for every open set V of Y.

3. *Continuous* if, it is both upper hemicontinuous and lower hemicontinuous.

Definition 15. A correspondence $F : X \mapsto 2^Y$ has:

1. *open lower sections* if, for every $y \in Y$ the set $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ is open in X.

2. *open upper sections* if, the set $F(x)$ is open in Y for every $x \in X$.

If correspondence F is a function (i.e., only selects singletons in 2^Y), both upper and lower hemicontinuity properties reduce to the standard continuity of functions.

We will also require the following fundamental theorems from real analysis and general topology.

Theorem 1.3.2. *(Tychonoff Theorem) The product of any collection of compact topological spaces is compact with respect to the product topology.*

Theorem 1.3.3. *A closed subset of a compact set in a topological space is compact.*

Theorem 1.3.4. *If $\{f_n\}$ is a sequence of continuous functions on X, and if $f_n \longrightarrow f$ uniformly, then f is continuous.*

Theorem 1.3.5. *If $\sum a_n$ is absolutely convergent and its value is s, then any rearrangement of $\sum a_n$ will also have a value of s.*

Theorem 1.3.6. *(Bolzano–Weierstrass Theorem) Each bounded sequence in R^n has a convergent subsequence.*

Theorem 1.3.7. *(Extreme Value Theorem) If a real-valued function is continuous on the closed interval, then must attain a maximum and a minimum, each at least once.*

Definition 16. A real-valued function f defined on a convex subset $C \subset \mathbb{R}^n$ is said to be *quasiconcave* if for all real $\alpha \in \mathbb{R}$, the set $\{x \in C : f(x) \geq \alpha\}$ is convex.

Definition 17. Let Δ_{n-1} be an $(n-1)$-dimensional simplex with n vertices labeled as $1, \ldots, n$. A *Knaster-Kuratowski-Mazurkiewicz covering* is defined as a set $C_1, \ldots, C_n$ of closed sets such that for any $I \subseteq \{1, \ldots, n\}$, the convex hull of the vertices corresponding to I is covered by $\bigcup C_{i \in I}$.

Theorem 1.3.8. *(Knaster-Kuratowski-Mazurkiewicz Theorem) In every KKM covering, the common intersection of all* ***n*** *sets is nonempty, i.e:*

$$\bigcap_{i=1}^{n} C_i \neq \emptyset.$$

1.3.4 Kakutani's Fixed Point Theorem

Theorem 1.3.9. *(Kakutani's Fixed Point Theorem) Let X be a non-empty, compact and convex subset of a finite-dimensional Euclidean space and F :*

$X \to 2^X$ *be a non-empty, convex correspondence, such that* $F(x)$ *is a subset* X *and is upper hemicontinuous. Then,* F *has a fixed point, that is, there exists some* $x \in X$*, such that* $x \in F(x)$*.*

Proof. We prove the theorem for a non-degenerate simplex X in R^n. Let $X = [a_o, a_1, \ldots, a_n]$. Now, for each integer p we consider the p^{th} barycentric subdivision of X and a continuous function $f^{(p)}$ as follows. If x is the vertex of any cell in the subdivision, let y be an arbitrary point of $F(x)$ and set $f^{(p)}(x) = y \in F(x)$. If x is not such a vertex, then x lies in some cell of the subdivision, say $x \in \left[a_0^{(p)}, \ldots a_n^{(p)}\right]$. Then x is a convex combination of these vertices, say

$$x = \sum_{j=0}^{n} \lambda_j^{(p)} a_j^{(p)}; \lambda_j^{(p)} \geq 0; \sum_{j=0}^{n} \lambda_j^{(p)} = 1,$$

and we get,

$$f^{(p)}(x) = \sum_{j=0}^{n} \lambda_j^{(p)} f^{(p)} \left(a_j^{(p)}\right).$$

Since the barycentric coordinates of points are unique, if x lies on a common face, the two definitions coincide on the common face. Now it is clear that the various maps $f^{(p)}$ are continuous maps of the simplex X onto itself. Hence, the Theorem 1.3.1 guarantees that each has a fixed point, say a point $x_*^{(p)}$ such that $f^{(p)}\left(x_*^{(p)}\right) = x_*^{(p)}$. Suppose that any of these fixed points is a vertex. Therefore, it is a fixed point of F by construction, and the proof is complete. On the other hand, if none of these points are vertices, then, for a given p, we have

$$x_*^{(p)} = \sum_{j=0}^{n} \lambda_j^{(p)} a_j^{(p)}.$$

So, using the definition of $f^{(p)}$ and the fact that $x_*^{(p)}$ is its fixed point, we have

$$x_*^{(p)} = \sum_{j=0}^{n} \lambda_j^{(p)} y_j^{(p)},$$

where

$$y_j^{(p)} = f^{(p)}\left(a_j^{(p)}\right) \in F\left(a_j^{(p)}\right), j = 0, 1, 2, \ldots, n.$$

We now have $2(n+1)$ sequences all of which lie in compact subsets of $\mathbb{R}^n$, namely the sequence of fixed points $\left\{x_*^{(p)}\right\}_{p=1}^{\infty}$, the n sequences of their barycentric coordinates $\left\{\lambda_j^{(p)}\right\}_{p=1}^{\infty}$ for each $j = 1, \ldots, n$ and the n sequences $\left\{y_j^{(p)}\right\}_{p=1}^{\infty}$ for each $j = 1, \ldots, n$. The first and last of these lie in the simplex X which is closed and bounded. All the sequences of the barycentric coordinates lie in the simplex of $\mathbb{R}^n$. By a direct application of Theorem 1.3.6, we may assume that all thee sequences converge as $p \to \infty$. Thus,

$$x_*^{(p)} \to x_* \text{ as } p \to \infty,$$

$$\lambda_j^{(p)} \to \lambda_j \text{ as } p \to \infty, j = 1, \ldots, n, , and$$

$$y_j^{(p)} \to y_j \text{ as } p \to \infty, j = 1, \ldots, n.$$

Now, as the diameter of the subcells approach 0 as $p \to \infty$, the convergence of the fixed points to x_* implies that the vertices $a_j^{(p)} \to x_*$ as $p \to \infty$ for all $j = 1, \ldots, n$. Moreover we must have

$$x_* = \sum_{j=0}^{n} \lambda_j y_j.$$

As F is upper hemicontinuous, we have

$$y_j^{(p)} \in F\left(a_j^{(p)}\right), \text{ and } a_j^{(p)} \to x_*, y_j^{(p)} \in y_j.$$

Hence we must have $y_j \in F(x_*)$. But $F(x_*)$ is convex and we have x_* is a convex combination of the y_j. Hence, $x_* \in F(x_*)$. □

1.4 Nash Equilibrium Existence Theorem

In 1950, John Nash, the mathematician later featured in the book and film "A Beautiful Mind," — wrote a two-page paper that transformed the theory of economics. Nash's equilibrium concept, which earned him a Nobel Prize in economics in 1994, offers a unified framework for understanding strategic behavior in economics and psychology, evolutionary biology, and a host of other fields. We will prove Nash Equilibrium using Theorem 1.3.9.

Proof. (of Theorem 1.2.1) Recall that a mixed-strategy profile s^* is a Nash equilibrium if

$$u_i\left(s_i^*, s_{-i}\right) \geq u_i\left(s_i, s_{-i}\right) \forall s_i \in S_i.$$

In other words, s^* is a Nash equilibrium if and only if $s^* \in BR_i\left(s_{-i}^*\right)$, where $BR_i\left(s_{-i}^*\right)$ is the best response of player i, given that the other players' strategies are $\left(s_{-i}^*\right)$.

We define the Best Response Correspondence $B : S \to 2^S$ such that for all $s \in S$, we have

$$B(s) = \left[BR_i\left(s_{-i}\right)\right]_{i \in N}$$

We will now prove that a mixed-strategy profile $s^* \in S$ is a Nash equilibrium if and only if it is a fixed point of the best-response correspondence B, $s^* \in B(s^*)$. To find the fixed point of the correspondence B, we apply Theorem 1.3.9 to the best response correspondence $B : S \to 2^S$. We start by showing that $B(s)$ indeed satisfies all the conditions of Theorem 1.3.9. By definition,

$$S = \prod_{i \in N} S_i,$$

where each S_i is a simplex of dimension $|S_i| - 1$. Thus each S_i is closed and bounded, and thus compact. Their product set is also compact by Theorem 1.3.2. Also, S is the convex hull of the set of pure strategies. Hence, S is convex. As S_i is non-empty and compact, and u_i is linear in x. Hence, u_i is continuous, and by Theorem 1.3.7, $B(s)$ is non-empty.

$B(s) \subset S$ is convex if and only if $BR_i(s_{-i})$ is convex for all i. Let $s_i', s_i'' \in BR_i(s_{-i})$. Then, we have:

$$u_i(s_i', s_{-i}) \geq u_i(\tau_i, s_{-i}) \text{ for all } \tau_i \in S_i$$

and

$$u_i(s_i'', s_{-i}) \geq u_i(\tau_i, s_{-i}) \text{ for all } \tau_i \in S_i.$$

Thus, for all $\lambda \in [0, 1]$, we have:

$$\lambda u_i(s_i', s_{-i}) + (1 - \lambda) u_i(s_i'', s_{-i}) \geq u_i(\tau_i, s_{-i}) \, \forall \tau_i \in S_i.$$

By the linearity of u_i, we have,

$$u_i\left(\lambda s_i' + (1-\lambda)s_i'', s_{-i}\right) \geq u_i\left(\tau_i, s_{-i}\right) \quad \forall \tau_i \in S_i.$$

Therefore, $\lambda s_i' + (1-\lambda)s_i'' \in B_i(s-i)$, showing that $B(s)$ is convex-valued.

Lastly, we need to show that $B(s)$ is upper hemicontinuous. Suppose we assume on the contrary that $B(s)$ not be upper hemicontinuous. Then, there exists a sequence $(s^n, \hat{s}^n) \to (s, \hat{s})$ with $\hat{s}^n \in B\left(s^n\right)$, but $\hat{s}_i \notin BR_i\left(s_{-i}\right)$ for some i. This implies that there exists some $s_i' \in S_i$ and some $\epsilon > 0$ such that

$$u_i\left(s_i', s_{-i}\right) > u_i\left(\hat{s}_i, s_{-i}\right) + 3\epsilon.$$

By continuity of u_i and the fact that $s_{-i}^n \to s_{-i}$, we have for sufficiently large n,

$$u_i\left(s_i', s_{-i}^n\right) \geq u_i\left(s_i', s_{-i}\right) - \epsilon.$$

Combining the preceding two relations and using the continuity of the u_i, we obtain, we obtain

$$u_i\left(s_i', s_{-i}^n\right) > u_i\left(\hat{s}_i, s_{-i}\right) + 2\epsilon \geq u_i\left(\hat{s}_i^n, s_{-i}^n\right) + \epsilon.$$

This contradicts the assumption that $\hat{s}_i^n \in BR_i\left(s_{-i}^n\right)$. The existence of the fixed point then follows from Theorem 1.3.9. □

Chapter 2

Nash Equilibrium in Generalized Games

Most existence theorems for Nash equilibrium in generalized games require the strategic spaces to be the Euclidean space or a locally convex Hausdorff space. In this chapter, we explore the results in the paper [10] to derive Nash equilibrium existence theorems for strategy spaces that are noncompact subsets of a Hausdorff topological vector space.

2.1 Browder Fixed Point theorems

We will start by proving Browder fixed point theorem [2] on Hausdorff topological vector spaces and later prove that one of its generalizations is equivalent to Fan-Knaster-Kuratowski-Mazurkiewicz theorem (FKKM) [5].

Theorem 2.1.1. *(Browder's fixed point theorem [2]) Let K be a nonempty compact convex subset of a Hausdorff topological vector space E. Suppose that*

$F : K \mapsto 2^K$ is a correspondence such that:

1. *for each $x \in K, F(x)$ is a nonempty convex subset of K and*
2. *for each $y \in K, F^{-1}(y) = \{x \mid x \in K, y \in F(x)\}$ is open in K (i.e., F has open lower sections in K).*

Then there exists $x^* \in K$ such that $x^* \in F(x^*)$.

Proof. For each y in $K, F^{-1}(y)$ is an open subset of K, and each $x \in K$ lies in at least one of these open subsets, by our hypothesis. Since K is compact there exists a finite family $\{y_1, \ldots, y_n\}$ such that $K = \cup_{j=1}^{n} F^{-1}(y_j)$. Let $\{\beta_1, \ldots, \beta_n\}$ be a partition of unity corresponding to this covering, i.e. each β_j is a continuous mapping of K into R^1 which vanishes outside of $F^{-1}(y_j), 0 \leq \beta_j(x) \leq 1$ for all x in K and all j with $1 \leq j \leq n$, while $\sum_{j=1}^{n} \beta_j(x) = 1$ for all x in K.

We define a continuous mapping f of K into K by setting

$$f(x) = \sum_{j=1}^{n} \beta_j(x) y_j.$$

Since each y_j lies in K and $f(x)$ is a convex linear combination of the points y_j, $f(x)$ lies in K. Moreover, for each j such that $\beta_j(x) \neq 0, x$ lies in $F^{-1}(y_j)$ so that $y_j \in F(x)$. Hence, $f(x)$ is a convex linear combination of points in the convex set $F(x)$ and therefore, $f(x) \in F(x)$ for each x in K.

Let K_0 be the finite dimensional simplex spanned by the n points $\{y_1, \ldots, y_n\}$. K_0 is homeomorphic to a Euclidean ball. f maps K_0 into K_0, and by the Theorem 1.3.1, f has a fixed point x_0 in K_0. For this point, we have

$$x_0 = f(x_0) \in F(x_0).$$

□

Lemma 2.1.1. *[4] Let X be an arbitrary set in a topological vector space E. To each $x \in X$, let a closed set $F(x)$ in E be given such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $\cup_{i=1}^{n} F(x_i)$. If $F(x)$ is compact for at least one $x \in X$, then $\cap_{x \in X} F(x) \neq \emptyset$.*

Proof. We just need to show that $\cap_{i=1}^{n} F(x_i) \neq \emptyset$ for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X. Given $\{x_1, x_2, \ldots, x_n\} \subset X$, consider the $(n-1)$-simplex $S = v_1, v_2, \ldots, v_n$ with vertices $v_1 = (1, 0, 0, \ldots, 0)$, $v_2 = (0, 1, 0, \ldots, 0), \ldots, v_n = (0, 0, \ldots, 0, 1)$.

We define a continuous mapping $\varphi : S \to Y$ such that $\varphi(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{i=1}^{n} \alpha_i x_i$ for $\alpha_i \geqq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Consider the n closed subsets $G_i = \varphi^{-1}(F(x_i))$ for $1 \leq i \leq n$ of S. For any indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, the $(k-1)$-simplex $v_{i_2} v_{i_z} \ldots v_{i_k}$ is contained in $G_{i_1} \cup G_{i_z} \cup \cdots \cup G_{i_k}$. By Theorem 1.3.8, we have $\cap_{i=1}^{n} G_i \neq \emptyset$ and therefore, $\cap_{i=1}^{n} F(x_i) \neq \emptyset$. □

The following theorem, popularly known as the Fan-Knaster-Kuratowski-Mazurkiewicz (FKKM) theorem [5], is an extension of the Brower's fixed point theorem. Tarafdar showed an equivalent statement of this theorem in [11].

Theorem 2.1.2. *(FKKM theorem) [5] In a Hausdorff topological vector space, let Y be a nonempty convex subset and $\emptyset \neq X \subseteq Y$. Let $F : X \mapsto 2^Y$ be a correspondence such that:*

1. *for each $x \in X, F(x)$ is a relative closed subset of Y,*
2. *the convex hull of each finite subset $\{x_1, x_2, \ldots, x_n\}$ of Y is contained in the corresponding union $\cup_{i=1}^{n} F(x_i)$ and*

3. *there is a nonempty subset X_0 of X such that X_0 is contained in a compact convex subset of Y and the intersection $\cap_{x\in X_0} F(x)$ is compact.*

Then $\cap_{x\in X} F(x) \neq \emptyset$.

We now restate Theorem 2.1.2 in its contrapositive form by considering the complement of $F(x)$ in Y.

Theorem 2.1.3. *([5]) In a topological vector space, let Y be a convex set and let $\emptyset \neq X \subset Y$. For each $x \in X$, let $A(x)$ be a relatively open subset of Y such that $\cup_{x\in X} A(x) = Y$. If there exists a non-empty subset X_0 of X such that $Y \backslash \cup_{x\in X_0} A(x)$ is compact or empty, and X_0 is contained in a compact convex subset C of Y, then there exists a non-empty finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that the convex hull of $\{x_1, x_2, \ldots, x_n\}$ contains a point of the corresponding intersection $\cap_{i=1}^{n} A(x_i)$.*

Proof. Let $F(x) = Y \backslash A(x)$ and let $G(x) = C \cap F(x)$. Suppose that the above result is false. Consider the case $\cup_{x\in X_0} A(x) = Y$. The convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is thus contained in $Y \backslash \cap_{i=1}^{n} A(x_i) = \cup_{i=1}^{n} F(x_i)$. Consequently, the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in $C \cap \cup_{i=1}^{n} F(x_i) = \cup_{i=1}^{n} G(x_i)$. For each $x \in X, F(x)$ is closed in Y (since $A(x)$ be a relatively open subset of Y.) As C is a compact subset of Y, $G(x) = C \cap F(x)$ is compact. Then, by Lemma 2.1.1, we would have $\cap_{x\in X} G(x) \neq \emptyset$, and therefore, $\cap_{x\in X} F(x) \neq \emptyset$. But this means that $\cup_{x\in X} A(x) \neq Y$, contradicting our hypothesis.

Now, for the case when the complement of $\cup_{x\in X_0} A(x)$ in Y is non-empty. Let $D = \cap_{x\in X_0} F(x) = Y \backslash \cup_{x\in X_0} A(x)$. Then D is nonempty and compact.

Suppose that the conclusion of the theorem is false, i.e., suppose that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in $Y \setminus \cap_{i=1}^{n} A(x_i)$ $= \cup_{i=1}^{n} F(x_i)$. From this assumption, we are going to show that $\cap_{x \in X} F(x) \neq \emptyset$, which means $\cup_{x \in X} A(x) \neq Y$, a contradiction. Consider an arbitrary finite subset $\{x_1, x_2, \ldots, x_n\}$ of X. Let $X_1 = X_0 \cup \{x_1, x_2, \ldots, x_n\}$ and let K be the convex hull of $C \cup \{x_1, x_2, \ldots, x_n\}$. Since C is compact convex, K is compact. Also, since $C \cup X$ is contained in the convex set Y, we have $K \subset Y$. For $y \in X_1$, let $G(y) = K \cap F(y)$. As $F(y)$ is closed in Y and K is a compact subset of $Y, G(y)$ is compact. The convex hull of every finite subset $\{y_1, y_2, \ldots, y_m\}$ of X_1 is contained in $K \cap \left(\cup_{j=1}^{m} F(y_j)\right) = \cup_{j=1}^{m} G(y_j)$. By Lemma 2.1.1, we have $\cap_{y \in X_1} G(y) \neq \emptyset$. As

$$D \cap \left(\cap_{i=1}^{n} F(x_i)\right) \supset K \cap \left(\cap_{x \in X_0} F(x)\right) \cap \left(\cap_{i=1}^{n} F(x_i)\right) = \cap_{y \in X_1} G(y),$$

it follows that $\cap_{i=1}^{n} [D \cap F(x_i)] \neq \emptyset$ for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X. Since D is compact, so is also $D \cap F(x)$. Hence, $\cap_{x \in X} [D \cap F(x)] \neq \emptyset$ and therefore, $\cap_{x \in X} F(x) \neq \emptyset$. Equivalently, we have $\cup_{x \in X} A(x) \neq Y$. Thus, the assertion follows. □

Definition 18. Let X and Y be two topological spaces. A correspondence $F : X \mapsto 2^Y$ is said to have:

1. *transfer closed valued* if for every $x \in X$ and $y \in F(x)$, there exists $x' \in X$ such that $y \notin cl\left(F(x')\right)$.

2. *transfer open valued* if for every $x \in X$ and $y \in F(x)$, there exists $x' \in X$ such that $y \in \operatorname{int}\left(F(x')\right)$.

Remark 2.1.1. A correspondence F has open lower sections implies that F^{-1} is transfer open valued; a correspondence is transfer closed-valued if it is closed-valued, and correspondence is transfer open-valued if it is open-valued. It is easy to see that a correspondence $F : X \mapsto 2^Y$ is transfer closed valued if and only if the correspondence $T : X \mapsto 2^Y$ defined by $T(x) = Y \backslash F(x)$ for each $x \in X$ is transfer open valued.

Definition 19. Let Y be a convex subset of E and let $\emptyset \neq X \subseteq Y$. A correspondence $F : X \mapsto 2^Y$ is said to be *FS-convex* on X if for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X the convex hull of $\{x_1, x_2, \ldots, x_n\}$ is contained in $\cup_{i=1}^{n} (F(x_i))$.

Remark 2.1.2. The FS-convexity of F implies that every point $x \in X$ is a fixed point of $F(x)$, i.e., $x \in F(x)$.

We denote the closure of a set A by $cl(A)$ and the convex hull of a set A by $co(A)$. The following theorem is an extension of Theorem 2.1.2.

Theorem 2.1.4. *([12]) In a Hausdorff topological vector space, let Y be a nonempty convex subset and $\emptyset \neq X \subseteq Y$. Let $F : X \mapsto 2^Y$ be a correspondence such that:*

1. *F is transfer closed valued,*
2. *the convex hull of each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $\cup_{i=1}^{n} \mathrm{cl}(F(x_i))$ or it is FS-convex on X, and*
3. *there is a nonempty subset X_0 of X such that X_0 is contained in a compact convex subset of Y and the intersection $\cap_{x \in X_0} cl(F(x))$ is compact.*

Then $\cap_{x\in X}F(x)\neq\emptyset$.

Proof. First, we prove that $\cap_{x\in X}\text{cl}_Y F(x)=\cap_{x\in X}F(x)$. It is clear that $\cap_{x\in X}F(x)\subset\cap_{x\in X}\text{cl}_Y F(x)$. It suffices $\cap_{x\in X}\text{cl}_Y F(x)\subset\cap_{x\in X}F(x)$. Suppose, by way of contradiction, that there is some y in $\cap_{x\in X}\text{cl}_Y F(x)$ but not in $\cap_{x\in X}F(x)$. Then $y\notin F(x)$ for some $x\in X$. By our hypothesis, there is some $x'\in X$ such that $y\notin\text{cl}_Y F(x')$, a contradiction. For $x\in X$, let $K(x)=\text{cl}_Y F(x)$. Then $K(x)$ satisfies all conditions of Theorem 2.1.1 and thus, $\cap_{x\in X}F(x)=\cap_{x\in X}K(x)\neq\emptyset$. □

The following fixed point theorem generalizes the Theorem 2.1.1. We use A^c to denote the complement of a set A.

Theorem 2.1.5. *Let* X *be a nonempty convex subset of a Hausdorff topological vector space. Let* $F: X\mapsto 2^X$ *be a correspondence such that:*

1. *for each* $x\in X, F(x)$ *is a nonempty convex subset of* X*,*
2. F^{-1} *is transfer open valued, and*
3. *there is a nonempty subset* X_0 *of* X *such that* X_0 *is contained in a compact convex subset of* X *and the intersection* $\cap_{x\in X_0}\text{cl}\left(\left(F^{-1}(x)\right)^c\right)$ *is compact.*

Then there exists an $x^*\in X$ *such that* $x^*\in F(x^*)$.

Proof. For each $x\in X$, define $G(x)=\left(F^{-1}(x)\right)^c$. Then G is transfer closed valued as F^{-1} is transfer open valued. Since $\cup_{x\in X}F^{-1}(x)=X$, it follows that $\cap_{x\in X}G(x)=\cap_{x\in X}\left(F^{-1}(x)\right)^c=\emptyset$. By Theorem 2.1.4, we must have a finite subset $\{x_1,\ldots,x_n\}$ such that $co(x_1,\ldots,x_n)$ is not contained in the union $\cup_{i=1}^n cl\left(G\left(x_i\right)\right)$. Thus, there exists $\lambda_i\geq 0$ for $i=1,2,\ldots,n$ with $\sum_{i=1}^n\lambda_i=1$

such that $x_0 = \sum_{i=1}^{n} \lambda_i x_i \notin \cup_{i=1}^{n} cl\left(G\left(x_i\right)\right)$, which implies that $x_0 \notin G\left(x_i\right)$ for each $i = 1, 2, \ldots$, n. It follows that $x_0 \in F^{-1}\left(x_i\right)$ for each $i = 1, 2, \ldots, n$, that is, $x_i \in F\left(x_0\right)$ for each $i = 1, 2, \ldots, n$. Since $F(x)$ is convex for every $x \in X$, we have $x_0 = \sum_{i=1}^{n} \lambda_i x_i \in F\left(x_0\right)$. Therefore, the assertion follows. □

By taking $X_0 = X$ in Theorem 2.1.5, we obtain the following corollary.

Corollary 2.1.1. *Let X be a nonempty convex compact subset of a Hausdorff topological vector space. Let $F : X \mapsto 2^X$ be a correspondence such that:*

1. *for each $x \in X, F(x)$ is a nonempty convex subset of X and*

2. *F^{-1} is transfer open valued.*

Then there exists an $x^* \in X$ such that $x^* \in F\left(x^*\right)$.

Proof. Since $\mathrm{cl}\left(\left(F^{-1}(x)\right)^c\right)$ of $\left(F^{-1}(x)\right)^c$ is a closed subset of X for each $x \in X, \cap_{x\in X} cl\left(\left(F^{-1}(x)\right)^c\right)$ is a closed subset of X. Since X is compact, $\cap_{x\in X} cl\left(\left(F^{-1}(x)\right)^c\right)$ is compact by Theorem 1.3.3. Thus, by taking $X_0 = X$, we see that the hypothesis of Theorem 2.1.5 is satisfied. Therefore, it follows from Theorem 2.1.5 that there exists $x^* \in X$ such that $x^* \in F\left(x^*\right)$. □

Another useful consequence of Theorem 2.1.5 is the following corollary.

Corollary 2.1.2. *Let X be a nonempty convex subset of a Hausdorff topological vector space. Let $F : X \mapsto 2^X$ be a correspondence such that:*

1. *for each $x \in X, F(x)$ is a nonempty convex subset of X,*

2. *F has open lower sections, and*

3. *there is a nonempty subset X_0 of X such that X_0 is contained in a compact convex subset of X and the intersection $\cap_{x \in X_0} \left(F^{-1}(x)\right)^c$ is compact.*

Then there exists $x^* \in X$ such that $x^* \in F\left(x^*\right)$.

Proof. We claim that a correspondence $F : X \mapsto 2^Y$ having open lower sections implies that $F^{-1} : Y \mapsto 2^X$ is transfer open valued. Indeed if, F has open lower sections then, $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ is open in X for every $y \in Y$. To show that F^{-1} is transfer open valued, by Definition 18, we need to show that for every $y \in Y$ and $x \in F^{-1}(y)$ and there exists $a \in Y$ such that $x \in \operatorname{int}\left(F^{-1}\left(a\right)\right)$. This clearly holds by taking $a = y$ since $F^{-1}(y)$ is open, and the assertion follows. One can see from the claim that the hypothesis of Corollary 2.1.2 implies the hypothesis of Theorem 2.1.5. Thus, our assertion follows from Theorem 2.1.5. □

Since a correspondence F has open lower sections implies that F^{-1} is transfer open valued as shown above. Thus, Corollary 2.1.1 implies Theorem 2.1.1. The following example shows that Corollary 2.1.1 is an extension of the Browder fixed point theorem.

Example 2.1.1. Let $X = [0, 1]$. Define the correspondence $F : X \mapsto 2^X$ by $F(x) = [0, x]$. Then $F(x)$ is nonempty convex for each $x \in X$. For each $y \in X, F^{-1}(y) = [y, 1]$, which is closed in X. Thus, Theorem 2.1.1 cannot be applied here since $F^{-1}(y)$ is not an open subset of X for any $y > 0$. However, it is easy to see that $F^{-1}(y)$ is transfer open for each $y \in X$. Thus, $F^{-1}(y)$ is transfer open valued. If follows from Corollary 2.1.1 that F has a fixed point.

Definition 20. A correspondence $\varphi : X \mapsto 2^Y$ is said to have the *inclusion open lower sections* at x if there exists an open neighborhood O_x of x and a correspondence $\phi_x : X \mapsto 2^Y$ such that ϕ_x has open lower sections and $\emptyset \neq \phi_x(z) \subseteq \varphi(z)$ for any $z \in O_x$. We say that φ has inclusion open lower sections if it has inclusion open lower sections at every $x \in X$.

If a correspondence $\varphi : X \mapsto 2^Y$ has open lower sections, then it has inclusion open lower sections as one can choose $\phi_x = \varphi$ for all $x \in X$.

Lemma 2.1.2. *[14] Let X, Y be linear topological spaces and $f : X \to 2^Y$ be a correspondence with open lower sections. Define the correspondence $\psi : X \to 2^Y$ by $\psi(x) = co(f(x))$ for all $x \in X$. Then ψ has open lower sections.*

Proof. Let $y_0 \in Y$ and $x_0 \in \psi^{-1}(y_0)$. We shall exhibit an open set U in X such that $x_0 \in U \subset \psi^{-1}(y_0)$. Since $y_0 \in \psi(x_0) = co(f(x_0))$, we can find $y_1, \ldots, y_n$ in $f(x_0)$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_i \geqq 0, \sum_{i=1}^n a_i = 1$ and $y_0 = \sum_{i=1}^n a_i y_i$. For each $i = 1, \ldots, n$; $f^{-1}(y_i)$ is open in X and $x_0 \in f^{-1}(y_i)$. Define $U = \cap_{i=1}^n f^{-1}(y_i)$. Then $x_0 \in U$, for an open U in X.

To complete the proof we must show that $U \subset \psi^{-1}(y_0)$. Let $x \in U$, then $x \in f^{-1}(y_i)$ or $y_i \in f(x)$ for all $i = 1, \ldots, n$. Hence, $y_0 = \sum_{i=1}^n a_i y_i \in \psi(x)$, i.e., $x \in \psi^{-1}(y_0)$. Consequently, $x_0 \in U \subset \psi^{-1}(y_0)$. □

The following theorem is a generalization of Theorem 2.1.1 and Corollary 2.1.1. As in Definition 20, when a correspondence F has inclusion open lower sections at x, we use O_x for the open neighborhood of x and ϕ_x for the correspondence having open lower sections such that $\emptyset \neq \phi_x(z) \subseteq \varphi(z)$ for any $z \in O_x$.

Theorem 2.1.6. *Let X be a nonempty convex subset of a Hausdorff topological vector space. Suppose that $F : X \mapsto 2^X$ is a correspondence such that:*

1. *for each $x \in X, F(x)$ is a nonempty convex subset of X,*

2. *F has inclusion open lower sections, and*

3. *there is a nonempty subset X_0 of X such that X_0 is contained in a compact convex subset of X and the intersection $\cap_{x\in X_0} (\phi_x^{-1}(x))^c$ is compact.*

Then there exists an $x^* \in X$ such that $x^* \in F(x^*)$.

Proof. Since F has inclusion open lower sections, for each $x \in X$, there exists an open neighborhood O_x of x and a correspondence $\phi_x : X \mapsto 2^X$ such that ϕ_x has open lower sections and $\emptyset \neq \phi_x(z) \subseteq F(z)$ for any $z \in O_x$. Clearly, the collection $\{O_x \mid x \in X\}$ is an open cover of X. For each $x \in X$, let $I(x) = \{x_i \mid x \in O_{x_i}\}$ and define $\varphi(x) = co((\cup_{x_i\in I(x)} co(\phi_{x_i}(x))))$. Clearly, $\varphi(x)$ is nonempty convex for each $x \in X$. Since F is convex valued and $x \in O_{x_i}$ for each $x_i \in I(x)$, we have $\varphi(x) \subseteq F(x)$ for each $x \in X$.

We claim that φ has open lower sections. By Lemma 2.1.2, it suffices to show that $\cup_{x_i\in I(x)}\ co(\phi_{x_i})$ has open lower sections. For any $x \in X$, since ϕ_{x_i} has open lower sections, $co(\phi_{x_i})$ has open lower sections for each $x_i \in I(x)$. It follows that for each $y \in X$, $\left(\cup_{x_i\in I(x)} co(\phi_{x_i})\right)^{-1}(y) = \cup_{x_i\in I(x)} \left(co(\phi_{x_i})\right)^{-1}(y)$ is open. Thus, $\cup_{x_i\in I(x)} co(\phi_{x_i})$ has open lower sections and so φ has open lower sections. Note that $\phi_x \subseteq \varphi(x)$ for each $x \in X$. We have $\cap_{x\in X_0} (\varphi^{-1}(x))^c \subseteq \cap_{x\in X_0} (\phi_x^{-1}(x))^c$. Since φ has open lower sections and $\cap_{x\in X_0} (\phi_x^{-1}(x))^c$ is compact by assumption 3, $\cap_{x\in X_0} (\varphi^{-1}(x))^c$ is a closed subset of a compact set.

It follows that $\cap_{x \in X_0} \left(\varphi^{-1}(x)\right)^c$ is compact. The assertion now follows from Corollary 2.1.2 with F being replaced by φ. $\square$

Definition 21. [8] A correspondence $\varphi : X \mapsto 2^Y$ has the *local intersection property* at some $x \in X$ if there exists an open set O_x such that $x \in O_x$ and $\cap_{x' \in O_x} \varphi(x') \neq \emptyset$, and φ is said to have the *local intersection property* if, this property holds for every $x \in X$.

It is clear that φ has the local intersection property implies that φ has inclusion open lower sections, as one can define ϕ_x for each $x \in X$ as follows: $\phi_x(z) = \{y\}$ with $y \in \cap_{x' \in O_x} \varphi(x')$ for each $z \in O_x$ and $\phi_x(z) = \emptyset$ for any $z \in X \backslash O_x$. An immediate consequence of Theorem 2.1.6 is the following.

Theorem 2.1.7. *Let X be a nonempty convex subset of a Hausdorff topological vector space. Suppose that $F : X \mapsto 2^X$ is a correspondence such that:*

1. *for each $x \in X, F(x)$ is a nonempty convex subset of X,*
2. *F has the local intersection property, and*
3. *there is a nonempty subset X_0 of X such that X_0 is contained in a compact convex subset of X and the intersection $\cap_{x \in X_0} (O_x)^c$ is compact.*

Then there exists an $x^* \in X$ such that $x^* \in F(x^*)$.

Theorem 2.1.7 can be equivalently stated as follows.

Theorem 2.1.8. *[11] Let X be a nonempty convex subset of a Hausdorff topological vector space. Suppose that $F : X \mapsto 2^X$ is a correspondence such that:*

1. *for each $x \in X, F(x)$ is a nonempty convex subset of X,*

2. for each $y \in X, F^{-1}(y) = \{x \in X : y \in F(x)\}$ *contains a relatively open subset* O_y *of* X *(*O_y *may be empty for some* y*),*

3. $U_{x\in X} O_x = X$*, and*

4. there exists a nonempty $X_0 \subset X$ *such that* X_0 *is contained in a compact convex subset* X_1 *of* X *and the set* $D = \cap_{x\in X_0} O_x^c$ *is compact, (D could be empty and as before* O_x^c *denotes the complement of* O_x *in* X *).*

Then there exists an $x^* \in X$ *such that* $x^* \in F(x^*)$*.*

2.2 Fan-Knaster-Kuratowski-Mazurkiewicz Equivalent theorems

In this section, we will prove equivalence of Theorems 2.1.5, 2.1.7, and 2.1.2. It is easy to see that Theorem 2.1.2 implies Theorem 2.1.5, as Theorem 2.1.4 is an extension of the FKKM theorem, and it implies Theorem 2.1.5.

Proof. (Theorem 2.1.5 $\implies$ Theorem 2.1.2) Assume that the conditions of Theorem 2.1.2 hold. We assume on the contrary, that $\cap_{x\in X} F(x) = \emptyset$. Define the set valued mapping $f : Y \mapsto 2^Y$ by $f(y) = \{x \in X \mid y \notin F(x)\}$. Clearly, $f(y)$ is a nonempty subset of Y for each $y \in Y$. Note that $f^{-1}(x) = (F(x))^c_Y$ (complement of $F(x)$ with respect to Y) for each $x \in X$. Since F is closed valued, f^{-1} is open valued on Y. Since $f^{-1}(x) \cap X$ is open relative to X if $f^{-1}(x)$ is open relative to Y, f^{-1} is open valued on X when we view f as a correspondence $f : X \mapsto 2^X$.

Now, for each $x \in X$, define the correspondence $G(x) = co(f(x))$. Then for any $x, y \in X$, $f(x) \subseteq G(x)$ and $f^{-1}(y) \subseteq G^{-1}(y)$. We claim that G^{-1} is transfer open valued. Let $x \in G^{-1}(y)$. Then $y \in G(x) = co(f(x))$. Thus, there exist $x_1, x_2, \dots, x_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $x_i \in f(x)$ for all $1 \leq i \leq n$, $\sum_{i=1}^{n} \lambda_i = 1$, and $x = \sum_{i=1}^{n} \lambda_i x_i$. So $x \in f^{-1}(x_i)$ for each $i = 1, 2, \dots, n$. Since f^{-1} is open valued, $f^{-1}(x_1)$ is open which implies that $x \in \text{int}\,(f^{-1}(x_1)) \subseteq \text{int}\,(G^{-1}(x_1))$. Thus, G^{-1} is transfer open valued. For each $x \in X$, since $f(x)$ is nonempty, $G(x)$ is nonempty convex. Since $f^{-1}(x) = (F(x))^c$ (in X), $(f^{-1}(x))^c = F(x)$, which is closed in X. Recall that $f^{-1}(x) \subseteq G^{-1}(x)$ for each $x \in X$, $(G^{-1}(x))^c \subseteq (f^{-1}(x))^c$ and so $cl\left((G^{-1}(x))^c\right) \subseteq cl\left((f^{-1}(x))^c\right) = (f^{-1}(x))^c$ for each $x \in X$. By Theorems 1.3.3 and 2.1.2, we have that

$$\cap_{x \in X_0} cl\left(\left(G^{-1}(x)\right)^c\right) \subseteq \cap_{x \in X_0} cl\left(\left(f^{-1}(x)\right)^c\right) = \cap_{x \in X_0} F(x)$$

is compact. By Theorem 2.1.5, there exists $x_0 \in X$ such that $x_0 \in G(x_0) = co((f(x_0)))$. It follows that there exist $x_1, x_2, \dots, x_n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $x_i \in f(x_0)$ for all $i \leq n$, $\sum_{i=1}^{n} \lambda_i = 1$, and $x_0 = \sum_{i=1}^{n} \lambda_i x_i$. This implies that $x_0 \notin F(x_i)$ for all $1 \leq i \leq n$, that is, $x_0 = \sum_{i=1}^{n} \lambda_i x_i \notin \cup_{i=1}^{n} F(x_i)$, contradicting assumption 2 of Theorem 2.1.2. Thus, $\cap_{x \in X} F(x) \neq \emptyset$. □

We will now show that:

Proof. (Theorem 2.1.2 $\iff$ Theorem 2.1.7.) Let us assume that the conditions of Theorem 2.1.2 hold. If possible, suppose that $\cap_{x \in X} F(x) = \emptyset$. Then we can define a set valued mapping $f : Y \to 2^Y$ by $f(y) = \{x \in X : y \notin F(x)\}$. Clearly $f(y)$ is a nonempty subset of Y for each $y \in Y$. It also follows that for each

$x \in Y, f^{-1}(x) = (F(x))^{\mathrm{c}} = O_x$ is a relatively open set in Y. Let $g : Y \to 2^Y$ be the set valued mapping defined by $g(y) = co(f(y))$ for each $y \in Y$. Thus, for each $y \in Y$, $g(y)$ is a nonempty convex subset of Y and for each $x \in Y$, $g^{-1}(x) \supset f^{-1}(x) = O_x$. Also, $\cap_{x \in X} F(x) = \emptyset$ implies $\cup_{x \in X} O_x = Y$ and hence, $U_{x \in Y} O_x = Y$. Finally, $\cap_{x \in X_0} O_x^{\mathrm{c}} = \cap_{x \in X_0} F(x) = D$ is compact. Hence, by Theorem 2.1.8, there exists a point $x_0 \in X$ such that $x_0 \in g(x_0) = co(f(x_0))$. This implies that there exist points $y_1, y_2, \ldots, y_m$ in K such that $y_i \in f(x_0)$ for $i = 1, 2, \ldots, m$, where $x_0 = \sum_{i=1}^{n} \lambda_i y_i, \lambda_i \geqslant 0$ for $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} \lambda_i = 1$. This means that $x_0 \notin F(y_i)$ for $i = 1, 2, \ldots, m$, i.e., $x_0 = \sum_{i=1}^{m} \lambda_i y_i \notin \cup_{i=1}^{m} F(y_i)$, which contradicts our established fact that the convex hull of each finite subset $\{y_1, y_2, \ldots, y_n\}$ of X is contained in the corresponding union $\cup_{i=1}^{m} F(y_i)$. Thus, we have proved that $\cap_{x \in K} F(x) \neq \emptyset$.

Assume that the conditions of Theorem 2.1.8 hold. For each $x \in X, F(x) = O_x^{\mathrm{c}}$, which is a relatively closed set in X. Let us first consider the case when $D = \emptyset$. Then by taking $Y = X$ in Theorem 2.1.2 we must have a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that the convex hull of $\{x_1, x_2, \ldots, x_n\}$ is not contained in the corresponding union $\cup_{i=1}^{n} F(x_i)$, for otherwise D will be nonempty by the first part of Theorem 2.1.2. This means that $x_0 = \sum_{i=1}^{n} \lambda_i x_i \notin F(x_i) = O_{x_i}^{\mathrm{c}}$ for each $i = 1, 2, \ldots, n$ and for some $\lambda_i \geqslant 0$, $i = 1, 2, \ldots, n$, with $\sum_{i=1}^{n} \lambda_i = 1$. Thus, $x_0 \in O_{x_i} \subset f^{-1}(x_i)$, i.e., $x_i \in f(x_0)$ for each $i = 1, 2, \ldots, n$. Hence, $x_0 \in f(x_0)$ as $f(x_0)$ is convex and Theorem 2.1.8 is proved in this case.

Finally, let $D \neq \emptyset$. If the convex hull of each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $U_{i=1}^{n} F(x_i)$, then by Theorem 2.1.2 , $\cap_{x \in X} O_x^{\mathrm{c}} = \cap_{x \in X} F(x) \neq \emptyset$, which contradicts the condition 3 of Theorem

2.1.8. Hence, there must exist a finite subset $\{x_1, x_2, \ldots, x_r\}$ of X such that the convex hull of $\{x_1, x_2, \ldots, x_r\}$ is not contained in the corresponding union $\cup_{i=1}^{r} F(x_i)$. Now repeating the same argument as in the first case, we obtain a point $x_0 \in X$ such that $x_0 \in f(x_0)$. This completes the proof. $\square$

2.3 Existence of Equilibrium for Noncompact Stratergy Spaces

We now apply previously derived fixed point theorems to derive Nash equilibrium theorems in generalized games with noncompact strategy sets on a Hausdorff topological vector spaces. These existence theorems generalize some well-known Nash equilibrium existence theorems like the existence theorem by Cubiotti [3] and by Arrow and Debreu [1]. Arrow and Debreu's theorem for the abstract economy is a weaker but valuable version of Debreu's social equilibrium existence theorem.

Theorem 2.3.1. *(Arrow and Debreu's theorem [1]) Let N agents be characterized by a strategic space X_i for player $i, 1 \leq i \leq N$, and $X = X_1 \times X_2 \times \cdots \times X_N$. For each player i, let $u_i : X \longrightarrow \mathbb{R}$ be a payoff function and $F_i(x_{-i})$ be a restricted strategic space given other player actions x_{-i}. If for all agents i, we have:*

1. *X_i is nonempty, convex and compact subset of a Euclidean space,*
2. *$F_i : X_{-i} \mapsto 2^{X_i}$ is both upper and lower hemicontinuous in X_{-i},*
3. *for any $x_{-i} \in X_{-i}, F_i(x_{-i})$ is nonempty, convex, and closed,*

4. u_i *is continuous, and*

5. *for any* $x \in X, x_i \longrightarrow u_i(x_i, x_{-i})$ *is quasiconcave on* $F_i(x_{-i})$*.*

Then there exists a Nash equilibrium in the generalized game.

Remark 2.3.1. We will denote fixed point set as $\Delta = \{x \in X \mid x \in F(x)\}$, where $F(x) = \prod_{i \in I} F_i(x)$.

When X is compact that is, if each X_i is compact, F_i being upper hemicontinuous for each $i \in I$ implies that Δ is closed.

Instead of proving Arrow and Debreu's theorem 2.3.1 and Cubiotti's theorem [3], we prove a generalization of both by replacing finite-dimensional Euclidean spaces $\mathbb{R}^n$ with locally convex (possibly infinite-dimensional) Hausdorff topological vector spaces. While also allowing for uncountable infinitely many players.

Theorem 2.3.2. *Let* $\Gamma = (X_i, F_i, u_i)_{i \in I}$ *be a generalized game such that for each* $i \in I$ *(possibly uncountable):*

1. X_i *is a nonempty, convex, compact, and metrizable subset of a locally convex Hausdorff topological vector space,*

2. $F_i : X_{-i} \mapsto 2^{X_i}$ *is lower hemicontinuous with nonempty convex values,*

3. u_i *is continuous and* $u_i(x_i, x_{-i})$ *is quasiconcave on* $F_i(x_{-i})$ *and*

4. *the fixed point set* $\Delta = \{x \in X \mid x \in F(x)\}$ *is closed.*

Then Γ *has a Nash equilibrium.*

In Theorem 2.3.2, we let go of the condition that $F_i : X_{-i} \mapsto 2^{X_i}$ is upper hemicontinuous as compared to Theorem 2.3.1. An example given by Cubiotti [3] provides a generalized game which satisfies the conditions in Theorems 2.3.1 and 2.3.2, but the $F_i's$ are not all upper hemicontinuous.

Example 2.3.1. [3] Let there be 2 players, $N = 1, 2$; $X_1 = X_2 = [0, 1]$. Let $F_1(x_2) = \left[0, \frac{1}{2}\right]$ and

$$F_2(x_1) = \begin{cases} \left[x_1, \frac{1}{2}\right] & \text{if } x_1 \in \left[0, \frac{1}{2}\right] \\ [0, 1] & \text{if } x_1 \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Take $u_1(x_1, x_2) = u_2(x_1, x_2) = x_1 + x_2$. Then all conditions for Theorems 2.3.1 and 2.3.2 are satisfied. Here, both F_1 and F_2 are lower hemicontinuous and

$$\Delta = \left\{(x_1, x_2) \in [0, 1] \times [0, 1] \mid x_1 \in \left[0, \frac{1}{2}\right], x_2 \in \left[x_1, \frac{1}{2}\right]\right\}$$

is closed. However, F_2 is not upper hemicontinuous. Therefore, we can only apply Theorem 2.3.2

We need the following lemma to prove Theorem 2.3.2.

Lemma 2.3.1. *Suppose that $F : X \mapsto 2^Y$ and $G : X \mapsto 2^Y$ are two correspondences such that F is lower hemicontinuous and G has open upper sections. Then $H = F \cap G$ is lower hemicontinuous.*

Proof. We just need to show for any open subset V in Y, $\{x \in X \mid (F \cap G)(x) \cap$

$V \neq \emptyset\}$ is open F. We see that:

$$
\begin{aligned}
&\{x \in X \mid (F \cap G)(x) \cap V \neq \emptyset\} \\
&\quad = \{x \in X \mid (F(x) \cap G(x)) \cap V \neq \emptyset\} \\
&\quad = \{x \in X \mid F(x) \cap (G(x) \cap V) \neq \emptyset\}.
\end{aligned}
$$

Since $G(x)$ and V are open in $Y, G(x) \cap V$ is open in Y. Moreover, since F is lower hemicontinuous $\{x \in X \mid F(x) \cap (G(x) \cap V) \neq \emptyset\}$ is open in Y. Thus, $\{x \in X \mid (F \cap G)(x) \cap V \neq \emptyset\}$ is open F and $H = F \cap G$ is lower hemicontinuous. □

Proof. (of Theorem 2.3.2) X_i is compact for each $i \in I$. By Theorem 1.3.2), if we define $X = \prod_{i \in I} X_i$ we get that X is compact. Let $\Delta U_i(x, y) = u_i(y_i, x_{-i}) - u_i(x)$ for each $i \in I$ and define G_i by

$$G_i(x) = \{y_i \in X_i \mid \Delta U_i(x, y) > 0\}. \tag{2.1}$$

Since, $\Delta U_i(x, x) = u_i(x_i, x_{-i}) - u_i(x) = 0$, $x_i \notin G_i(x)$ for each $x \in X$. By assumption 3, $u_i(x_i, x_{-i})$ is quasiconcave which would imply that for each $x \in X$, $G_i(x)$ is convex.

Therefore, for each $x \in X$, $x_i \notin G_i(x) = co((G_i(x)))$. Thus, $x_i \notin co((F_i(x) \cap G_i(x)))$. In order to show that Γ has a Nash equilibrium, it suffices to show that there exists $x^* \in X$ such that $x^* \in F(x^*)$ and $F_i(x^*) \cap G_i(x^*) = \emptyset$ for each $i \in I$. Moreover, $G_i(x)$ is open for each $x \in X$ for each $i \in I$, as u_i is continuous by assumption 3. Implying that G_i has open upper sections. Applying Lemma 2.3.1, we see that $H_i = F_i \cap G_i$ is lower hemicontinuous since, by assumption

2, F_i is lower hemicontinuous.

Let, for each $\varepsilon > 0$,

$$G_i^{\varepsilon}(x) = \{y_i \in X_i \mid \Delta U_i(x, y) > \varepsilon\}. \tag{2.2}$$

It can easily be seen that G_i^{ε} is convex, and has upper lower sections and by similar arguments as above, we have $H_i^{\varepsilon} = F_i \cap G_i^{\varepsilon}$ is lower hemicontinuous.

$$\operatorname{cl}\left(co((G_i^{\varepsilon}(x)))\right) = \operatorname{cl}\left(G_i^{\varepsilon}(x)\right) = \{y_i \in X_i \mid \Delta U_i(x, y) \geq \varepsilon\}.$$

Thus, for each $x \in X$, $x_i \notin cl\left(co((G_i^{\varepsilon}(x)))\right)$, which implies that $x_i \notin \operatorname{cl}\left(co((F_i \cap G_i^{\varepsilon}(x)))\right) \subseteq \operatorname{cl}\left(co((G_i^{\varepsilon}(x)))\right)$. Using Theorem 5 from [13] with $A_i = B_i = F_i$ and $D_i = X_i$ for the game $(X_i, F_i, G_i^{\varepsilon})_{i \in I}$, for any $\varepsilon > 0$, there exists a Nash equilibrium $x^{\varepsilon} \in X$. Note that $x^{\varepsilon} \in F\left(x^{\varepsilon}\right)$ and $F_i\left(x^{\varepsilon}\right) \cap G_i^{\varepsilon}\left(x^{\varepsilon}\right) = \emptyset$ for each $i \in I$ and for any $\varepsilon > 0$.

Let $\varepsilon = \frac{1}{m}$ for $m \geq 1$, that is, $x^{\frac{1}{m}} \in \Delta$ for each $m \geq 1$. By assumption 4, Δ is a closed, and it is subset of compact set X. Thus, Δ is compact by Theorem 1.3.3 and the sequence $\left\{x^{\frac{1}{m}}\right\}_{m \geq 1}$ of Δ has a convergent subsequence, say $x^{\frac{1}{m}} \longrightarrow x^*$. Thus, $x^* \in \Delta$, as Δ is closed, that is, $x^* \in F\left(x^*\right)$.

It remains to be shown that $F_i\left(x^*\right) \cap G_i\left(x^*\right) = \emptyset$ for all $i \in I$. Suppose we assume on the contrary that $F_i\left(x^*\right) \cap G_i\left(x^*\right) \neq \emptyset$ for some $i \in I$, such that $y_i \in F_i\left(x^*\right) \cap G_i\left(x^*\right)$. Then $y_i \in G_i\left(x^*\right)$ and $y_i \in F_i\left(x^*\right)$. In (2.1), $y_i \in G_i\left(x^*\right)$ implies that $\Delta U_i\left(x^*, y\right) > 0$ with $y = (y_i, y_{-i})$. As u_i is continuous and $x^{\frac{1}{m}} \longrightarrow x^*$, there exists integer $m' > 0$ such that $\Delta U_i\left(x^{\frac{1}{m}}, y\right) > \frac{1}{m}$ for all

$m \geq m'$. Thus, $y_i \in P^{\frac{1}{m}}\left(x^{\frac{1}{m}}\right)$. Since F_i is lower hemicontinuous, $y_i \in F_i\left(x^*\right)$, and $x^{\frac{1}{m}} \longrightarrow x^*$, which implies that for $b = y_i \in F_i\left(x^*\right)$. So, there exists a subsequence $\left\{x^{\frac{1}{m_n}}\right\}$ of the sequence $\left\{x^{\frac{1}{m}}\right\}$ such that there exist $b_n \in F_i\left(x^{\frac{1}{m_n}}\right)$ for $n \geq 1$ with $\lim_{n\to\infty} b_n = b = y_i$. Since $\Delta U_i\left(x^{\frac{1}{m}}, y\right) > \frac{1}{m}$ for all $m \geq m'$ and $\lim_{n\to\infty} b_n = y_i$, there exists $n_0 \geq 0$ such that $m_{n_0} \geq m'$ and $\Delta U_i\left(x^{\frac{1}{m_n}}, y^n\right) > \frac{1}{m_n}$ (where $y_i^n = b_n$), which implies that $b_n \in G_i^{\frac{1}{m_n}}\left(x^{\frac{1}{m_n}}\right)$ for $n \geq n_0$. Since $b_n \in F_i\left(x^{\frac{1}{m_n}}\right)$ for $n \geq 1$, it follows that $b_n \in F_i\left(x^{\frac{1}{m_n}}\right) \cap G_i^{\frac{1}{m_n}}\left(x^{\frac{1}{m_n}}\right)$ for $n \geq n_0$, contradicting the fact $F_i\left(x^\varepsilon\right) \cap G_i^\varepsilon\left(x^\varepsilon\right) = \emptyset$ for each $i \in I$ and for all $\varepsilon = \frac{1}{m}$ with $m \geq 1$. Thus, we must have $F_i\left(x^*\right) \cap G_i\left(x^*\right) = \emptyset$ for all $i \in I$, that is, x^* is a Nash equilibrium for the game.

□

Next, we state the existence of Nash equilibrium where the strategy set is a subset of any Hausdroff's space instead of a Euclidean space or a locally convex Hausdorff vector space and relax the compactness condition on X by allowing only countable players.

Theorem 2.3.3. *Let $\Gamma = (X_i, F_i, u_i)_{i\in I}$ be a generalized game such that for each $i \in I$ (countable):*

1. *X_i is nonempty, convex subset of a Hausdorff topological vector space,*
2. *F_i has nonempty convex values and $F(x) = \prod_{i\in I} F_i(x)$ has open lower sections,*
3. *the fixed point set $\Delta = \{x \in X \mid x \in F(x)\}$ is closed and compact,*
4. *u_i is bounded, continuous in x, and concave in x_i, and*

5. *there exists a nonempty* $X_0 \subseteq X$ *such that* X_0 *is contained in a compact convex subset* X' *of* X *and the set* $D = \cap_{x \in X_0} \left(F^{-1}(x)\right)^c$ *is compact.*

Then Γ *has a Nash equilibrium.*

To prove the Theorem 2.3.3, we need the following lemma, which is an infinite analog to solving a quasi-equilibrium problem with the Nikaido-Isoda aggregate function.

Let $\Gamma = (X_i, F_i, u_i)_{i \in I}$ be a generalized game such that $|u_j(x)| \leq M_j$ for each $j \in I$ and for all $x \in X = \prod_{i \in I} X_i$. For each $i \in I$, let

$$\phi_i(x, y) = u_i\left(y_i, x_{-i}\right) - u_i(x) \tag{2.3}$$

and

$$\phi(x, y) = \sum_{i \in I} \frac{1}{2^i M_i} \phi_i(x, y). \tag{2.4}$$

Lemma 2.3.2. *Let* $\Gamma = (X_i, F_i, u_i)_{i \in I}$ *be a generalized game with* I *being countable and* $F(x) = \prod_{i \in I} F_i(x)$*. Then* $x^* \in X$ *is a Nash equilibrium of* Γ *if and only if* $x^* \in F(x^*)$ *and*

$$\phi\left(x^*, y\right) \leq 0 \text{ for all } y \in F\left(x^*\right).$$

Proof. $x^* \in X$ is a Nash equilibrium if and only if $\phi_i\left(x^*, y\right) = u_i\left(y_i, x_{-i}^*\right) - u_i\left(x^*\right) \leq 0$ for all $y \in F\left(x^*\right)$ and every $i \in I$ by (1.1). For all $y \in F\left(x^*\right)$, if $\phi_i\left(x^*, y\right) \leq 0$ for every $i \in I$ then it is obvious to see that $\phi\left(x^*, y\right) \leq 0$. Assume $\phi\left(x^*, y\right) \leq 0$ and take $y_{-i} = x_{-i}^*$. Thus, $\phi_j\left(x^*, y\right) = u_j\left(y_j, x_{-j}^*\right) - u_j\left(x^*\right) = 0$ for all $j \neq i$ and $\phi_i\left(x^*, y\right) = u_i\left(y_i, x_{-i}^*\right) - u_i\left(x^*\right) \leq 0$. □

Proof. (of Theorem 2.3.3.) By assumption 2, F_i has nonempty convex values for each $i \in I$ and $F(x) = \prod_{i \in I} F_i(x)$. Thus, it is clear that $F(x)$ has nonempty convex values. By assumption 4, u_i is bounded, say by $M_i > 0$ such that $|u_i(x)| \leq M_i$ for all $x \in X$. Define $\phi_i(x, y)$ as in (2.3) for each $i \in I$ and $\phi(x, y)$ as in (2.4). Then $\phi(x, y) = \sum_{i \in I} \frac{1}{2^i M_i} \phi_i(x, y)$ is bounded and converges uniformly and absolutely. Assumption 4 also gives us that u_i is continuous for each $i \in I$, and from Theorem 1.3.4 we conclude that $\phi(x, y)$ is continuous. By Lemma 2.3.2, $x^* \in X$ is a Nash equilibrium of Γ if and only if $x^* \in F(x^*)$ and $\phi(x^*, y) \leq 0$ for all $y \in F(x^*)$.

Let $P : X \mapsto 2^X$ be a correspondence such that $P(x) = \{y \in X : \phi(x, y) > 0\}$ for each $x \in X$. Clearly, $x \notin P(x)$ for all $x \in X$. Then P has open lower sections, since $P^{-1}(y)$ is open for any $y \in X$ as $\phi(x, y)$ is continuous. Since, $u_i(x)$ is concave in x_i for each $i \geq 1$. Assumption 4 implies $\phi_i(x, y)$ is concave in y_i. To prove $P(x)$ is convex for each $x \in X$, let $a, b \in P(x)$. Thus, $\phi(x, a) > 0$ and $\phi(x, b) > 0$. For each $i \in I$, $\phi_i(x, y) = u_i(y_i, x_{-i}) - u_i(x) = \phi_i(x, (y_i, x_{-i}))$.

For some t such that $0 \leq t \leq 1$, using Theorem 1.3.5 and (2.4), we have:

$$
\begin{aligned}
&\phi\left(x, ta+(1-t)b\right) \\
&\quad = \sum_{i \in I} \frac{1}{2^i M_i} \phi_i\left(x, ta+(1-t)b\right) \\
&\quad = \sum_{i \in I} \frac{1}{2^i M_i} \phi_i\left(x, \left(ta_i+(1-t)b_i, x_{-i}\right)\right) \\
&\quad \geq \sum_{i \in I} \frac{1}{2^i M_i} \left[t\phi_i\left(x, \left(a_i, x_{-i}\right)\right)+(1-t)\phi_i\left(x, \left(b_i, x_{-i}\right)\right)\right] \\
&\quad = t\sum_{i \in I} \frac{1}{2^i M_i} \phi_i\left(x, \left(a_i, x_{-i}\right)\right)+(1-t)\sum_{i \in I} \frac{1}{2^i M_i} \phi_i\left(x, \left(b_i, x_{-i}\right)\right) \\
&\quad = t\phi\left(x, a\right)+(1-t)\phi\left(x, b\right) > 0.
\end{aligned}
$$

Thus, $ta+(1-t)b \in P(x)$ and so $P(x)$ is convex. $x^* \in X$ is a Nash equilibrium of Γ if and only if $x^* \in F\left(x^*\right)$ and $\phi\left(x^*, y\right) \leq 0$ for all $y \in F\left(x^*\right)$ by Lemma 2.3.2.

Define $\varphi : X \mapsto 2^X$ by

$$\varphi(x) = F(x) \cap P(x) \text{ for all } x \in X.$$

Since $F(x)$ and $P(x)$ are convex for each $x \in X$, φ has convex values. F and P has open lower sections. Implies for every $x \in X$, $F^{-1}(x)$ and $P^{-1}(x)$ is open in X and $F^{-1}(x) \cap P^{-1}(x)$ is open in X and $\varphi^{-1}(x)$ is open in X. Hence, φ has open lower sections.

Let $G : X \mapsto 2^X$ be a correspondence such that

$$G(x) = \begin{cases} F(x), & \text{if } x \in X \backslash \Delta \text{ and} \\ \varphi(x), & \text{if } x \in \Delta. \end{cases}$$

Then, G is convex valued. Since $\varphi(x) \subseteq F(x)$ for each $x \in X$, we have $\varphi^{-1}(y) \subseteq F^{-1}(y)$ for each $y \in X$. For any $y \in X$, if $y \in \varphi(x) \subseteq F(x)$, then $G^{-1}(y) = \varphi^{-1}(y) \cup (F^{-1}(y) \cap (X \backslash \Delta))$; if $y \in F(x) \backslash \varphi(x)$, then $G^{-1}(y) = F^{-1}(y) \cap (X \backslash \Delta)$. By our hypothesis, Δ is closed and compact, implying that $X \backslash \Delta$ is open in X. Since F and φ have open lower sections, G has open lower sections. For each $y \in X, (G^{-1}(y))^c = (\varphi^{-1}(y))^c \cap (F^{-1}(y))^c \cup \Delta\big)$ or $(F^{-1}(y))^c \cup \Delta$, which implies that $(G^{-1}(y))^c \subseteq (F^{-1}(y))^c \cup \Delta$. By assumptions our hypothesis, $\cap_{x \in X_0} (F^{-1}(x))^c$ and Δ is compact.

$$\cap_{x \in X_0} cl \left((G^{-1}(x))^c \right) \subseteq \cap_{x \in X_0} cl \left((F^{-1}(x))^c \cup \Delta \right) = \left(\cap_{x \in X_0} cl \left((F^{-1}(x))^c \right) \right) \cup \Delta.$$

Thus, $\cap_{x \in X_0} cl \left((G^{-1}(x))^c \right)$ is compact by Theorem 1.3.3. Let $G(x) \neq \emptyset$ for each $x \in X$. Then by Corollary 2.1.2 there exists $x^* \in X$ such that $x^* \in G(x^*)$. Since $G(x^*) \subseteq F(x^*)$, we have $x^* \in \Delta \cap \varphi(x^*)$, which contradicts the fact $x \notin P(x)$ for all $x \in X$. Thus, $G(x') = \emptyset$ for some $x' \in X$. Since $F(x)$ is nonempty for every $x \in X$, we have $x' \in \Delta$ and $\varphi(x') = \emptyset$. Thus, $x' \in F(x')$ and $F(x') \cap P(x') = \emptyset$. □

Remark 2.3.2. For each $y \in X$, $F = \prod_{i \in I} F_i, F^{-1}(y) = \cap_{i \in I} F_i^{-1}(y_i)$. If F_i has open lower sections for each $i \in I = \{1, 2, \ldots, n\}$, then F has open lower sections.

Finally, as a consequence Theorem 2.3.3 we obtain the following generalization of the Nash equilibrium as desired.

Corollary 2.3.1. *Let* $\Gamma = (X_i, F_i, u_i)_{i\in I}$ *be a generalized game such that for each* $i \in I = \{1, 2, \ldots, n\}$,

1. X_i *is nonempty, convex subset of a Hausdorff topological vector space,*
2. F_i *is nonempty convex valued and has open lower sections,*
3. *the fixed point set* $\Delta = \{x \in X \mid x \in F(x)\}$ *is closed and compact,*
4. u_i *is continuous in* x *and concave in* x_i*, and*
5. *there exists a nonempty* $X_0 \subseteq X$ *such that* X_0 *is contained in a compact convex subset* X' *of* X *and the set* $D = \cap_{x\in X_0} \left(F^{-1}(x)\right)^c$ *is compact.*

Then Γ *has a Nash equilibrium.*

We conclude the thesis with an example that motivates the need for Theorem 2.3.3.

Example 2.3.2. Let $I = \mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers and $X_i = [0, 1]$ for each $i \in I$. Then $X = \prod_{i\in I} X_i$ is a compact subset of $\mathbb{R}^\infty$ by Theorem 1.3.2. For each $i \in I$, define F_i as follows:)

$$F_i\left(x_i, x_{-i}\right) = \begin{cases} \left[\frac{1}{3}, 1\right] & \text{if } x_{-i} \in \prod_{j\neq i} X_j \text{ and } x_{-i} \neq 0 \\ \left(\frac{1}{2}, 1\right] & \text{if } x_{-i} = 0 \end{cases}$$

Then each F_i has open lower sections since for any $y \in X_i = [0, 1]$,

$$F_i^{-1}(y) = \begin{cases} \emptyset & \text{if } y \in \left[0, \frac{1}{3}\right) \\ \left(\prod_{j \in I} X_j\right) \backslash \{0\} & \text{if } y \in \left[\frac{1}{3}, \frac{1}{2}\right] \\ \prod_{j \in I} X_j & \text{if } y \in \left(\frac{1}{2}, 1\right] \end{cases}$$

which is open in $X = \prod_{j \in I} X_j$. It follows that for $F = \prod_{i \in I} F_i, F^{-1}(y) = \cap_{i \in I} F_i^{-1}(y_i) = \emptyset$ or $\left(\prod_{j \in I} X_j\right) \backslash \{0\}$ or $\prod_{j \in I} X_j$ which is open in $X = \prod_{j \in I} X_j$ for each $y \in X$. However, it is easy to see that F_i is not upper hemicontinuous at 0 . For each $i \in I$, take

$$u_i(x) = \sum_{j=1}^{\infty} \frac{1}{2^j} x_j.$$

Then it follows from Theorem 1.3.4 that each $u_i(x)$ is continuous in $x \in X$. Moreover, it is clear that each u_i is bounded and concave in x_i. Note that

$$\Delta = \prod_{j=1}^{\infty} \left[\frac{1}{3}, 1\right],$$

which is closed and compact. Thus, this game satisfies the hypothesis of Theorem 2.3.3, and so the game has an equilibrium. However, Theorem 2.3.1 cannot be applied here as F_i is not upper hemicontinuous, and the theorem given by Cubiotti [3] cannot work either since we have infinitely many players.

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