A Study of Hyperbolic Geometry

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for the degree of

Master of Science

in

Mathematics

by

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DECLARATION

I hereby declare that the dissertation entitled **"A Study of Hyperbolic Geometry"** is a genuine record of an advanced reading project carried out by me and no part of this dissertation has been submitted to any university or institution for the award of any degree or diploma.

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CERTIFICATE

This is to certify that the dissertation entitled **"A Study of Hyperbolic Geometry"** is based on an advanced reading project carried out by Ms. Prakruti Kishor Kalsaria under my guidance and supervision at Indian Institute of Science Education and Research Bhopal, India.

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To Whom It May Concern:

Sub: Certificate of completion of advanced reading project entitled "A Study of Hyperbolic Geometry".

This is to certify that the dissertation entitled "A Study of Hyperbolic Geometry" is based on an advanced reading project carried out by Ms. Prakruti Kishor Kalsaria under my guidance and supervision at Indian Institute of Science Education and Research Bhopal, India.

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-Prakruti Kalsaria

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ABSTRACT

This dissertation delves into the captivating field of hyperbolic geometry, a non-Euclidean geometry that involves exploring curved spaces characterized by a constant negative curvature. We will begin by examining the upper half plane model $\mathbb{H}$. The geometry of the hyperbolic plane will be explored by analyzing Möbius transformations. Subsequently, we will develop a metric in $\mathbb{H}$ and look at its group of isometries. The concepts of convex sets and hyperbolic polygons will then be introduced, after which the Gauss-Bonnet theorem will be stated. This wonderful theorem shows how the area of a hyperbolic polygon depends only on its angles. Further, we define Fuchsian groups as discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ (the group of orientation-preserving isometries of $\mathbb{H}$). We will then show that a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is discrete if and only if it acts properly discontinuously on $\mathbb{H}$.

The notion of fundamental regions is then defined, as they play a fundamental role in understanding the structure of hyperbolic surfaces. A fundamental region represents a subset of the hyperbolic plane that encapsulates the essential geometry of a Fuchsian group. It serves as a crucial element in the study of tessellations, group actions, and the construction of quotient spaces. We then delve into defining side pairing transformations, which carry one side of the fundamental region to another. These transformations help to establish congruent vertices, thus facilitating the definition of elliptic cycles. Moreover, we discuss cocompact Fuchsian groups and then introduce the concept of the signature of a Fuchsian group. Following that, we present proof of Poincaré's theorem, a significant result in hyperbolic geometry. Overall, this study explores the captivating aspects of hyperbolic geometry, unraveling its unique properties, and providing valuable insights into its applications across various fields.

Contents

List of Figures

List of Tables

Chapter 1

Hyperbolic Geometry

The aim of this chapter is to provide a brief history of hyperbolic geometry, a description of its development, and an outline of the fundamental work that goes into the creation of a new branch of mathematics. The first section outlines the initial work done and provides a number of references for the same, while the subsequent sections are based on [2], [6, Chapter 1] and [8, Chapter 1-3].

1.1 How it began

It all began with *The Elements*, a mathematical treatise composed of 13 books written in Alexandria around 300 BC by a great Greek mathematician named Euclid. For millennia to come, this laid the foundation for geometry (see [5] for a translated version). Although geometry has been practiced since 3000 BC, Euclid's Elements is the first rigorous mathematical axiomatic system for geometry. It starts with 23 definitions before moving on to a list of five postulates and five common notions. The known Euclidean geometry is then logically derived from these.

For centuries, mathematicians struggled to prove the seemingly derivable fifth postulate, often called *the parallel postulate*. The original statement of this postulate reads: 'If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines if produced indefinitely, will meet on that side on which the angles are less than two right angles.' Whereas, a more modern paraphrased version of it says 'For every line l and for every point P that does not lie on l, there exists a unique line m through P that is parallel to l.' [5]

Mathematicians like Lobachevski [7], Bolyai [3], and Gauss [4, Chapter 3] first proposed the idea of hyperbolic geometry as a mathematical concept in the nineteenth century, but like any other newly created theory, it was initially received with criticism from the mathematical community [4, Chapter 4]. By challenging the notion that Euclidean geometry was the sole acceptable type of geometry, the revolutionary idea of hyperbolic geometry opened the door for the development of additional non-Euclidean geometries, such as spherical geometry. The reason why these are known as non-Euclidean geometries is because they fail to comply with the postulates upon which Euclidean geometry is based. A brief overview of the three main geometries that are studied widely is given in Table 1.1 (based on [9]).

Properties \ Geometries	Euclidean	Hyperbolic	Spherical
Sum of angles of a triangle	$= 180$	< 180	> 180
Number of parallel lines passing through a point outside the given line	One	Infinite	None
Curvature	Zero	-1	1
Models	Euclidean plane	Upper half-plane, Poincaré dis, Klein model, etc	Sphere

Table 1.1: Difference between Euclidean, hyperbolic, and spherical geometries.

Before proceeding any further, it becomes necessary for us to define terms such as line and parallelism. Even though we could do it without one (see [1]), a model makes it easier for us to define basic concepts like lines or points.

1.2 The upper half-plane model $\mathbb{H}$

The two models that are most frequently used are the upper half-plane and the Poincaré disk models.

Consider the set defined as

$$\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\},$$

where $\Im$ denotes the imaginary part of z. As the underlying space of this model is the upper half-plane of the complex plane $\mathbb{C}$, it is widely known as the *upper half-plane*. There are two types of *hyperbolic lines* that appear to be distinct when defined as objects in the Euclidean plane however they are not any different in hyperbolic geometry.

Definition 1.2.1. As Euclidean objects, *hyperbolic lines* are

(i) Vertical half-line: The intersection of $\mathbb{H}$ with a Euclidean line in $\mathbb{C}$ perpendicular to the real axis $\mathbb{R}$ in $\mathbb{C}$.

(ii) Semi-circle orthogonal to $\mathbb{R}$: The intersection of $\mathbb{H}$ with a Euclidean circle centered on the real axis $\mathbb{R}$.

Given any two distinct points in $\mathbb{H}$, there is a unique hyperbolic line passing them.

The set $\mathbb{H}$ when considered with these hyperbolic lines forms a model for hyperbolic geometry which is known as the *Poincaré half-plane model* or the *upper half-plane model*.

Definition 1.2.2. Two hyperbolic lines in $\mathbb{H}$ are *parallel* if they are disjoint.

Remark 1. The extended real line $\overline{\mathbb{R}}$ is referred to as the *boundary at infinity of* $\mathbb{H}$, and points of $\overline{\mathbb{R}}$ as *points at infinity*. The reason is that any point on $\overline{\mathbb{R}}$ is at an infinite hyperbolic distance from any point in $\mathbb{H}$.

Given two parallel lines in the hyperbolic space, either their boundaries intersect at infinity or they do not, in which case they are known as *ultraparallel* lines.

Figure 1.2.1: Parallel (left) and ultraparallel (right) hyperbolic lines.

Theorem 1.2.3. *Let ℓ be a hyperbolic line in $\mathbb{H}$ and p be a point in $\mathbb{H}$ not on ℓ. Then, there exist infinitely many distinct hyperbolic lines through p that are parallel to ℓ.*

Proof. First, we consider the case when ℓ is contained in a Euclidean line L as shown in Figure 1.2.2. For a point p not on ℓ, there exists a Euclidean line K parallel to L which passes through p. As L is a vertical straight line and K is parallel to L, K is a vertical straight line. Therefore, the vertical half-line $k = \mathbb{H} \cap K$ is a hyperbolic line through p that is parallel to ℓ. For a point x on $\mathbb{R}$ between L and K, we construct the perpendicular bisector of the Euclidean line segment whose endpoints are x and p. Let this perpendicular bisector meet $\mathbb{R}$ in c. Since $Re(x) \neq Re(p)$, there is a Euclidean circle A centered at c and passing through x also passes through p. As A and L are disjoint, by the construction of A, there exists a hyperbolic line $\mathbb{H} \cap A$ passing through p and parallel to ℓ. There are infinitely many distinct hyperbolic lines through p that are parallel to ℓ because there are infinitely many distinct points on $\mathbb{R}$ between L and K.

Now we consider the case when ℓ is contained in an Euclidean circle A as shown in Figure 1.2.3. Let B be an Euclidean circle passing through p and concentric to A. As concentric circles are disjoint, the hyperbolic line $\mathbb{H} \cap B$ is parallel to ℓ and passes through p. As earlier, consider any point x on $\mathbb{R}$ between A and B such that $Re(x) \neq Re(p)$. There exists an Euclidean circle C passing through x and p. By construction, A and C are disjoint and therefore, there exists a second hyperbolic line $\mathbb{H} \cap C$ passing through p and parallel to ℓ. Since there are infinitely many distinct points on $\mathbb{R}$ between A and B, there are infinitely many distinct hyperbolic lines through p that are parallel to ℓ. $\square$

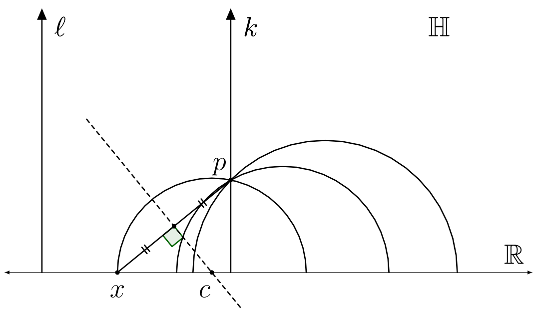


Figure 1.2.2: Lines parallel to a vertical half-line.

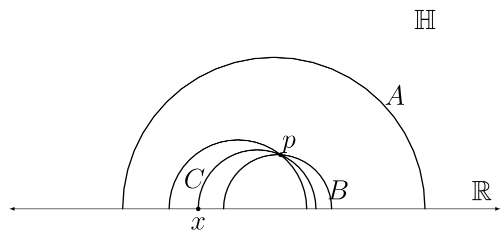


Figure 1.2.3: Lines parallel to a semi-circle orthogonal to ℝ.

1.3 Group of isometries

The set of real matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } ad - bc = 1$$

forms a group under matrix multiplication known as *special linear* (or *unimodular*) *group* over the reals denoted by $\mathrm{SL}(2, \mathbb{R})$.

Definition 1.3.1. A *Möbius* (or a *linear fractional*) *transformation* $T : \mathbb{C} \to \mathbb{C}$ is defined as

$$T(z) = \frac{az + b}{cz + d} \text{ where } a,\ b,\ c,\ d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

The *Möbius group* $\mathrm{Möb}(\mathbb{C})$ is the group of Möbius transformations under composition.

Theorem 1.3.2. *All biholomorphisms that preserve* $\mathbb{H}$ *are Möbius transformations.*

Let us define a map $f : \mathrm{GL}(2, \mathbb{C}) \to \mathrm{Möb}(\mathbb{C})$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto T(z) = \frac{az+b}{cz+d}$ which is a surjective homomorphism. Then the kernel $\mathrm{Ker}(f) = \{k\mathrm{Id} : k \in \mathbb{C}\}$, where Id is the 2×2 identity matrix. Therefore, $\mathrm{Möb}(\mathbb{C})$ is isomorphic to $\mathrm{GL}(2, \mathbb{C})/\mathrm{Ker}(f) = \mathrm{SL}(2, \mathbb{C})/\pm\mathrm{Id} = \mathrm{PSL}(2, \mathbb{C})$. The set of all Möbius transformations such that $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ form a group under composition which preserves the upper half-plane is denoted by $\text{Möb}^+(\mathbb{H})$. Proceeding as above, we can show that $\text{Möb}^+(\mathbb{H}) \cong \text{PSL}(2,\mathbb{R})$. Thus, we can use $\text{PSL}(2,\mathbb{R})$ and Möbius transformation interchangebly. Each of these transformations can be represented by a pair of matrices

$$T = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proposition 1.3.3. *The group* $\text{PSL}(2,\mathbb{R})$ *acts on* $\mathbb{H}$ *via Möbius transformation.*

Proof. For $z \in \mathbb{H}$ and $T \in \text{PSL}(2,\mathbb{R})$, we show that $\Im T(z) > 0$. Let $T(z) = \frac{az+b}{cz+d} = w$ (say), where $a,\ b,\ c,\ d \in R$ and $ad - bc = 1$. Then, substituting the value of

$$w = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}$$

we have,

$$\Im w = \frac{w - \bar{w}}{2i} = \frac{(az+b)(c\bar{z}+d) - \overline{(az+b)(c\bar{z}+d)}}{2i|cz+d|^2} = \frac{z - \bar{z}}{2i|cz+d|^2}.$$

Since $\Im z > 0$, we have $\Im w = \frac{\Im z}{|cz+d|^2} > 0$. Thus, any transformation of $\text{PSL}(2,\mathbb{R})$ maps $\mathbb{H}$ to itself. $\square$

For a piecewise C^1 path $f : [a,b] \to \mathbb{H}$, we define the *hyperbolic length* of f to be

$$\ell_{\mathbb{H}}(f) = \int_f \frac{1}{\Im z}|\mathrm{d}z| = \int_a^b \frac{1}{\Im f(t)}\,|f'(t)|\,\mathrm{d}t.$$

For $x, y \in \mathbb{H}$, consider the function

$$\mathrm{d}_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$$

defined by

$$\mathrm{d}_{\mathbb{H}}(x,y) = \inf\left\{\ell_{\mathbb{H}}(f)\right\},$$

where the infimum is taken over all f joining x and y in $\mathbb{H}$. The function $\mathrm{d}_{\mathbb{H}}(x,y)$ is referred to as the *hyperbolic distance* between x and y.

Definition 1.3.4. A homeomorphism $T : \mathbb{H} \to \mathbb{H}$ is said to be an *isometry* of $\mathbb{H}$ if

$$\mathrm{d}_{\mathbb{H}}(T(x), T(y)) = \mathrm{d}_{\mathbb{H}}(x, y), \text{ for every } x,\ y \in \mathbb{H}.$$

The set of isometries of $\mathbb{H}$ forms a group under composition and is denoted by $\mathrm{Isom}(\mathbb{H})$.

Theorem 1.3.5. *The group* $\mathrm{PSL}(2, \mathbb{R})$ *is a subgroup of* $\mathrm{Isom}(\mathbb{H})$.

Proof. By Proposition 1.3.3, it follows that any transformation of $\mathrm{PSL}(2, \mathbb{R})$ maps $\mathbb{H}$ onto itself. Now, we show that they preserve the hyperbolic distance. Equivalently, for a piecewise C^1-path $f : \mathrm{I} = [0, 1] \to \mathbb{H}$ and $T \in \mathrm{PSL}(2, \mathbb{R})$, we show that $\mathrm{l}_{\mathbb{H}}(T(f)) = \mathrm{l}_{\mathbb{H}}(f)$. Let $T \in \mathrm{PSL}(2, \mathbb{R})$ such that $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a,\ b,\ c,\ d \in \mathbb{R}$ and $ad - bc = 1$. For $z = x + iy \in \mathbb{H}$, let $T(z) = w = u + iv$. Since $ad - bc = 1$, we have

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{a(cz + d) - c(az + d)}{(cz + d)^2} = \frac{1}{(cz + d)^2}.$$

Since $\Im w = \frac{\Im z}{|cz+d|^2}$, we have $|\frac{\mathrm{d}w}{\mathrm{d}z}| = \frac{v}{y}$, and therefore,

$$\mathrm{l}_{\mathbb{H}}(T(f)) = \int_0^1 \frac{|\frac{\mathrm{d}w}{\mathrm{d}t}|\mathrm{d}t}{v(t)} = \int_0^1 \frac{|\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t}|\mathrm{d}t}{v(t)} = \int_0^1 \frac{|\frac{\mathrm{d}z}{\mathrm{d}t}|\mathrm{d}t}{y(t)} = \mathrm{l}_{\mathbb{H}}(f).$$

Hence, any transformation in $\mathrm{PSL}(2, \mathbb{R})$ is an isometry of $\mathbb{H}$. □

In other words, Theorem 1.3.5 states that the distance $\mathrm{d}_{\mathbb{H}}(x, y)$ remains invariant under $\text{Möb}^+(\mathbb{H})$, i.e.,

$$\mathrm{d}_{\mathbb{H}}(x, y) = \mathrm{d}_{\mathbb{H}}(T(x), T(y)),\ T \in \text{Möb}^+(\mathbb{H}).$$

We state the some proposition without proving them.

Proposition 1.3.6. *The hyperbolic distance between any two points is equal to the hyperbolic length along the unique hyperbolic line passing them.*

Proposition 1.3.7. *A Möbius transformation maps hyperbolic lines to hyperbolic lines in* $\mathbb{H}$.

Theorem 1.3.8. *For $z, w \in \mathbb{H}$, the following results hold.*

(i) $\mathrm{d}_{\mathbb{H}}(z, w) = \ln \dfrac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}$

(ii) $\cosh \mathrm{d}_{\mathbb{H}}(z, w) = 1 + \dfrac{|z - w|^2}{2\Im z \Im w}$

(iii) $\sinh \left[\frac{1}{2}\mathrm{d}_{\mathbb{H}}(z, w)\right] = \dfrac{|z - w|}{2(\Im z \Im w)^{1/2}}$

(iv) $\cosh \left[\frac{1}{2}\mathrm{d}_{\mathbb{H}}(z, w)\right] = \dfrac{|z - \bar{w}|}{2(\Im z \Im w)^{1/2}}$

(v) $\tanh \left[\frac{1}{2}\mathrm{d}_{\mathbb{H}}(z, w)\right] = \left|\dfrac{z - w}{z - \bar{w}}\right|$

Theorem 1.3.9. *The group of Möbius transformations is generated by $z \mapsto az + b$ for $a, b \in \mathbb{C}, a \neq 0$ and $z \mapsto \frac{1}{z}$. Moreover, Möbius transformations together with the transformation $z \mapsto -\bar{z}$ generates* $\mathrm{Isom}(\mathbb{H})$.

Definition 1.3.10. If two smooth curves C and D in $\mathbb{C}$ intersect at a point z, then the *angle* $\angle(C, D)$ between these two curves at z is defined to be the angle between the tangent lines to C and D at z, measured from C to D.

Definition 1.3.11. If the absolute value of the angle between curves remains invariant under a homeomorphism of $\mathbb{C}$, then it is said to be *conformal*.

In other words, T is conformal if $\angle(C, D) = \angle(T(C), T(D))$.

Definition 1.3.12. An *orientation-preserving* homeomorphism is the one that preserves the angle between curves along with its sign, while an *orientation-reversing* homeomorphism is the one under which the sign of the angle changes.

Theorem 1.3.13. *The elements of* $\mathrm{Isom}(\mathbb{H})$ *are conformal.*

1.4 Poincaré disc model

There is another model that is commonly used to study hyperbolic geometry besides the upper half-plane model. Consider the set defined as

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\},$$

whose underlying space is the unit disc centered at the origin of the complex plane $\mathbb{C}$. Let the boundary of $\mathbb{D}$ be denoted by $\partial\mathbb{D}$, that is, $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$. To move from $\mathbb{D}$ to $\mathbb{H}$ we introduce the following Möbius transformation,

Definition 1.4.1. The Möbius transformation

$$C : z \mapsto \frac{z-i}{z+i}$$

is known as the *Cayley transformation*.

Lemma 1.4.2. *The Cayley transformation is a conformal transformation $C : \mathbb{H} \longrightarrow \mathbb{D}$.*

Proof. From the Definition 1.4.1, $C(1) = -i, C(0) = -1$, and $C(\infty) = 1$. Thus, C maps the extended real line to the unit circle centered at the origin. Hence, C maps the two connected components of $\hat{\mathbb{C}} \backslash (\mathbb{R} \cup \infty)$ to the two connected components of $\hat{\mathbb{C}} \backslash \partial\mathbb{D}$. As $C(i) = 0$, the upper half-plane is mapped to the interior of the circle. □

A *hyperbolic line in* $\mathbb{D}$ is defined to be the image of a hyperbolic line in $\mathbb{H}$ under the map C. The set $\mathbb{D}$ considered with these hyperbolic lines forms a model known as the *Poincaré disc model.*

We know that the boundary of $\mathbb{H}$ is $\mathbb{R} \cup \infty$ and the boundary of $\mathbb{D}$ is $\partial\mathbb{D}$. The *Euclidean closure of* $\mathbb{H}$ is defined as $\tilde{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \infty$ and the *Euclidean closure of* $\mathbb{D}$ is defined as $\tilde{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$.

1.5 Classification of Möbius transformations

Definition 1.5.1. Two elements $S_1, S_2 \in \text{Möb}^+(\mathbb{H})$ are said to be *conjugate* in $\text{Möb}^+(\mathbb{H})$ if there exists $T \in \text{Möb}^+(\mathbb{H})$ such that $S_2 = TS_1T^{-1}$. It will be denoted by $S_1 \sim_T S_2$.

Equivalently, $S_1 \sim_T S_2$ means that the map S_2 acts like S_1 after T is applied to $\mathbb{H}$ as shown in the following commutative diagram.

$$\begin{tikzcd} \mathbb{H} \arrow[r, "S_1"] \arrow[d, "T"'] & \mathbb{H} \arrow[d, "T"] \\ \mathbb{H} \arrow[r, "S_2"] & \mathbb{H} \end{tikzcd}$$

Definition 1.5.2. For $T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{PSL}(2, \mathbb{R})$, we define the *trace* of T denoted by $\operatorname{tr} T$ as $(\operatorname{tr} T) = |\alpha + \delta|$.

For $S, T \in \text{Möb}^+(\mathbb{H})$, since $\operatorname{tr}(ST) = \operatorname{tr}(TS)$, we have

$$\operatorname{tr} STS^{-1} = \operatorname{tr} TSS^{-1} = \operatorname{tr} T.$$

This outcome is crucial since it shows that the trace is still invariant under conjugation.

Proposition 1.5.3. *If T is any non-identity element of $\text{Möb}^+(\mathbb{H})$, then T has either one or two fixed points.*

Proof. Consider $T(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. Suppose that T fixes p. Then

$$T(p) = p \Leftrightarrow \frac{\alpha p + \beta}{\gamma p + \delta} = p \Leftrightarrow \gamma p^2 + (\delta - \alpha)p - \beta = 0 \Leftrightarrow p = \frac{\alpha - \delta \pm \sqrt{(\delta - \alpha)^2 + 4\beta\gamma}}{2\gamma}.$$

Since $\alpha\delta - \beta\gamma = 1$, we can conclude that $p = \frac{\alpha-\delta\pm\sqrt{(\operatorname{tr} T)^2-4}}{2\gamma}$. □

Now, we classify the Möbius transformations by investigating their fixed points. Apparently, the trace is sufficient to classify every transformation. We will investigate each case individually in such a way that the fixed point(s) are either 0 or ∞. Consider $S_1 \in \text{Möb}^+(\mathbb{H})$.

Case-1: T has a unique fixed point.

Assume that S_1 has exactly one fixed point p. By Proposition 1.5.3, this is possible if and only if $\operatorname{tr} S_1 = \pm 2$ and $S_1 \neq \text{Id}$. There exists transformation $T \in \text{Möb}^+(\mathbb{H})$ such that $T(p) = \infty$. Then $S_2 = TS_1T^{-1}$ fixes ∞ as

$$S_2(\infty) = TS_1T^{-1}(\infty) = TS_1(p) = T(p) = \infty.$$

Let $S_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with ∞ as a fixed point. This implies that $\gamma = 0$ and $\alpha\delta = 1$. Thus, $S_2(z) = \frac{\alpha z+\beta}{\alpha^{-1}} = az + b$, where $a, b \in \mathbb{R}$. Since $\operatorname{tr} S_2 = \alpha + \alpha^{-1} = \pm 2$, we get $\alpha = \pm 1$. Hence, $S_2(z) = z \pm b$, which is an Euclidean translation. Such Möbius transformations are called *parabolic* transformations.

Case-2: T has two fixed points.

Now, we consider the case when S_1 has two fixed points, say p and q. There exists $T \in \text{Möb}^+(\mathbb{H})$ such that $T(p) = 0$ and $T(q) = \infty$. Then $S_2 = TS_1T^{-1}$ fixes 0 and ∞. Suppose that

$$S_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad S_2(0) = 0,\ S_2(\infty) = \infty,\ \text{and } \alpha\delta - \beta\gamma = 1.$$

It follows that $\beta = 0, \gamma = 0$, and $\alpha\delta = 1$. Therefore, $S_2(z) = \alpha^2 z$. Since $\operatorname{tr} S_2 = \alpha + \alpha^{-1} \neq \pm 2$, we have $\alpha \neq \pm 1$. For $\alpha^2 = \lambda$, we get $S_2(z) = \lambda z$ with $\lambda \neq 1$, where λ is known as the *multiplier of* S_2. Based on the multiplier, we consider the following subcases.

Case-2(i): When $\lambda > 0$, S_2 is said to be a *hyperbolic transformation*. If $\lambda > 1$, it is an expansion in $\mathbb{H}$ along hyperbolic lines joining the fixed points as shown in Figure 1.5.1a. Whereas, if $\lambda < 1$, it is a contraction in $\mathbb{H}$ as shown in Figure 1.5.1b.

(a) $\lambda > 1$ (Expansion) (b) $\lambda < 1$ (Contraction)

Figure 1.5.1: Hyperbolic transformations in $\mathbb{H}$.

Case-2(ii): When $|\lambda| = 1$, S_2 is said to be an *elliptic transformation*. These are rotations about the fixed points in $\mathbb{D}$ as in Figure 1.5.2.

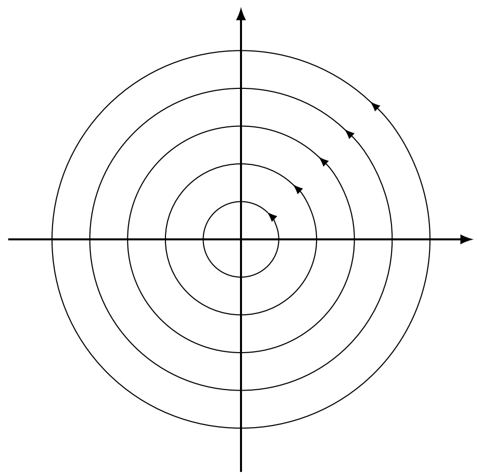


Figure 1.5.2: An elliptic transformation in $\mathbb{D}$.

Case-2(iii): A Möbius transformation is said to be *loxodromic* if it is neither hyperbolic nor elliptic. Points move along spiral arms in this transformation, which combines the hyperbolic and elliptic cases. In fact, in Greek loxodromic means "running obliquely" in the parlance of cartographers.

To determine the type of a Mobius transformation, we conjugate it into a standard form and then decide its type based on the standard form we obtain. The tabular form of the above discussion is given as Table 1.2.

Type	Fixed points	Multiplier	Standard form
Identity	Infinite	-	$m(z) = z$
Parabolic	One	-	$m(z) = az + b$
Hyperbolic	Two	$\lambda > 0$	$m(z) = r^2 z, \ \ r^2 \neq 1 \in \mathbb{R}$
Elliptic	Two	$\lvert\lambda\rvert = 1$	$m(z) = e^{2i\theta} z, \ \ \theta \in (0, \pi)$
Loxodromic	Two	$\lvert\lambda\rvert \neq 1$ and $\lambda \not> 0$	$m(z) = r^2 e^{2i\theta} z, \ \ \theta \in [0, \pi], \ \ r^2 \neq 1 \in \mathbb{R}$

Table 1.2: Classification of Möbius transformations.

1.6 Convexity

Definition 1.6.1. A subset $X \subset \mathbb{H}$ is said to be *convex* if for every pair of distinct points x and y in X, the closed hyperbolic line segment ℓ_{xy} that joins x to y is contained in X.

The complement of any hyperbolic line ℓ in the hyperbolic plane contains two components, which are the two open half-planes determined by ℓ. The union of any of these two open half-planes with ℓ is called a *closed half-plane.*

Proposition 1.6.2. *Hyperbolic lines, hyperbolic rays, hyperbolic line segments, open half-planes, and closed half-planes are all convex.*

Proof. Let ℓ be a hyperbolic line, and x and y be any two points on ℓ. There is a unique hyperbolic line joining x and y, namely, ℓ. Thus $\ell_{xy} \subset \ell$, and hence hyperbolic lines are convex. A similar argument shows that hyperbolic rays, hyperbolic line segments, open half-planes, and closed half-planes are all convex. □

Just like all polygons in Euclidean geometry are convex, we shall define polygons in the hyperbolic plane analogously. To describe specific terms associated with hyperbolic polygons, we now provide several definitions.

Definition 1.6.3. Consider a collection of half planes $\mathcal{P} = \{P_\alpha\}_{\alpha \in A}$ in $\mathbb{H}$ with ℓ_α as the bounding hyperbolic lines of the half-plane P_α. Let $B_r(z)$ be an open hyperbolic disc of hyperbolic radius r and hyperbolic center z. The collection $\mathcal{P}$ is said to be *locally finite* if for each point $z \in \mathbb{H}$, there exists some $\varepsilon > 0$ such that only finitely many bounding lines ℓ_α of the half-planes in $\mathcal{P}$ intersect the $B_\varepsilon(z)$.

Definition 1.6.4. A *hyperbolic polygon* is a convex set $P \subset \mathbb{H}$ such that P is the intersection of a locally finite collection of closed half-planes.

Definition 1.6.5. A hyperbolic polygon is said to be *nondegenerate*, if it has a nonempty interior, otherwise it is called *degenerate*.

Definition 1.6.6. Suppose P is a hyperbolic polygon and ℓ is a hyperbolic line, such that $P \cap \ell \neq \varnothing$ and P is a subset of a closed half-plane determined by ℓ. If the intersection of P and ℓ is a point, we call it a *vertex* of P. Otherwise, if the intersection of P and ℓ is either a closed hyperbolic line segment, a closed hyperbolic ray, or all of ℓ, we call it a *side* of P.

Definition 1.6.7. Suppose P is a hyperbolic polygon with two of its sides s_1 and s_2 intersecting at a vertex A. Assuming ℓ_i to be the hyperbolic line containing s_i for $i = 1, 2$, the union $\ell_1 \cup \ell_2$ partitions the hyperbolic plane into four components, one of which contains P. The angle between ℓ_1 and ℓ_2, measured in the component of the complement of $\ell_1 \cup \ell_2$ containing P, is called the *interior angle* of P at A.

Definition 1.6.8. If there are two adjacent sides of P that are either closed hyperbolic rays or hyperbolic lines such that they have a common endpoint at infinity, say the vertex A, then we call it A as an *ideal vertex* of P.

Definition 1.6.9. If the boundary of P has a part of the boundary at infinity as a component with positive Euclidean length, we call this a *free side* of P.

Definition 1.6.10. The unique hyperbolic line passing through the mid-point w of the line segment $[p_1, p_2]$ is called the *perpendicular bisector* of the line segment $[p_1, p_2]$.

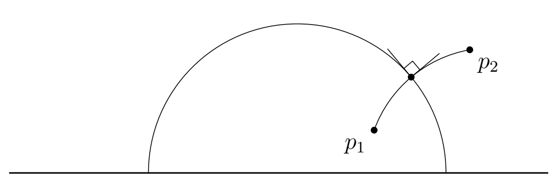


Figure 1.6.1: Perpendicular bisector of $[p_1, p_2]$.

Lemma 1.6.11. *The perpendicular bisector of the line segment* $[p_1, p_2]$ *is a line whose equation is given by*

$$\mathrm{d}_{\mathbb{H}}(z, p_1) = \mathrm{d}_{\mathbb{H}}(z, p_2).$$

Proof. Since Isom($\mathbb{H}$) acts transitively on the set of hyperbolic lines, without loss of generality, we assume that $p_1 = i, p_2 = ir^2$, where $r > 0,\ r \neq 1$. We show that the midpoint of the line segment $[p_1, p_2]$ is $w = ir$ and the equation of perpendicular bisector is given by $|z| = r$. From Theorem 1.3.8, we have

$$\cosh\left[\mathrm{d}_{\mathbb{H}}(z, w)\right] = 1 + \frac{|z - \bar{w}|^2}{2(\Im z \Im w)},$$

and hence $\mathrm{d}_{\mathbb{H}}(z, p_1) = \mathrm{d}_{\mathbb{H}}(z, p_2)$ is equivalent to

$$\frac{|z - p_1|^2}{2\Im z \Im p_1} = \frac{|z - p_2|^2}{2\Im z \Im p_2}.$$

Since $p_1 = i$ and $p_2 = ir^2$, we have

$$|z - i|^2 = \frac{|z - ir^2|^2}{r^2}$$

impyling $(r^2 - 1)z\bar{z} - (r^2 - 1)r^2 = 0$. As $r \neq 1$, we get $|z|^2 = r^2$ or $|z| = r$. □

1.7 Area and the Gauss-Bonnet theorem

We have seen that the hyperbolic length of a piecewise C^1-path is obtained by integrating the hyperbolic element of arc-length $\frac{1}{\Im z}dz$ along the path. Similarly, the hyperbolic area of a region in $\mathbb{H}$ can be obtained by integrating the square of the hyperbolic element of arc-length over the region.

Definition 1.7.1. The *hyperbolic area* $\mathrm{area}_{\mathbb{H}}(P)$ of a set $P \subset \mathbb{H}$ is defined as

$$\mathrm{area}_{\mathbb{H}}(P) = \int_P \frac{1}{\mathrm{Im}(z)^2}\ \mathrm{d}x\ \mathrm{d}y = \int_P \frac{1}{y^2}\ \mathrm{d}x\ \mathrm{d}y,$$

where $z = x + iy$, provided that this integral exists.

Proposition 1.7.2. *The hyperbolic area in* $\mathbb{H}$ *is invariant under the action of* $\mathrm{PSL}(2, \mathbb{R})$.

The following result, known as the Gauss-Bonnet theorem, relates the hyperbolic area of a polygon to its interior angles at vertices.

Theorem 1.7.3. *Suppose that a hyperbolic triangle T have θ_1, θ_2, and θ_3 as the interior angles. Then,*

$$\mathrm{area}_{\mathbb{H}}(T) = \pi - (\theta_1 + \theta_2 + \theta_3).$$

Proof. We begin by considering the case when at least one of the vertices of the triangle T is ideal, and hence the angle at this vertex is zero. Up to conjugation, we consider the triangle as shown in Figure 1.7.1. The side AB is a line segment which is a part of the unit circle centered at origin. Since C is an ideal vertex, the sides AC and BC are part of vertical lines through A and B, respectively.

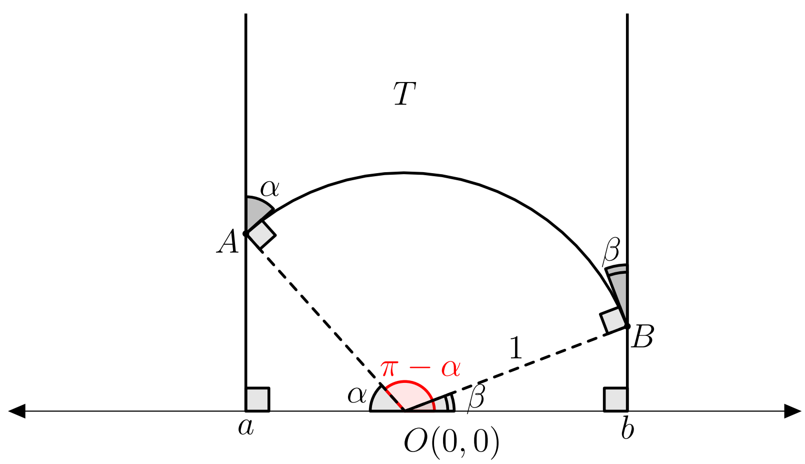


Figure 1.7.1: Triangle T with vertex C at infinity.

We have

$$\begin{aligned}
\mathrm{area}_{\mathbb{H}}(T) &= \iint_T \frac{1}{y^2} dx dy = \int_a^b \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dy \right) dx \\
&= \int_a^b \left(\lim_{R \to \infty} \left[\frac{-1}{y} \right]_{\sqrt{1-x^2}}^{R} \right) dx = \int_a^b \frac{1}{\sqrt{1-x^2}} dx \\
&= \int_{\pi-\alpha}^{\beta} -1 d\theta \text{ (substituting } x = cos\theta) \\
&= \pi - (\alpha + \beta).
\end{aligned}$$

Now consider the case when T has no ideal vertices. Let the vertices of T be A, B, and C, with internal angles α, β, and γ, respectively. Up to conjugation, we consider the triangle as shown in Figure 1.7.2.

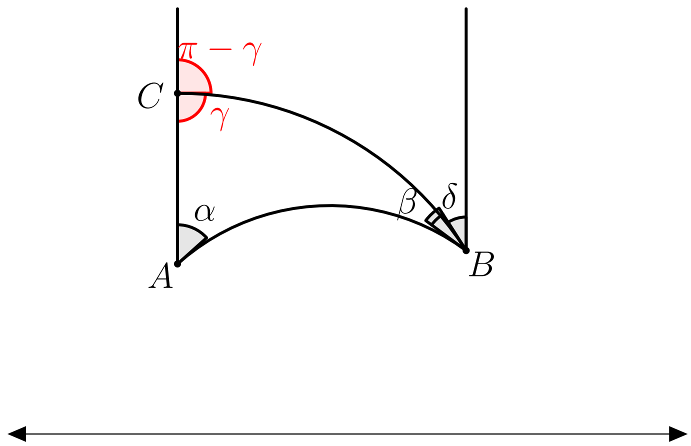


Figure 1.7.2: Triangle T with no ideal vertex.

Let δ be the angle at B between the side CB and the vertical half-line passing through B. Thus, the triangle T can be divided into two triangles each with an ideal vertex: triangle $AB\infty$ and triangle $CB\infty$.

$$\begin{aligned}
\text{area}_{\mathbb{H}}(T) &= \text{area}_{\mathbb{H}}(ABC) \\
&= \text{area}_{\mathbb{H}}(AB\infty) - \text{area}_{\mathbb{H}}(CB\infty) \\
&= [\pi - (\alpha + (\beta + \delta))] - [\pi - ((\pi - \gamma) + \delta)] \\
&= \pi - (\alpha + \beta + \gamma).
\end{aligned}$$

□

Theorem 1.7.4. *(Gauss-Bonnet theorem for hyperbolic polygons) Suppose that a hyperbolic polygon P have θ_1, $\theta_2, \ldots, \theta_n$ as the interior angles at vertices $v_1, v_2, \ldots, v_n$, respectively. Then,*

$$\text{area}_{\mathbb{H}}(P) = (n-2)\pi - (\theta_1 + \theta_2 + \cdots + \theta_n).$$

Proof. Consider Q to be a hyperbolic quadrilateral with vertices A, B, C, D (labelled, say, clockwise) and corresponding internal angles $\alpha, \beta, \gamma, \delta$, respectively. The line segment BD creates two triangles ABD (with internal angles $\alpha, \beta_1, \delta_2$) and BCD (with internal angles $\beta_2, \gamma, \delta_1$), where $\beta_1 + \beta_2 = \beta$ and $\delta_1 + \delta_2 = \delta$.

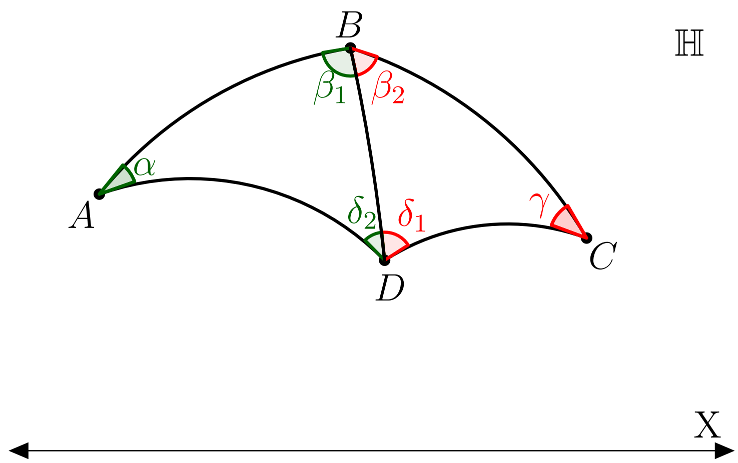


Figure 1.7.3: A quadrilateral Q.

By Theorem 1.7.3, we have

$$\begin{aligned} \text{area}_{\mathbb{H}}(Q) &= \text{area}_{\mathbb{H}}(ABD) + \text{area}_{\mathbb{H}}(BCD) \\ &= [\pi - (\alpha + \beta_1 + \delta_2)] + [\pi - (\beta_2 + \gamma + \delta_1)] \\ &= 2\pi - (\alpha + \beta + \gamma + \delta). \end{aligned}$$

Extending to n-gon, the statement follows. □

Chapter 2

Fuchsian Groups

In this chapter, before defining Fuchsian groups, we first define the discrete and properly discontinuous group actions and prove that these actions are equivalent on the hyperbolic plane. Although we will largely be working on the upper half-plane model, the same can be done on the Poincaré disc model with some modifications. We then show that abelian Fuchsian groups are precisely the cyclic groups. This chapter is based on [6, Chapter 2] and [8, Chapter 4].

2.1 Discrete and properly discontinuous groups

Before proceeding, we first define topological groups.

Definition 2.1.1. If a group $\mathcal{G}$ is a topological space with

i) the multiplication map, i.e., $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(a, b) \mapsto a \cdot b$ and

ii) the inverse map, i.e., ${}^{-1} : \mathcal{G} \to \mathcal{G}$, $a \mapsto a^{-1}$

being continuous, then $\mathcal{G}$ is called a *topological group*.

Example 2.1.1. i) Euclidean n-space $\mathbb{R}^n$ under addition.

ii) $\mathrm{GL}(n, \mathbb{R})$ under matrix multiplication.

Definition 2.1.2. Let $\mathcal{G}$ be a topological group. A subgroup $\mathcal{S} \subset \mathcal{G}$ with no limit points in $\mathcal{G}$ is called *discrete*.

Lemma 2.1.3. *For a subgroup $\mathcal{S} \subset \mathrm{SL}(2,\mathbb{R})$, the following are equivalent:*

(i) *$\mathcal{S}$ does not contain any limit points.*

(ii) *$\mathcal{S}$ is discrete, i.e. no limit points of $\mathcal{S}$ are in* $\mathrm{SL}(2,\mathbb{R})$.

(iii) *The identity* Id *of $\mathcal{S}$ is an isolated point.*

Proof. We will first show that the negation of statement (ii) implies the negation of statement (i). If $\mathcal{S}$ is not discrete, then there exists a sequence $S_n \to S$ for some $S \in \mathrm{SL}(2,\mathbb{R})$. Therefore, $S_n S_{n+1} \to SS^{-1} = \mathrm{Id}$. Since, $\mathrm{Id} \in \mathcal{S}$, there are limit points in $\mathcal{S}$.

Now, we will show that the negation of statement (iii) implies the negation of statement (ii). If the identity Id of $\mathcal{S}$ is not an isolated point, $\mathcal{S}$ is not discrete.

We finally show that statement (iii) implies statement (i). If the identity Id of $\mathcal{S}$ is an isolated point, we choose a neighbourhood $\mathcal{N}$ of Id such that $\mathcal{S} \cap \mathcal{N} = \mathrm{Id}$. For every $S \in \mathcal{S}$, we get $\mathcal{S} \cap S\mathcal{N} = S$, i.e., each element of $\mathcal{S}$ is isolated and hence $\mathcal{S}$ does not contain any limit points. □

Lemma 2.1.4. *Any non-trivial discrete subgroup of $\mathbb{R}$ is infinite cyclic.*

Proof. Let $\mathcal{S}$ be a non-trivial discrete subgroup of $\mathbb{R}$. Since $0 \in \mathcal{S}$, and $\mathcal{S}$ is discrete, there exists a smallest positive real $s \in \mathcal{S}$. Consider the infinite cyclic subgroup $\mathcal{S}_s = \{ns : n \in \mathbb{Z}\}$ of $\mathcal{S}$. Let $t \in \mathcal{S}$ be such that $t \neq ns$ for some $n \in \mathbb{N}$. (If not, we consider $-t \in \mathcal{S}$.) There exists $m \in \mathbb{N}$ such that $ms < t < (m+1)s$, and so we have $t - ms < s$ and $t - ms \in \mathcal{S}$. This contradicts the definition of s. Hence, it follows that $\mathcal{S} = \mathcal{S}_s$. □

Lemma 2.1.5. *Any discrete subgroup of* $\mathrm{S}^1 = \{z \in \mathbb{C} : z = e^{i\theta}, \theta \in \mathbb{R}\}$ *is finite cyclic.*

Proof. Let $\mathcal{S}$ be a discrete subgroup of S^1. By discreteness of $\mathcal{S}$, there exists the smallest argument θ_0 such that $z = e^{i\theta_0} \in \mathcal{S}$ and $z = e^{ik} \notin \mathcal{S}$ for $0 < k < \theta_0$ so that for some $n \in \mathbb{Z}$, $n\theta_0 = 2\pi$. Thus, $\mathcal{S}$ is finite cyclic. □

Definition 2.1.6. Let $\mathcal{S}$ be a subgroup of $\mathrm{SL}(2,\mathbb{R})$. The action of $\mathcal{S}$ is *properly discontinuous* on $\mathbb{H}$ provided that for all compact subsets $C \subset \mathbb{H}$, $sC \cap C = \varnothing$ for all but finitely many $s \in \mathcal{S}$.

Several variations of the definition of acting properly discontinuously exist, and at times these variants are more helpful in proving theorems than the definition we just stated. In the following lemma, a few equivalent statements are listed.

Lemma 2.1.7. *For a subgroup $\mathcal{S}$ of* $\mathrm{SL}(2,\mathbb{R})$*, the following are equivalent.*

(i) The action of $\mathcal{S}$ is not properly discontinuous on $\mathbb{H}$.

(ii) Limit points of some $\mathcal{S}$-orbits in $\mathbb{H}$ are in $\mathbb{H}$.

(iii) Limit points of all $\mathcal{S}$-orbits in $\mathbb{H}$ are in $\mathbb{H}$ except, possibly, one orbit consisting of a single common fixed point of all elements in $\mathcal{S}$.

Proof. First we show that the statement (i) implies the statement (ii). If $\mathcal{S}$ does not act properly discontinuously on $\mathbb{H}$, then by definition, there is a compact set C such that $s_nC \cap C \neq \varnothing$ for infinitely many distinct $s_n \in \mathcal{S}$. Thus, there are points $z_n \in C$ such that $s_nz_n \in C$. Since C is compact, there is a subsequence of z_n, say z_{n_k} such that $z_{n_k} \longrightarrow w \in C$. As C is bounded, $C \subset \overline{B_r(w)}$ for some $r > 0$. Since $z_n \longrightarrow w \in C$, we have $d(z_n, w) < 1$ for large enough n and therefore, $d(s_nz_n, s_nw) < 1$. Since $s_nz_n \in C$ it follows that $s_nw \in \overline{B_{r+1}(w)}$. Hence, we obtain a convergent sequence showing that limit point of $\mathcal{S}$-orbit of w lies in $\mathbb{H}$.

Now, we show that the statement (ii) implies the statement (iii). We assume that the $\mathcal{S}$-orbit of z_0 has a limit point in $\mathbb{H}$. Therefore, there exist distinct points s_nz_0 such that $s_nz_0 \longrightarrow w_0 \in \mathbb{H}$ for $s_n \in \mathcal{S}$. For $z \in \mathbb{H}$, we have

$$\begin{aligned} \mathrm{d}_{\mathbb{H}}(s_nz, z) &\leq \mathrm{d}_{\mathbb{H}}(s_nz, s_nz_0) + \mathrm{d}_{\mathbb{H}}(s_nz_0, w_0) + \mathrm{d}_{\mathbb{H}}(w_0, z) \\ &= \mathrm{d}_{\mathbb{H}}(z, z_0) + \mathrm{d}_{\mathbb{H}}(s_nz_0, w_0) + \mathrm{d}_{\mathbb{H}}((w_0, z). \end{aligned}$$

Since $\mathrm{d}_{\mathbb{H}}(s_nz, z)$ is bounded independently of n, the points s_nz are all contained in a closed ball centered at z. If infinitely many of these points are distinct, the $\mathcal{S}$-orbit has a limit point. However, if s_nz coincides for infinitely many distinct n, then for infinitely many distinct elements $t_n \in \mathcal{S}$ we obtain $t_nz = z$. This implies all t_n fixes z and are thus elliptic transformations. Considering this as an exception, we claim that the $\mathcal{S}$-orbit of any other point $w \neq z$ has a limit point. Since t_n are all distinct elliptic transformations, they all rotate through different angles. Hence, there must be infinitely many distinct points $t_n w$ which all lie on a circle center z. Since this circle is compact, the claim follows.

Finally we show that statement (iii) implies statement (i). Let z be a point in $\mathbb{H}$ whose $\mathcal{S}$-orbit has a limit point, i.e., $s_n z \longrightarrow w$ for some $s_n \in \mathcal{S}$ and $w \in \mathbb{H}$. Let $r > 0$ be such that $z \in \overline{B_r(w)}$. Hence, $s_n z \in \overline{B_r(w)}$ for infinitely many s_n, so $s_n\overline{B_r(w)} \cap \overline{B_r(w)} \neq \varnothing$ for infinitely many s_n. Since $\overline{B_r(w)}$ is compact, the assertion follows. □

2.2 Fuchsian groups

Following the notions of discreteness and properly discontinuousness, we now define Fuchsian groups.

Definition 2.2.1. A discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is called a *Fuchsian group.*

Example 2.2.1. (i) $\mathcal{H} = \{T \in \mathrm{PSL}(2,\mathbb{R}) : T(z) = \lambda^n z, n \in \mathbb{Z}\}$ for a fixed $\lambda > 1$.

(ii) $\mathcal{P} = \{T \in \mathrm{PSL}(2,\mathbb{R}) : T(z) = z + na, n \in \mathbb{Z}\}$ for a fixed $a \in \mathbb{R}$.

(iii) $\mathcal{E} = \{T \in \mathrm{PSL}(2,\mathbb{R}) : T(z) = e^{ik\theta} z, 0 < k < n\}$ for $n \in \mathbb{N}$ and $\theta = 2\pi/n$.

(iv) $\mathrm{SL}(2,\mathbb{Z})$.

The following lemma will be used to prove the equivalence of Fuchsian groups and properly discontinuous groups.

Lemma 2.2.2. *Let $z_0 \in \mathbb{H}$ and $\mathcal{C} \subset \mathbb{H}$ be compact. Then the set*

$$\mathcal{D} = \{S \in \mathrm{PSL}(2,\mathbb{R}) : S(z_0) \in \mathcal{C}\}$$

is compact.

Proof. Since the topology on $\mathrm{PSL}(2,\mathbb{R})$ is induced from $\mathbb{R}^4$, we show that, $\mathcal{D}$ is closed and bounded.

Consider the continuous map

$$\gamma : \mathrm{PSL}(2,\mathbb{R}) \longrightarrow \mathbb{H}, \text{ defined as } S \mapsto S(z_0).$$

Since γ is continuous and $\mathcal{C}$ is closed, we have $\mathcal{D} = \gamma^{-1}(\mathcal{C})$ is closed.

Since $\mathcal{C}$ is compact, there exist $l > 0$ such that

$$|S(z_0)| = \left|\frac{az_0 + b}{cz_0 + d}\right| < l \text{ for every } S \in \mathcal{D}, \text{ where } S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Moreover, since $\mathcal{C}$ is compact, there exist $m > 0$ such that

$$\Im\left(\frac{az_0 + b}{cz_0 + d}\right) \geq m.$$

Since

$$\Im\left(\frac{az + b}{cz + d}\right) = \frac{(ad - bc)\Im z}{(cz + d)^2}$$

and $ad - bc = 1$, we get

$$\frac{\Im z_0}{(cz_0 + d)^2} \geq m \Rightarrow \Im z_0 > m\,|cz_0 + d|^2 \Rightarrow |cz_0 + d| < \sqrt{\frac{\Im z_0}{m}}$$

and

$$\left|\frac{az_0 + b}{cz_0 + d}\right| < l \Rightarrow |az_0 + b| < l\,|cz_0 + d| \Rightarrow |az_0 + b| < l\sqrt{\frac{\Im z_0}{m}}.$$

Thus, $|cz_0 + d|$ and $|az_0 + b|$ are bounded. If $|c| > n$, then $|cz_0 + d| > |n|\,|z_0| - |d| > k$, for some $k > 0$. Therefore, $|cz_0 + d| \to \infty$ as $|n| \to \infty$. Thus, $|c|$ must be bounded. Similarly, $|a|$, $|b|$, and $|d|$ are also bounded. Hence, $\mathcal{D}$ is bounded. □

Theorem 2.2.3. *Let $\mathcal{S} \subset \mathrm{PSL}(2, \mathbb{R})$. Then $\mathcal{S}$ is discrete if and only if it acts properly discontinuously on $\mathbb{H}$.*

Proof. If $\mathcal{S}$ is not discrete, then by Lemma 2.1.3 (iii), there exists a sequence $s_n \in \mathcal{S}$ such that $S_n \longrightarrow \mathrm{Id}$, and therefore, $S_n z \longrightarrow z$ for every $z \in \mathbb{H}$.

We choose $z_0 \in \mathbb{H}$ which is not fixed by $S_m^{-1} S_n$, for $n,\ m \in \mathbb{N}$. Therefore, all of $S_n z_0$ are distinct. Since $S_n z_0 \longrightarrow z_0$, we get that z_0 is a limit point of $\mathcal{S}$ in $\mathbb{H}$. Therefore, from Lemma 2.1.7, $\mathcal{S}$ does not act properly discontinuously on $\mathbb{H}$.

Now, we assume that $\mathcal{S}$ is discrete. We show that for any compact subset $\mathcal{C} \subset \mathbb{H}$, $S(\mathcal{C}) \cap \mathcal{C} = \varnothing$ for all but finitely many $S \in \mathcal{S}$. Without loss of generality, we take

$\mathcal{C} = \overline{B_r(i)}$, that is the closed hyperbolic disk with center i and radius r. For z_1, $z_2 \in \overline{B_r(i)}$, if $z_1 = S(z_2)$ then

$$d(i, S(i)) \leq d(i, S(z_2)) + d(S(z_2), S(i)) = d(i, z_1) + d(z_2, i) \leq 2r.$$

Define $\mathcal{D}_{\mathcal{S}} = \{S \in \mathrm{PSL}(2, \mathbb{R}) : S(i) \in \overline{B_{2r}(i)}\} \cap \mathcal{S}$. By Lemma 2.2.2, the set $\{S : S(i) \in \overline{B_{2r}(i)}\}$ is compact. Since $\mathcal{S}$ is discrete, $\mathcal{D}_{\mathcal{S}}$ is also discrete. Since $\mathcal{D}_{\mathcal{S}}$ is discrete and compact, it must be finite. Thus, $S\left(\overline{B_r(i)}\right) \cap \overline{B_r(i)} \neq \varnothing$ for all but finitely many $S \in \mathcal{S}$. Hence $\mathcal{S}$ acts properly discontinuously on $\mathbb{H}$. □

The following corollary is an immediate consequence of Lemma 2.1.7 and Theorem 2.2.3.

Corollary 2.2.4. *(a) If $\mathcal{S}$ is not a Fuchsian group, then every $\mathcal{S}$-orbit has limit points with the possible exception of one orbit.*

(b) If $\mathcal{S}$ is a Fuchsian group, then every $\mathcal{S}$-orbit is without limit points in $\mathbb{H}$.

(c) If Fuchsian group $\mathcal{S}$ is infinite, then the limit point of all orbits of $z_0 \in \mathbb{H}$ lies on $\partial\mathbb{H}$.

2.3 Abelian Fuchsian groups

In this section, we show that every abelian Fuchsian group is cyclic. Before proving this result, we first prove the following lemma.

Lemma 2.3.1. *Given $S, T \in \mathrm{PSL}(2, \mathbb{R})$, if S and T commutes then T maps the fixed-point set of S to itself injectively.*

Proof. Let p be a fixed point of S. Then

$$TS(p) = T(p) \text{ which implies } ST(p) = T(p).$$

Thus, $T(p)$ is also a fixed point of S. □

Theorem 2.3.2. *Two non-identity elements in* $\mathrm{PSL}(2,\mathbb{R})$ *have the same fixed-point set if and only if they commute.*

Proof. Consider two non-identity element $S, T \in \mathrm{PSL}(2,\mathbb{R})$. If S and T have the same fixed-point set then they are of the same type (either hyperbolic, parabolic, or elliptic). Thus, S and T are conjugate to one of the following

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

It can be shown two elements of the same form commute. Thus, S and T must commute as well.

To prove the converse, we first consider the case when S is either parabolic or elliptic. In either case, let the unique fixed point of S be p. Since S and T commute, from Lemma 2.3.1, $T(p)$ is a fixed point of S. As S has a unique fixed point, we have $T(p) = p$. Hence, p is a fixed point of T. If possible, let q be another fixed point of T such that $q \neq p$. Then, by Lemma 2.3.1, $S(q)$ is also a fixed point of T. Since S has a unique fixed point, $S(q) \neq q$. Moreover, $S(q) \neq p$ otherwise $\mathrm{d}_{\mathbb{H}}(p,q) = \mathrm{d}_{\mathbb{H}}(S(q), S(p)) = \mathrm{d}_{\mathbb{H}}(p,p) = 0$. Thus, T has three fixed points, namely p, q, and $S(q)$ which is impossible as T is non-identity.

Now, we consider the case when S is hyperbolic. By Lemma 2.3.1, it follows that T is also hyperbolic. We conjugate T by P such that the fixed points of PTP^{-1} are 0 and ∞. Thus, for some $\alpha, \beta, \gamma, \delta, \lambda \in \mathbb{R}$ such that $\lambda \neq 0, 1$, we have

$$PTP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } PSP^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Since S and T commutes, we have

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

which implies that
$$\begin{pmatrix} \lambda\alpha & \lambda\beta \\ \lambda^{-1}\gamma & \lambda^{-1}\delta \end{pmatrix} = \begin{pmatrix} \lambda\alpha & \lambda^{-1}\beta \\ \lambda\gamma & \lambda^{-1}\delta \end{pmatrix}.$$
Thus, we obtain $\lambda\beta = \lambda^{-1}\beta$ and $\lambda^{-1}\gamma = \lambda\gamma$. Since $\lambda \neq 0, 1$, we get $\beta = \gamma = 0$. Therefore PSP^{-1} fixes 0 and ∞. Hence, the fixed-point set of S and T are the same. □

Lemma 2.3.3. *Let $\mathcal{S}$ be a non-trivial Fuchsian group and $z_0 \in \mathbb{H}$ be a fixed point of some element of $\mathcal{S}$. Then there exists a neighbourhood $\mathcal{N}$ of z_0 such that no element of $\mathcal{S} - \{\mathrm{Id}\}$ fixes a point in $\mathcal{N} - \{z_0\}$. Equivalently, fixed points of elements of a Fuchsian group are isolated.*

Theorem 2.3.4. *A Fuchsian group is cyclic if all of its non-identity elements have the same fixed-point set.*

Proof. The fixed-point set of an element determines its type. Thus, the given Fuchsian group, say $\mathcal{S}$, contains elements of the same type.

First, consider the case when $\mathcal{S}$ contains only hyperbolic elements. Up to conjugation, we assume that for all $S \in \mathcal{S}$, S fixes 0 and ∞. Thus, $\mathcal{S}$ is a subgroup of $\mathcal{H} = \{T \in \mathrm{PSL}(2, \mathbb{R}) : T(z) = \lambda^n z, \lambda > 0\}$. The maps $f : \mathcal{H} \to \mathbb{R}$ defined as $f(T) = \ln \lambda$ is an isomorphism of topological groups. By Lemma 2.1.4, $\mathcal{S}$ is infinite cyclic.

Now, we consider the case when $\mathcal{S}$ contains only parabolic elements. Up to conjugation, we may assume that for all $S \in \mathcal{S}$, S fixes ∞. Thus, $\mathcal{S}$ is a subgroup of $\mathcal{P} = \{T \in \mathrm{PSL}(2, \mathbb{R}) : T(z) = z + na, a \in \mathbb{R}\}$. The map $f : \mathcal{P} \to \mathbb{R}$ defined as $f(T) = a$ is an isomorphism of topological groups. Again, by Lemma 2.1.4, $\mathcal{S}$ is infinite cyclic.

Finally, we consider the case when $\mathcal{S}$ contains only elliptic elements. Consider $\mathcal{S}$ to be a discrete subgroup of orientation-preserving isometries of $\mathbb{D}$. Up to conjugation, we assume that for all $S \in \mathcal{S}$, S fixes 0. Thus, $\mathcal{S}$ is a subgroup of $\mathcal{E} = \{T \in \mathrm{PSU}(1, 1) : T(z) = e^{ik\theta} z, \theta = 2\pi/n, n \in \mathbb{Z}, 0 \leq k < n\}$. Thus, there is a subgroup of S^1 which is isomorphic to $\mathcal{S}$ and if this subgroup of S^1 is discrete then $\mathcal{S}$ must be discrete as well.

By Lemma 2.1.5, $\mathcal{S}$ is cyclic. $\square$

Theorem 2.3.5. *Every abelian Fuchsian group is cyclic.*

Proof. Since the group is abelian, by Theorem 2.3.2 all non-identity elements have the same fixed-point set, and hence, are of the same type. By Theorem 2.3.4, this group is cyclic. $\square$

Chapter 3

Fundamental Region

In this chapter, we will introduce the notion of a fundamental region and understand the fundamental region of a Fuchsian group. In the sections that follow, we will define and look at a few examples of the Dirichlet region for a Fuchsian group. This chapter is based on [6, Chapter 3] and [8, Chapter 5].

3.1 Fundamental region

Consider a metric space X, and a group $\mathcal{S}$ acting properly discontinuously on X.

Definition 3.1.1. A *fundamental region* for $\mathcal{S}$ is defined to be a closed region $\mathfrak{F} \subset X$ such that

(i) $\displaystyle\bigcup_{S \in \mathcal{S}} S(\mathfrak{F}) = X$ and

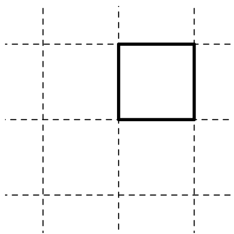
(ii) For all $S \in \mathcal{S} - \{\mathrm{Id}\}$, $\mathring{\mathfrak{F}} \cap S(\mathring{\mathfrak{F}}) = \varnothing$, where $\mathring{\mathfrak{F}}$ is the interior of $\mathfrak{F}$.

The *boundary of* $\mathfrak{F}$, denoted by $\partial\mathfrak{F}$, is defined as $\partial\mathfrak{F} = \mathfrak{F} - \mathring{\mathfrak{F}}$. A *tessellation* or tiling is the family $\{S(\mathfrak{F}) : S \in \mathcal{S}\}$. We observe that statement (i) of the definition above says that the translations $S(\mathfrak{F})$ cover X, while statement (ii) says that there are no non-trivial overlaps between these translations.

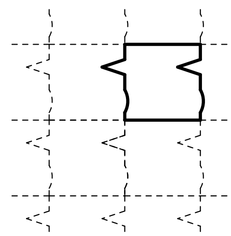
To better understand the fundamental regions, we provide the following examples.

Example 3.1.1. Consider the Euclidean plane $\mathbb{R}^2$ and the group $\mathcal{S}$ generated by translations $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x, y+1)$. Then the fundamental region for $\mathcal{S}$ is the unit square shown in Figure 3.1.1a.

As one can easily anticipate from the figure, a fundamental region of any Fuchsian group is not uniquely determined. Any arbitrary perturbation in a fundamental region leads to another fundamental region of the same group as shown in Figure 3.1.1b.



(a) Regular square tiling.



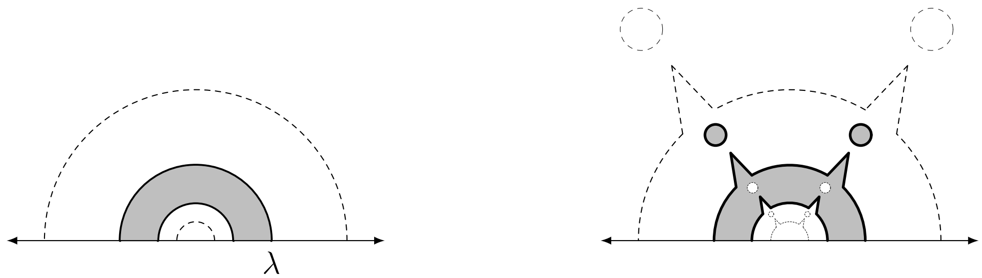
(b) Perturbed tiling.

Figure 3.1.1: Examples of tilings of the Euclidean plane.

This leads us to conclude that a group can not have a unique fundamental region. One approach to constructing different tilings involves removing a subset from the fundamental region and analyzing its orbit under the elements of the group. The images of the removed set under the group can then be used to generate a different tiling. Mathematically, if $\mathfrak{F}$ is a fundamental region for $\mathcal{S}$ and $\mathfrak{G}$ is some non-empty subset of $\mathfrak{F}$, then $(\mathfrak{F} \backslash \mathfrak{G}) \cup S(\mathfrak{G})$ is also a fundamental region of $\mathcal{S}$ for some non-trivial element S of $\mathcal{S}$. (see Figure 3.1.2b.)

Now, we provide an example of a fundamental region for a Fuchsian group.

Example 3.1.2. Consider the cyclic group $\mathcal{S}$ generated by a hyperbolic element $z \mapsto \lambda z$ for some fixed $\lambda > 1$. Then the fundamental region for $\mathcal{S}$ is the semi-annulus shown in Figure 3.1.2a.



(a) Tessellation by semi-annuli. (b) Tessellation by perturbed semi-annuli.

Figure 3.1.2: Examples of tilings of the upper half-plane.

As discussed above, a fundamental region of a group is not unique, however, the hyperbolic area of a fundamental region of a Fuchsian group is an invariant (if finite).

Theorem 3.1.2. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two fundamental regions for a Fuchsian group $\mathcal{S}$. Assume that the hyperbolic area of boundaries of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are zero. If the area of $\mathfrak{F}_1$ is finite then* $\mathrm{area}_{\mathbb{H}}(\mathfrak{F}_2) = \mathrm{area}_{\mathbb{H}}(\mathfrak{F}_1)$, *where* $\mathrm{area}_{\mathbb{H}}(\mathfrak{F}_i)$ *denotes the hyperbolic area of* $\mathfrak{F}_i$.

Proof. From the assumption that the area of boundaries is zero, we obtain

$$\mathrm{area}_{\mathbb{H}}(\mathring{\mathfrak{F}}_i) = \mathrm{area}_{\mathbb{H}}(\mathfrak{F}_i), i = 1, 2.$$

Since $\mathfrak{F}_1$ is a fundamental region, we get

$$\bigcup_{S\in\mathcal{S}} S(\mathfrak{F}_1) = \mathbb{H} \text{ which implies } \bigcup_{S\in\mathcal{S}} \left(S(\mathfrak{F}_1) \cap \mathring{\mathfrak{F}}_2\right) = \mathbb{H} \cap \mathring{\mathfrak{F}}_2 = \mathring{\mathfrak{F}}_2. \tag{3.1.1}$$

Further,

$$\mathfrak{F}_1 \supseteq \mathfrak{F}_1 \cap \left(\bigcup_{S\in\mathcal{S}} S(\mathring{\mathfrak{F}}_2)\right) = \bigcup_{S\in\mathcal{S}} \left(\mathfrak{F}_1 \cap S(\mathring{\mathfrak{F}}_2)\right).$$

Since elements of the set $\{\mathfrak{F}_1 \cap S(\mathring{\mathfrak{F}}_2) : \ S \in \mathcal{S}\}$ are pairwise disjoint, we get

$$\mathrm{area}_{\mathbb{H}}(\mathfrak{F}_1) \geq \sum_{S\in\mathcal{S}} \mathrm{area}_{\mathbb{H}} \left(\mathfrak{F}_1 \cap S(\mathring{\mathfrak{F}}_2)\right) = \sum_{S\in\mathcal{S}} \mathrm{area}_{\mathbb{H}} \left(S^{-1}(\mathfrak{F}_1) \cap \mathring{\mathfrak{F}}_2\right) = \sum_{S\in\mathcal{S}} \mathrm{area}_{\mathbb{H}} \left(S(\mathfrak{F}_1) \cap \mathring{\mathfrak{F}}_2\right).$$

Using (3.1.1), we obtain

$$\operatorname{area}_{\mathbb{H}}(\mathfrak{F}_1) \geq \sum_{S \in \mathcal{S}} \operatorname{area}_{\mathbb{H}}(S(\mathfrak{F}_1) \cap \mathring{\mathfrak{F}}_2) \geq \operatorname{area}_{\mathbb{H}}\left(\bigcup_{S \in \mathcal{S}} S(\mathfrak{F}_1) \cap \mathring{\mathfrak{F}}_2\right) = \operatorname{area}_{\mathbb{H}}(\mathring{\mathfrak{F}}_2).$$

Therefore, we get

$$\operatorname{area}_{\mathbb{H}}(\mathring{\mathfrak{F}}_2) = \operatorname{area}_{\mathbb{H}}(\mathfrak{F}_2).$$

Similarly, since $\mathfrak{F}_2$ is also a fundamental region, we obtain $\operatorname{area}_{\mathbb{H}}(\mathfrak{F}_2) \geq \operatorname{area}_{\mathbb{H}}(\mathfrak{F}_1)$. Hence, the statement holds. □

It is evident that a compact fundamental region has a finite area as seen in Example 3.1.1. A fundamental region of a Fuchsian group may have an infinite area as we saw Example 3.1.2. Furthermore, a non-compact fundamental region may have a finite area as in Figure 3.3.4.

One can ask the question: if a fundamental region of a Fuchsian group is known, can we determine a fundamental region for a finite index subgroup? The following result answers this question in the affirmative.

Theorem 3.1.3. *Let $\mathcal{S}$ be a Fuchsian group with fundamental region $\mathfrak{F}$, and $\mathcal{G}$ be a subgroup of $\mathcal{S}$ of index n. If a decomposition of $\mathcal{S}$ into $\mathcal{G}-$cosets is given as*

$$\mathcal{S} = \mathcal{G}S_1 \cup \mathcal{G}S_2 \cup \cdots \cup \mathcal{G}S_n$$

then

(i) $\mathfrak{F}_{\mathcal{G}} = S_1(\mathfrak{F}) \cup S_2(\mathfrak{F}) \cup \cdots \cup S_n(\mathfrak{F})$ *is a fundamental region for $\mathcal{G}$, and*

(ii) $\operatorname{area}_{\mathbb{H}}(\mathfrak{F}_{\mathcal{G}}) = n \operatorname{area}_{\mathbb{H}}(\mathfrak{F})$ *provided that* $\operatorname{area}_{\mathbb{H}}(\mathfrak{F}) < \infty$ *and* $\operatorname{area}_{\mathbb{H}}(\partial\mathfrak{F}) = 0$.

Proof. To prove that $\mathfrak{F}_{\mathcal{G}}$ is a fundamental region, it has to be closed and it must satisfy the two properties stated in Definition 3.1.1. Since $\mathfrak{F}$ is closed and $S_i \in \mathcal{S}$ are isometries of $\mathbb{H}$, $S_i(\mathfrak{F})$ are closed for all $i = 1, 2, \ldots, n$. As $\mathfrak{F}_{\mathcal{G}}$ is a finite union of closed regions, it is closed.

Now, we show that images of $\mathfrak{F}_{\mathcal{G}}$ under $\mathcal{G}$ cover $\mathbb{H}$. As $\mathfrak{F}$ is a fundamental region for $\mathcal{S}$, for $z \in \mathbb{H}$ there exists $w \in \mathfrak{F}$ and $S \in \mathcal{S}$ such that $z = S(w)$. From the coset decomposition of $\mathcal{S}$, it follows that $S = TS_i$, for some $T \in \mathcal{G}$ and some $i, 1 \leq i \leq n$. Therefore,

$$z = S(w) = TS_i(w) = T(S_i(w)).$$

Since $S_i(w) \in \mathfrak{F}_{\mathcal{G}}$ and $T \in \mathcal{G}$, it follows that, the union of the $\mathcal{G}-$images of $\mathfrak{F}_{\mathcal{G}}$ is $\mathbb{H}$.

For $z \in \mathring{\mathfrak{F}}_{\mathcal{G}}$ and $T \in \mathcal{G}$, we assume that $T(z) \in \mathring{\mathfrak{F}}_{\mathcal{G}}$. To prove that there are no overlaps, we need to show that $T = \mathrm{Id}$. This implies that, $\mathring{\mathfrak{F}}_{\mathcal{G}}$ contains precisely one point of each $\mathcal{G}$-orbit. Consider an open hyperbolic disc $B_\epsilon(z)(\subsetneq \mathring{\mathfrak{F}}_{\mathcal{G}})$ of radius $\epsilon > 0$ and centered at z. Then there exist k such that for exactly k images of $\mathring{\mathfrak{F}}$ under $S_1, S_2, \cdots, S_n$, say $S_{i_1}(\mathring{\mathfrak{F}}), \cdots, S_{i_k}(\mathring{\mathfrak{F}})$, $B_\epsilon(z) \cap S_{i_l}(\mathfrak{F}) \neq \varnothing$, where $1 \leq l \leq k$ and $1 \leq k \leq n$. Since T is an isometry, $B_\epsilon\left(T(z)\right) = T\left(B_\epsilon(z)\right)$. Assume that $B_\epsilon\left(T(z)\right) \cap S_j(\mathring{\mathfrak{F}}) \neq \varnothing$ for some j such that $1 \leq j \leq n$, which implies that

$$B_\epsilon(z) \cap T^{-1}S_j(\mathring{\mathfrak{F}}) \neq \varnothing, \text{ and therefore, } T^{-1}S_j = S_{i_l}, \text{ where } 1 \leq l \leq k.$$

Since $T \in \mathcal{G}$, we have $\mathcal{G}S_j = \mathcal{G}T^{-1}S_j = \mathcal{G}S_{i_l}$, and therefore, $S_j = S_{i_l}$. This shows that $T = \mathrm{Id}$. Hence, $\mathfrak{F}_{\mathcal{G}}$ is a fundamental region for $\mathcal{G}$.

We observe that $\mathrm{area}_{\mathbb{H}}\left(S(\mathfrak{F})\right) = \mathrm{area}_{\mathbb{H}}(\mathfrak{F})$ for all $S \in \mathrm{PSL}(2, \mathbb{R})$. Further, for $i \neq j$ $\mathrm{area}_{\mathbb{H}}\left(S_i(\mathfrak{F}) \cap S_j(\mathfrak{F})\right) = 0$ as the intersection is either empty or a subset of boundaries of tiles. Thus, from the coset decomposition of $\mathcal{S}$, we conclude that, $\mathrm{area}_{\mathbb{H}}(\mathfrak{F}_{\mathcal{G}}) = n\,\mathrm{area}_{\mathbb{H}}(\mathfrak{F})$. This proves the theorem. □

3.2 Dirichlet region

In this section, we discuss a general method of constructing a fundamental region of a Fuchsian group known as a Dirichlet region. The construction uses the idea of perpendicular bisectors of hyperbolic lines, discussed in Definition 1.6.10. Let $\mathcal{S}$ be an arbitrary non-trivial Fuchsian group and let $z_0 \in \mathbb{H}$ be such that no element of $\mathcal{S} - \{\mathrm{Id}\}$ fixes z_0. The existence of such a z_0 is due to Lemma 2.2.

Definition 3.2.1. The *Dirichlet region* for $\mathcal{S}$ centered at z_0 is defined as

$$\mathfrak{D}_{z_0}(\mathcal{S}) = \{z \in \mathbb{H} : \mathrm{d}_{\mathbb{H}}(z, z_0) \leq \mathrm{d}_{\mathbb{H}}(z, S(z_0)) \text{ for all } S \in \mathcal{S}\}.$$

Equivalently,

$$\mathfrak{D}_{z_0}(\mathcal{S}) = \{z \in \mathbb{H} : \mathrm{d}_{\mathbb{H}}(z, z_0) \leq \mathrm{d}_{\mathbb{H}}(S(z), z_0) \text{ for all } S \in \mathcal{S}\}.$$

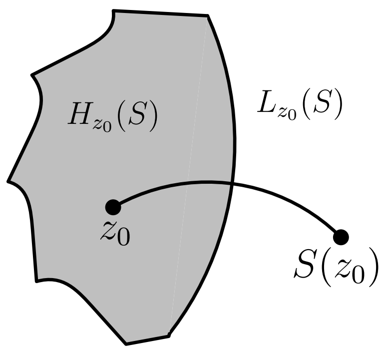


Figure 3.2.1: Dirichlet Region.

Thus, the Dirichlet region is a collection of all points in $\mathbb{H}$ whose images under $\mathcal{S}$ are farther from z_0, than themselves under the hyperbolic metric $\mathrm{d}_{\mathbb{H}}$. By our construction, the Dirichlet region $\mathfrak{D}_{z_0}(\mathcal{S})$ is the intersection of closed half-planes. Now, we show that the Dirichlet region is indeed a fundamental region.

Theorem 3.2.2. *The Dirichlet region $\mathfrak{D}_{z_0}(\mathcal{S})$ is a connected fundamental region for $\mathcal{S}$ provided that z_0 is not fixed by any element of $\mathcal{S} - \{\mathrm{Id}\}$.*

Proof. For $p \in \mathbb{H}$, by Corollary 2.2.4, the $\mathcal{S}-$orbit $\mathcal{S}p$ is discrete. Thus, there exists $p_0 \in \mathcal{S}p$ such that $\mathrm{d}_{\mathbb{H}}(p_0, z_0)$ is minimum, i.e., $\mathrm{d}_{\mathbb{H}}(p_0, z_0) \leq \mathrm{d}_{\mathbb{H}}(S(p_0), z_0)$ for all $S \in \mathcal{S}$. Therefore, by Definition 3.2.1, $p_0 \in \mathfrak{D}_{z_0}(\mathcal{S})$. Hence, at least one point from every $\mathcal{S}$-orbit lies in $\mathfrak{D}_{z_0}(\mathcal{S})$, that is, images of $\mathfrak{D}_{z_0}(\mathcal{S})$ under $\mathcal{S}$ cover $\mathbb{H}$.

Now, to show that there are no overlaps between any two $\mathcal{S}$-images of $\mathfrak{D}_{z_0}(\mathcal{S})$, we show that any two points, say p_1 and p_2, in the interior of $\mathfrak{D}_{z_0}(\mathcal{S})$ cannot lie in the same $\mathcal{S}$-orbit. If possible, assume that $S(p_1) = p_2$. Since $p_1 \in \mathfrak{D}_{z_0}(\mathcal{S})$, we get $d_{\mathbb{H}}(p_1, z_0) < d_{\mathbb{H}}(S(p_1), z_0) = d_{\mathbb{H}}(p_2, z_0)$. Further, since $p_2 \in \mathfrak{D}_{z_0}(\mathcal{S})$ and S is an isometry, we have $d_{\mathbb{H}}(p_2, z_0) < d_{\mathbb{H}}(S^{-1}(p_2), z_0) = d_{\mathbb{H}}(p_1, z_0)$. It follows that for $i = 1, 2$, $d_{\mathbb{H}}(p_i, z_0) < d_{\mathbb{H}}(p_i, z_0)$ which is impossible. Hence at most one point from every $\mathcal{S}$-orbit is contained in the interior of $\mathfrak{D}_{z_0}(\mathcal{S})$.

Since $\mathfrak{D}_{z_0}(\mathcal{S})$ is constructed as the intersection of closed half-planes, it follows that $\mathfrak{D}_{z_0}(\mathcal{S})$ is convex as well as closed. Now, the connectedness of $\mathfrak{D}_{z_0}(\mathcal{S})$ follows from its path-connectedness. □

We state the following properties of Dirichlet regions without proof.

Theorem 3.2.3. *Let $\mathfrak{D} = \mathfrak{D}_{z_0}(\mathcal{S})$ be a Dirichlet region for a Fuchsian group $\mathcal{S}$ centered at z_0.*

(i) The Dirichlet region $\mathfrak{D}$ is convex such that $\partial\mathfrak{D} \cap \mathbb{H}$ is a countable union of positive-length line segments, only a finite number of which intersects any compact set.

(ii) For any compact subset $\mathcal{C} \subset \mathbb{H}$, the set $\{S \in \mathcal{S} : S\mathring{\mathfrak{D}} \cap \mathcal{C} \neq \varnothing\}$ is finite. Equivalently, $\displaystyle\bigcup_{S \in \mathcal{S}} S(\mathring{\mathfrak{D}})$ is locally finite.

3.3 Examples of Dirichlet region

In this section, we provide Dirichlet regions for the Fuchsian groups defined in Examples 2.2.1.

(i) For a fixed $\lambda > 1$, consider the Fuchsian group $\mathcal{H} = \{T \in \mathrm{PSL}(2, \mathbb{R}) : T(z) = \lambda^n z,\ n \in \mathbb{Z}\}$ acting on $\mathbb{H}$. A Dirichlet region for $\mathcal{H}$ is given in Figure 3.3.1.

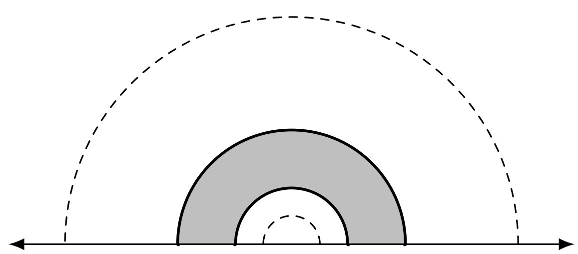


Figure 3.3.1: A Dirichlet region for $\mathcal{H}$.

(ii) For a fixed $a \in \mathbb{R}$, consider the cyclic Fuchsian group $\mathcal{P} = \{T \in \mathrm{PSL}(2,\mathbb{R}) : T(z) = z + na,\ n \in \mathbb{Z}\}$ acting on $\mathbb{H}$. A Dirichlet region for $\mathcal{P}$ centered at z_0 is given by the vertical strip:

$$\mathfrak{D} = \{z : \Re z_0 - a/2 < \Re z < \Re z_0 + a/2\},$$

as shown in Figure 3.3.2.

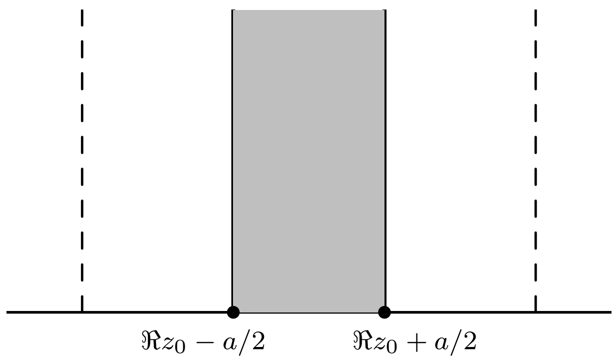


Figure 3.3.2: A Dirichlet region for $\mathcal{P}$.

(iii) For $n \in \mathbb{N}$ and $\theta = 2\pi/n$, consider the finite cyclic group $\mathcal{E} = \{T \in \mathrm{PSL}(2,\mathbb{R}) : T(z) = e^{ik\theta}z, 0 < k < n\}$ acting on $\mathbb{D}$. The Dirichlet region for $\mathcal{E}$ centered at $z_0 = r_0 e^{i\theta_0}$ is given by the circular sector

$$\mathfrak{D} = \{z = re^{i\theta} : \theta_0 - \pi/n < \theta < \theta_0 + \pi/n\},$$

as shown in Figure 3.3.3.

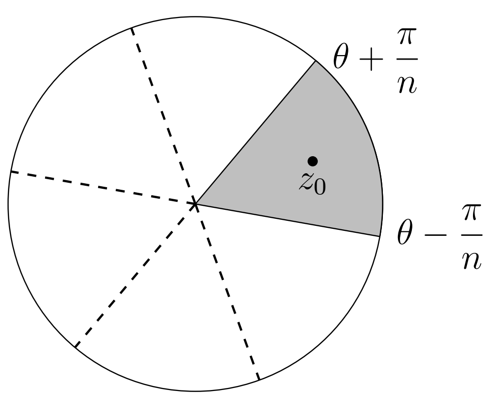


Figure 3.3.3: A Dirichlet region for $\mathcal{E}$.

(iv) A Dirichlet region for the modular group $\mathrm{PSL}(2,\mathbb{Z})$ centered at $2i$ is given by

$$\mathfrak{D} = \{z \in \mathbb{H} : |\Re z| < \frac{1}{2}, |z| > 1\},$$

as shown in Figure 3.3.4.

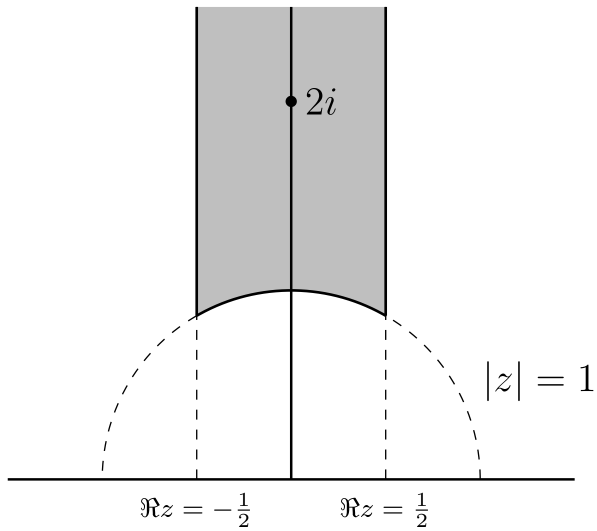


Figure 3.3.4: A Dirichlet region for $\mathrm{PSL}(2,\mathbb{Z})$.

Chapter 4

Side-pairing Transformations and Elliptic Cycles

In this chapter, we will learn to construct Fuchsian groups via side pairing transformations acting on hyperbolic polygons. This chapter is based on [6, Chapter 4], [8, Chapter 6] and [10, Chapter 21-22].

4.1 Side pairing

For a Fuchsian group $\mathcal{T}$, let $\mathfrak{D}$ be a finite-sided convex polygonal fundamental region for $\mathcal{T}$. In this chapter, we always work with such fundamental regions. In the tessellation of $\mathbb{H}$ by $\mathfrak{D}$, we observe that some copies of $\mathfrak{D}$ are adjacent to it either sharing a line segment of positive length with $\mathfrak{D}$ or meeting $\mathfrak{D}$ at a vertex. Assuming the notations used here, we define the following nomenclature.

Definition 4.1.1. A line segment $s \subset \partial\mathfrak{D}$ of positive length is said to be a *side* of $\mathfrak{D}$ if there exist some $T_s \in \mathcal{T}$ such that $T_s(s) = \mathfrak{D} \cap T_s\mathfrak{D}$, and the T_s is called a *side pairing* transformation for $\mathfrak{D}$ associated to side s.

It is evident from the definition of side pairings that a side may be paired with itself.

Remark 2. It is often beneficial to include information in a diagram as it allows the information to be easily referenced and better understood. As shown in Figure 4.1.1, we may indicate which sides of the fundamental region are paired or the action of side pairing transformations. In this figure, the sides with the same number of arrows (or lines) are paired, and the pairing preserves these directions, which indicates the orientation of the sides.

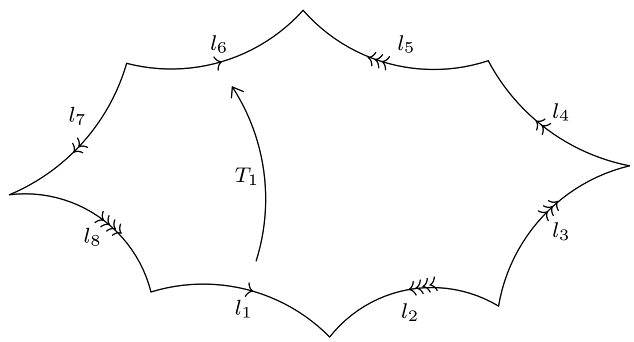


Figure 4.1.1: The side l_1 is mapped to the side l_6 by T_1. The sides with an equal number of arrowheads are paired.

Lemma 4.1.2. *Let T_s be a side-pairing transformation associated with the side s of $\mathfrak{D}$. Then,*

(i) $s = \mathfrak{D} \cap T_s^{-1}\mathfrak{D}$ and $T_s(s) = \mathfrak{D} \cap T_s\mathfrak{D}$;

(ii) If T_s is a side pairing, so is T_s^{-1}. Moreover, T_s^{-1} is the side-pairing associated with the side $T_s(s)$.

Suppose that a side-pairing transformation exists for all sides of $\mathfrak{D}$. Then we define

$$\mathcal{T}^* := \{T \in \mathcal{T} : T \text{ is a side pairing for } \mathfrak{D}\}.$$

Thus, if $\mathfrak{D}_1$ is adjacent to $\mathfrak{D}$, then they share a side and there exists $T \in \mathcal{T}^*$ such that $\mathfrak{D} = T(\mathfrak{D}_1)$. By Lemma 4.1.2, we observe that $T \in \mathcal{T}^*$, then $T^{-1} \in \mathcal{T}^*$, and therefore, $\mathcal{T}^*$ is symmetric.

Theorem 4.1.3. *Let $\mathcal{T}^*$ be the set of all side pairing transformations of a fundamental region $\mathfrak{D}$ for a Fuchsian group $\mathcal{T}$. Then $\mathcal{T}$ is generated by $\mathcal{T}^*$.*

Proof. Let $\mathcal{S}$ be the subgroup generated by $\mathcal{T}^*$. We show that $\mathcal{S} = \mathcal{T}$. For $S_1 \in \mathcal{S}$ and $S_2 \in \mathcal{T}$, if $S_2(\mathfrak{D})$ share an edge with $S_1(\mathfrak{D})$, then $S_1^{-1}S_2(\mathfrak{D})$ shares an edge with

$\mathfrak{D}$. Therefore, $S_1^{-1}S_2 \in \mathcal{T}$, and thus, $S_2 \in \mathcal{S}$. For $S_3 \in \mathcal{T}$, if $S_3(\mathfrak{D})$ meets $S_1(\mathfrak{D})$ at a vertex A, then $S_1^{-1}S_3(\mathfrak{D})$ meets $\mathfrak{D}$ at the vertex $B = S_1^{-1}(A)$. Since $\mathfrak{D}$ is locally finite (by Theorem 3.2.3), only finitely many faces meet at B. Now, we use a finite chain of adjacent faces to 'connect' $\mathfrak{D}$ to $S_1^{-1}S_3(\mathfrak{D})$. By repeatedly applying the above argument, (when the faces share an edge,) we get $S_3 \in \mathcal{S}$. Let $\mathfrak{A} = \bigcup_{S\in\mathcal{S}} S(\mathfrak{D})$ and $\mathfrak{B} = \bigcup_{S\in\mathcal{T}\setminus\mathcal{S}} S(\mathfrak{D})$. Clearly, $\mathfrak{A} \bigcap \mathfrak{B} = \varnothing$ and $\mathfrak{A} \cup \mathfrak{B} = \mathbb{H}$. Since $\mathbb{H}$ is connected, if we show that $\mathfrak{A}$ and $\mathfrak{B}$ are closed, then it follows that $\mathcal{S} = \mathcal{T}$.

We show that any union $\bigcup S_j(\mathfrak{D})$ of faces of the tessellation is closed. Let $z_n \in \bigcup S_j(\mathfrak{D})$ be a sequence such that $z_n \to z$, where $z \in \mathbb{H}$. There exists $T \in \mathcal{T}$ such that $z \in T(\mathfrak{D})$. By Theorem 3.2.3, there exists a neighborhood of z such that it intersects only finitely many $S_j(\mathfrak{D})$. Thus, among these finitely many faces, there exists a face $S_0(\mathfrak{D})$ containing a subsequence of z_n converging to z. Since $S_0(\mathfrak{D})$ is closed, $z \in S_0(\mathfrak{D}) \subseteq \bigcup S_j(\mathfrak{D})$. Hence, $\bigcup S_j(\mathfrak{D})$ is closed, and therefore, $\mathfrak{A}$ and $\mathfrak{B}$ are also closed. $\square$

4.2 Elliptic cycles

Let $\mathcal{T}$ be a Fuchsian group and $\mathfrak{D} = \mathfrak{D}_{z_0}(\mathcal{T})$ be a Dirichlet polygon for $\mathcal{T}$. Even though there are cases in which not all vertices of $\mathfrak{D}$ lie in $\mathbb{H}$, we won't be addressing those cases here. We know that sides are paired via unique side-pairing transformations. Analogously, each vertex A of $\mathfrak{D}$ is mapped to another vertex of $\mathfrak{D}$ via the side-pairing transformation associated with the side whose endpoint is A.

Every vertex A of $\mathfrak{D}$ is the end point of exactly two sides, say s and $\circledast s$, of $\mathfrak{D}$. For $\mathfrak{D}$, its vertex A together with the side s will be denoted by the pair (A, s) while vertex A together with the side $\circledast s$ will be denoted by $\circledast(A, s)$.

In the view of Figure 4.2.1, we consider the following algorithm:

Algorithm. (i) Suppose s_0 is a side of $\mathfrak{D}$ whose endpoint is A_0. Let T_1 be the side-pairing transformation mapping s_0 to another side $s_1 = T_1(s_0)$. We start with the pair (A_0, s_0).

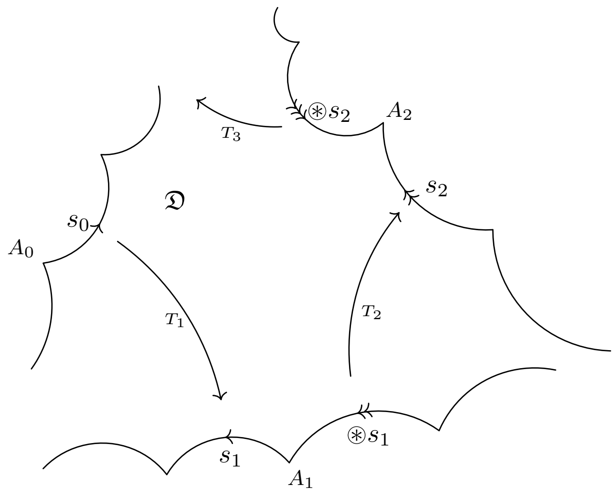


Figure 4.2.1: The pair $(A_0,\ s_0)$ is mapped to (A_1,s_1), which is mapped to $(A_1,\circledast s_1)$, which is mapped to (A_2,s_2), etc.

(ii) Let $A_1 = T_1\,(A_0)$, where A_1 is another vertex of $\mathfrak{D}$. Since A_0 is an endpoint of s_0, A_1 will be an endpoint of s_1. This results in a new pair (A_1, s_1).

(iii) Let $\circledast s_1$ be the other side of $\mathfrak{D}$ whose endpoint is also A_1. This gives us a new pair $\circledast\,(A_1, s_1)$.

(iv) Let T_2 be the side-pairing transformation associated to the side $\circledast s_1$ mapping $\circledast s_1$ to $s_2 = T_2(\circledast s_1)$, another side of $\mathfrak{D}$, A_1 to $A_2 = T_2(A_1)$, another vertex of $\mathfrak{D}$.

Repeating above mentioned steps, we get a sequence of pairs of vertices and sides as

$$\begin{aligned}
\begin{pmatrix} A_0 \\ s_0 \end{pmatrix} \xrightarrow{T_1} \begin{pmatrix} A_1 \\ s_1 \end{pmatrix} &\xrightarrow{\circledast} \begin{pmatrix} A_1 \\ \circledast s_1 \end{pmatrix} \\
\xrightarrow{T_2} \begin{pmatrix} A_2 \\ s_2 \end{pmatrix} &\xrightarrow{\circledast} \begin{pmatrix} A_2 \\ \circledast s_2 \end{pmatrix} \\
&\vdots \\
\xrightarrow{T_i} \begin{pmatrix} A_i \\ s_i \end{pmatrix} &\xrightarrow{\circledast} \begin{pmatrix} A_i \\ \circledast s_i \end{pmatrix} \\
\xrightarrow{T_{i+1}} \begin{pmatrix} A_{i+1} \\ s_{i+1} \end{pmatrix} &\xrightarrow{\circledast} \cdots .
\end{aligned}$$

This process will eventually return to the starting pair as there are only a finite number of pairings of vertices and sides. Let k be the smallest positive integer such that $(A_k, \circledast s_k) = (A_0, s_0)$.

Definition 4.2.1. An *elliptic cycle* $\mathcal{E}$ is the sequence of vertices $A_0 \to A_1 \to \cdots \to$

A_{k-1}. All the vertices in an elliptic cycle are considered *congruent* to each other. An *elliptic cycle transformation* is the composition $T_k T_{k-1} \cdots T_2 T_1$. Let D be a hyperbolic polygon and A be one of its vertices. The elliptic cycle transformation is denoted by $T_{A,s}$ if it is associated with the vertex A and the side s.

We observe that there are only finitely many elliptic cycles and elliptic cycle transformations as there are only finitely many pairs of vertices and sides.

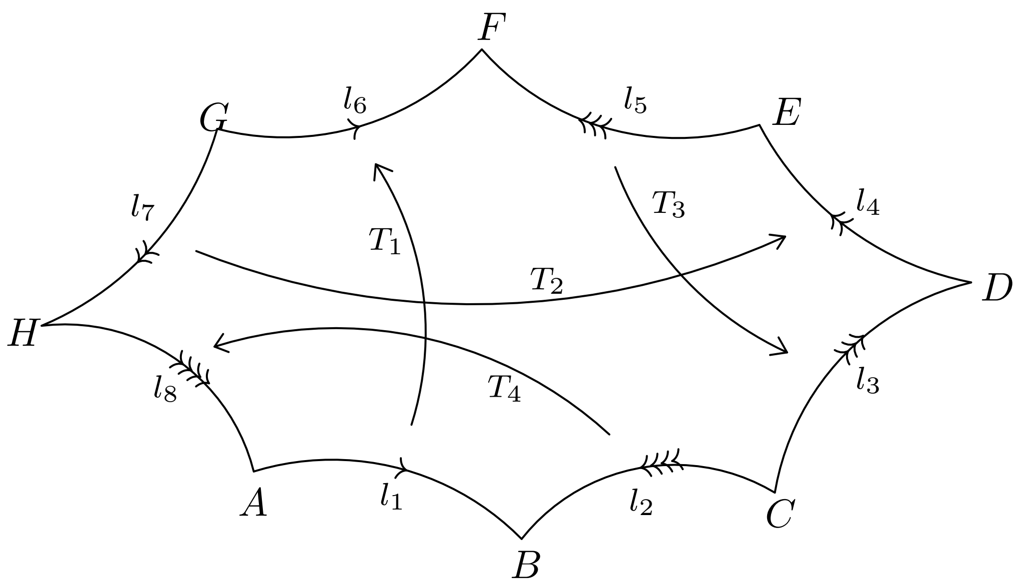


Figure 4.2.2: A polygon with sides, vertices, and side-pairing transformations.

We use Figure 4.2.2 to find elliptic cycles and elliptic cycle transformations. In the figure, T_3 maps the side $l_5 = EF$ to the side $l_3 = CD$. According to the orientation of these sides, T_3 maps the vertex E to the vertex C, and the vertex F to the vertex D. By following Algorithm 4.2 as described above, we get

$$\begin{aligned}
\begin{pmatrix} A \\ l_1 \end{pmatrix} &\xrightarrow{T_1} \begin{pmatrix} G \\ l_6 \end{pmatrix} \xrightarrow{\circledast} \begin{pmatrix} G \\ l_7 \end{pmatrix} \\
&\xrightarrow{T_2} \begin{pmatrix} D \\ l_4 \end{pmatrix} \xrightarrow{\circledast} \begin{pmatrix} D \\ l_3 \end{pmatrix} \\
&\xrightarrow{T_3^{-1}} \begin{pmatrix} F \\ l_5 \end{pmatrix} \xrightarrow{\circledast} \begin{pmatrix} F \\ l_6 \end{pmatrix} \\
&\xrightarrow{T_1^{-1}} \begin{pmatrix} B \\ l_1 \end{pmatrix} \xrightarrow{\circledast} \begin{pmatrix} B \\ l_2 \end{pmatrix} \\
&\xrightarrow{T_4} \begin{pmatrix} A \\ l_8 \end{pmatrix} \xrightarrow{\circledast} \begin{pmatrix} A \\ l_1 \end{pmatrix}.
\end{aligned}$$

Thus, we obtain the elliptic cycle transformation $T_4T_1^{-1}T_3^{-1}T_2T_1$ associated with the elliptic cycle $A \to G \to D \to F \to B$. We observe that this elliptic cycle does not contain all vertices, therefore, there must be another elliptic cycle. Again, as above, we get

$$\begin{aligned} \begin{pmatrix} H \\ l_8 \end{pmatrix} &\xrightarrow{T_4^{-1}} \begin{pmatrix} C \\ l_2 \end{pmatrix} \xrightarrow{\circledast} \begin{pmatrix} C \\ l_3 \end{pmatrix} \\ &\xrightarrow{T_3^{-1}} \begin{pmatrix} E \\ l_5 \end{pmatrix} \xrightarrow{\circledast} \begin{pmatrix} E \\ l_4 \end{pmatrix} \\ &\xrightarrow{T_2^{-1}} \begin{pmatrix} H \\ l_7 \end{pmatrix} \xrightarrow{\circledast} \begin{pmatrix} H \\ l_8 \end{pmatrix}. \end{aligned}$$

Now, we get the other elliptic cycle transformation $T_4^{-1}T_3^{-1}T_2^{-1}$ associated with the elliptic cycle $H \to C \to E$.

At this point, one might wonder what would have happened had we started with the other side of the chosen vertex or a different vertex altogether. We explain these situations in the following remark.

Remark 3. If we start with $(A_0, \circledast s_0)$, instead of (A_0, s_0), then one can verify that $T_{A_0,\circledast s_0} = T_{A_0,s_0}^{-1}$.

Further, if we start with $(A_i, \circledast s_i)$ instead of (A_0, s_0), then we get

$$T_{A_i,\circledast s_i} = T_iT_{i-1}\cdots T_1T_n\cdots T_{i+2}T_{i+1}.$$

One can verify that

$$T_{A_i,\circledast s_i} = (T_i\cdots T_1)\, T_{A_0,s_0}\, (T_i\cdots T_1)^{-1}.$$

Hence, $T_{A_i,\circledast s_i}$ and T_{A_0,s_0} are conjugate.

Theorem 4.2.2. *Let $\mathcal{S}$ be a Fuchsian group and $\mathfrak{D}$ be a fundamental region for $\mathcal{S}$. The elliptic cycles of $\mathfrak{D}$ are in one-to-one correspondence with the conjugacy classes of non-trivial maximal finite cyclic subgroups of $\mathcal{S}$.*

Proof. For $p \in \mathbb{H}$, the non-trivial stabilizer $\operatorname{stab}_{\mathcal{S}}(p)$ of p in $\mathcal{S}$ is a discrete subgroup. Since $p \in \mathbb{H}$, $\operatorname{stab}_{\mathcal{S}}(p)$ contains only elliptic elements and therefore, by Theorem 2.3.4,

$\mathrm{stab}_{\mathcal{S}}(p)$ is finite cyclic. By Lemma 2.3.1, it follows that the $\mathrm{stab}_{\mathcal{S}}(p)$ is a maximal finite cyclic subgroup of $\mathcal{S}$. Conversely, every maximal finite cyclic subgroup of $\mathcal{S}$ is the stabilizer of some point in $\mathbb{H}$. From Remark 3, it follows that elliptic cycles determine these maximal cyclic subgroups up to conjugacy. $\square$

Example 4.2.1. Consider the Dirichlet region $\mathfrak{D}$ of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ (see Figure 3.3.4). The region $\mathfrak{D}$ has three vertices, namely $w = \frac{-1+i\sqrt{3}}{2}$, $w+1 = \frac{1+i\sqrt{3}}{2}$, and i. The stabilizer subgroup of these vertices is generated by $z \mapsto \frac{-z-1}{z}$, $z \mapsto \frac{z-1}{z}$, and $z \mapsto \frac{-1}{z}$, respectively. Since these generators have the order 3, 3, and 2, respectively, any transformation in the stabilizer also has the same order. Now, w and $w+1$ belong to the same elliptic cycle, and some elliptic element of order 3 fixes them. However, i is fixed by an elliptic element of order 2 and thus, its elliptic cycle has a unique vertex. From Theorem 4.2.2, there are two conjugacy classes of the maximal finite cyclic subgroups in $\mathrm{PSL}(2,\mathbb{Z})$, of orders 2 and 3.

Definition 4.2.3. The *periods of* $\mathcal{S}$ are defined by the orders of non-conjugate maximal finite cyclic subgroups of $\mathcal{S}$.

The number of conjugacy classes of maximal finite cyclic subgroups of particular order determines how many times each period is repeated. In Example 4.2.1, $\mathrm{PSL}(2,\mathbb{Z})$ has periods 2, 2, and 3.

In the following result, we see the relationship between the order of an elliptic cycle and the sum of all the angles at the vertices of that elliptic cycle.

Theorem 4.2.4. *Suppose $\mathfrak{D}$ is a Dirichlet region for $\mathcal{S}$ and the internal angles at all congruent vertices of $\mathfrak{D}$ are $\alpha_1, \alpha_2, \ldots, \alpha_t$. If the order of the stabilizer of one of these vertices is n, then $\alpha_1 + \ldots + \alpha_t = 2\pi/n$.*

Proof. Consider an elliptic cycle containing vertices $A_1, A_2, \ldots, A_t$ with internal angles $\alpha_1, \alpha_2, \ldots, \alpha_t$, respectively. Since the stabilizers of any two vertices in an elliptic cycle are conjugate, without loss of generality, we assume that the stabilizer of A_1 in $\mathcal{S}$ be

$$\mathcal{E} = \left\{\mathrm{Id}, T, T^2, \ldots, T^{n-1}\right\},$$

where T is the generator of the stabilizer. For $0 \leq m < n$, $T^m(\mathfrak{D})$ has a vertex at A_1 whose internal angle is α_1. For some $\mathrm{S}_k \in \mathcal{S}$, such that $\mathrm{S}_k(A_k) = A_1$, we obtain a coset $\mathcal{E}S_k$ containing n elements mapping A_k to A_1. Thus, $T^m\mathrm{S}_k(\mathfrak{D})$ has one of its vertex as A_1 with internal angle α_k. For $R \in \mathcal{S}$, if A_1 is one vertex of $R(\mathfrak{D})$, then $R^{-1}(A_1)$ is a vertex of $\mathfrak{D}$, therefore, $R^{-1}(A_1) = A_i$, for some i such that $1 \leq i \leq t$, implying $R \in \mathcal{E}S_i$. Thus, we conclude that there are nt regions surrounding the vertex A_1. All of these regions are distinct, because if $T^mS_k(\mathfrak{D}) = T^lS_j(\mathfrak{D})$, then $T^mS_k = T^lS_j$, and therefore, $m = l$ and $k = j$. Hence, we obtain

$$n\left(\alpha_1 + \ldots + \alpha_t\right) = 2\pi.$$

$\square$

If the stabilizer $\mathcal{E}$ is trivial, then $n = 1$ and $\alpha_1 + \ldots + \alpha_t = 2\pi$.

Chapter 5

Poincaré's Theorem

In this chapter, we will define quotient orbifolds and cocompact Fuchsian groups. We show that a Fuchsian group is cocompact if and only if its associated quotient orbifold is compact. Then after defining the signature of a Fuchsian group, we prove the Poincaré's Theorem. Depending on the usefulness of a particular model, we will switch between the upper half-plane $\mathbb{H}$ and the Poincaré disc $\mathbb{D}$, whenever required. This chapter is based on [6, Chapter 4] and [10, Chapter 21-22] .

5.1 Cocompact Fuchsian groups

Definition 5.1.1. For a Fuchsian group $\mathcal{S}$, the orbit space $\mathbb{H}/\mathcal{S}$ is called the *quotient orbifold* associated with $\mathcal{S}$.

Definition 5.1.2. A Fuchsian group $\mathcal{S}$ is said to be *cocompact* if its associated quotient orbifold is compact.

Theorem 5.1.3. *A Fuchsian group $\mathcal{S}$ does not have any parabolic elements if it has a compact Dirichlet region.*

Proof. Consider a compact Dirichlet region $\mathfrak{D}$ for $\mathcal{S}$. For $z \in \mathbb{H}$, we define

$$\gamma(z) := \inf\{\mathrm{d}_{\mathbb{H}}(z, S(z)) \mid S \in \mathcal{S} - \{\mathrm{Id}\},\ S \text{ is not elliptic}\}.$$

Since S is continuous and $\mathcal{S}$-orbits are discrete, γ is continuous. Further, $\gamma(z) > 0$ as $\mathcal{S}$ is discrete. Since $\mathfrak{D}$ is compact, there exists $\gamma > 0$ such that $\gamma = \inf\{\gamma(z) \mid z \in \mathfrak{D}\}$ is attained. For every $z \in \mathbb{H}$, there exists $T \in \mathcal{S}$ such that $w = T(z) \in \mathfrak{D}$. Thus, for a non-elliptic element $S_0 \in \mathcal{S} - \{\mathrm{Id}\}$,

$$\mathrm{d}_{\mathbb{H}}(z, S_0(z)) = \mathrm{d}_{\mathbb{H}}(T(z), T(S_0(z))) = \mathrm{d}_{\mathbb{H}}(w, TS_0T^{-1}(w)) \geq \gamma.$$

Hence,

$$\inf\{\mathrm{d}_{\mathbb{H}}(z, S_0(z)) \mid z \in \mathbb{H},\ S_0 \text{ is not elliptic}\} = \gamma > 0.$$

If possible, let $S_1 \in \mathcal{S}$ be a parabolic element. For $P \in \mathrm{PSL}(2,\mathbb{R})$, if $\mathcal{S}_\infty = P\mathcal{S}P^{-1}$ then $P(\mathfrak{D})$ is a compact fundamental region for $\mathcal{S}_\infty$. Therefore, without loss of generality, we can assume that $S_1(z) = z + 1$. Then, by Theorem 1.3.8, as $\Im z \to \infty$, $\mathrm{d}_{\mathbb{H}}(z, S_1(z)) \to 0$ which is impossible as $\mathrm{d}_{\mathbb{H}}(z, S_1(z)) \geq \gamma > 0$. Hence, the assertion follows. □

Definition 5.1.4. Let $\mathcal{S}$ be a Fuchsian group. If there exists a finite-sided convex fundamental region of $\mathcal{S}$, then $\mathcal{S}$ is called geometrically finite.

Theorem 5.1.5. (Siegel's Theorem) *For a Fuchsian group $\mathcal{S}$, if the area of its quotient orbifold $\mathbb{H}/\mathcal{S}$ is finite, then $\mathcal{S}$ is geometrically finite.*

Theorem 5.1.6. *Let $\mathcal{S}$ be a Fuchsian group with a non-compact Dirichlet region $\mathfrak{D}$. Then the following statements hold.*

(i) The quotient orbifold associated to $\mathcal{S}$ is not compact.

(ii) If the hyperbolic area of $\mathfrak{D}$ is finite, then it has at least one ideal vertex.

Proof. Let $\mathfrak{D} = \mathfrak{D}_{z_0}(\mathcal{S})$ be a non-compact Dirichlet region of $\mathcal{S}$ centered at z_0(see Figure 5.1.1).

We take into account all oriented hyperbolic rays from the point z_0. Every hyperbolic ray is uniquely determined by its direction d at the point z_0. Since $\mathfrak{D}$ is convex, any ray either completely lies inside $\mathfrak{D}$ or intersects $\partial\mathfrak{D}$ in a unique point, as shown in the figure. We define $\lambda(d)$ as the length of a line segment inside $\mathfrak{D}$ in the direction d,

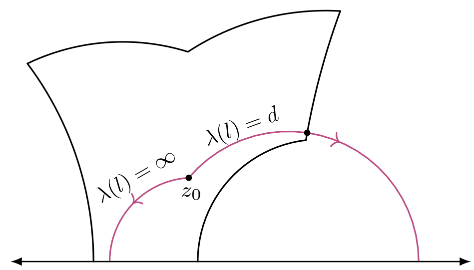


Figure 5.1.1: A non-compact Dirichlet region.

where $\lambda(d) = \infty$ if the hyperbolic ray lies completely inside $\mathfrak{D}$. If $\lambda(d)$ is finite for all directions d, then $\lambda(d)$ is bounded and λ is continuous. Hence, $\mathfrak{D}$ must be compact. Since $\mathfrak{D}$ is non-compact, there exist a d such that $\lambda(d) = \infty$. By identifying congruent points of $\partial\mathfrak{D}$, we obtain a non-compact quotient orbifold $\mathbb{H}/\mathcal{S}$, and therefore, the statement (i) holds.

Consider a direction d_0 such that $\lambda(d_0) = \infty$. The hyperbolic ray from z_0 in the direction d_0 intersects $\partial\mathbb{H}$, this point of intersection belong to $\tilde{\partial}\mathfrak{D}$. Since the hyperbolic area of $\mathfrak{D}$ is finite, by Theorem 5.1.5, $\mathfrak{D}$ is geometrically finite, and so $\tilde{\partial}\mathfrak{D}$ can have only finitely many free sides and vertices at infinity. Since the hyperbolic area of $\mathfrak{D}$ is finite, $\tilde{\partial}\mathfrak{D}$ can not contain any free sides. Thus, $\tilde{\partial}\mathfrak{D}$ consists of only vertices at infinity. Hence, $\mathfrak{D}$ has at least one ideal vertex, and therefore, the statement (ii) holds. □

The following corollary immediately follows from the above theorem.

Corollary 5.1.7. *A Fuchsian group $\mathcal{S}$ is cocompact if and only if any Dirichlet region for $\mathcal{S}$ is compact.*

We may consider a parabolic element as an elliptic element of infinite order with its fixed point in $\partial\mathbb{H}$. If there exists a non-trivial stabilizer in $\mathcal{S}$ for a point in $\mathbb{R}\cup\{\infty\}$, then the stabilizer is a maximal (cyclic) parabolic subgroup of $\mathcal{S}$. Conversely, every maximal parabolic subgroup of $\mathcal{S}$ is the stabilizer of a point in $\mathbb{R}\cup\{\infty\}$.

Let $\mathcal{S}$ be a Fuchsian group containing parabolic elements and $\mathfrak{D}$ be its Dirichlet region. In this case $\mathfrak{D}$ is not compact. If the area of $\mathfrak{D}$ is finite in addition to being non-compact, then $\mathfrak{D}$ has at least one ideal vertex. There is a one-to-one correspondence between conjugacy classes of maximal parabolic subgroups of $\mathcal{S}$ and non-congruent vertices at infinity of $\mathfrak{D}$. If infinite periods are allowed, the number of occurrences of the period ∞ equals the number of conjugacy classes of maximal parabolic subgroups and this number is known as the *parabolic class number* of $\mathcal{S}$.

Again, consider the modular group $\mathrm{PSL}(2,\mathbb{Z})$ and its Dirichlet region as in Figure 3.3.4. Every parabolic element is conjugate to $z \to z+n$ for some $n \in \mathbb{Z}$, resulting in the periods $2, 3$, and ∞ for the modular group. We know that the angle at an ideal vertex is zero. This convention yields the Dirichlet region for the modular group to have a vertex at ∞ whose angle is $\frac{\pi}{\infty} = 0$.

5.2 The signature of a Fuchsian group

Consider a fundamental region $\mathfrak{D}$ for a cocompact Fuchsian group $\mathcal{S}$. Since $\mathfrak{D}$ is compact, it has finitely many sides, and thus, finitely many vertices. Therefore, $\mathfrak{D}$ has finitely many elliptic cycles and a finite number of periods, say $p_1, p_2, \ldots, p_r$ counting multiplicities.

Definition 5.2.1. For a cocompact Fuchsian group $\mathcal{S}$, if the quotient orbifold associated to $\mathcal{S}$ is an oriented surface of genus g with r marked points, then the *signature* of $\mathcal{S}$ is defined by the tuple $(g; p_1, p_2, \ldots, p_r)$.

Theorem 5.2.2. *For a cocompact Fuchsian group $\mathcal{S}$, let $(g; p_1, p_2, \ldots, p_r)$ be its signature. Then we have*

$$\mathrm{area}_{\mathbb{H}}(\mathbb{H}/\mathcal{S}) = 2\pi \left[(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) \right].$$

Proof. Since $\mathfrak{D}$ is a Dirichlet region of $\mathcal{S}$, we have $\mathrm{area}_{\mathbb{H}}(\mathbb{H}/\mathcal{S}) = \mathrm{area}_{\mathbb{H}}(\mathfrak{D})$. By Theorems 4.2.2 and 4.2.4, there are r elliptic cycles of vertices in $\mathfrak{D}$ and the sum of internal angles at all elliptic vertices is $\sum_{i=1}^{r} {}^{2\pi}\!/_{p_i}$. If there are q non-elliptic cycles, then by Theorem 4.2.2, the sum of internal angles at these vertices is $2\pi q$. Hence, the sum of all the internal angles of $\mathfrak{D}$ is

$$2\pi \left[\left(\sum_{i=1}^{r} \frac{1}{p_i} \right) + q \right].$$

If $\mathfrak{D}$ has $2n$-sides, then, after identification, we get a decomposition of $\mathbb{H}/\mathcal{S}$ into $(r+q)$-vertices, n-edges, and a simply connected face. By the Euler formula, we get

$$2 - 2g = (r + q) - n + 1. \tag{5.2.1}$$

By the Gauss-Bonnet theorem, we get

$$\text{area}_{\mathbb{H}}(\mathfrak{D}) = (2n-2)\pi - 2\pi \left[(\sum_{i=1}^{r} \frac{1}{p_i}) + q \right]. \tag{5.2.2}$$

From Equations (5.2.1) and (5.2.2), we have

$$\begin{aligned}
\text{area}_{\mathbb{H}}(\mathfrak{D}) &= (2(r+q+2g-1)-2)\pi - 2\pi \left[(\sum_{i=1}^{r} \frac{1}{p_i}) + q \right] \\
&= 2\pi \left[(r+q+2g-1-1) - \left((\sum_{i=1}^{r} \frac{1}{p_i}) + q \right) \right] \\
&= 2\pi \left[(2g-2) + \sum_{i=1}^{r} 1 - \left(\sum_{i=1}^{r} \frac{1}{p_i} \right) \right] \\
&= 2\pi \left[(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{p_i} \right) \right].
\end{aligned}$$

□

It is rather remarkable that the converse of Theorem 4.3.1 is also true, that is, there exists a Fuchsian group with a given signature. This result is known as Poincaré Theorem or Poincaré polygon Theorem.

5.3 Poincaré theorem

In this section, we state and prove the Poincaré polygon theorem.

Theorem 5.3.1. [6, Theorem 4.3.2] *If $g \geq 0,\ r \geq 0,\ p_i \geq 2,\ for 1 \leq i \leq r$ are integers such that*

$$(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{p_i} \right) > 0,$$

then there exists a Fuchsian group with signature $(g; p_1, \dots, p_r)$.

Proof. For this proof, we work with the Poincaré unit disc model $\mathbb{D}$. We divide the disc into equal sectors with the help of $(4g + r)$ radii. (see Figure 5.3.1 when $g = 2$, $r = 4$, where $p_i > 2$ for $i = 1, 2, 3$, and $p_4 = 2$.) On each radius, we choose points at Euclidean distance t from the origin, where $0 < t < 1$. We form a regular hyperbolic polygon $P(t)$ by joining the points on consecutive edges. Then, we construct r external isosceles hyperbolic triangles on the first r sides of $P(t)$, such that the angle between the equal sides of the triangles are $\frac{2\pi}{p_1}, \frac{2\pi}{p_2}, \dots, \frac{2\pi}{p_r}$.

We get a hyperbolic polygon $Q(t)$ having $4g+2r$ sides labeled as $\alpha_1, \beta_1^{'}, \alpha_1^{'}, \beta_1, \dots,$ $\alpha_g, \beta_g^{'}, \alpha_g^{'}, \beta_g$ with orientation as in figure. From our construction, the sides in pairs $(\alpha_i, \alpha_i^{'})$, $(\beta_i, \beta_i^{'})$, and $(\xi_i, \xi_i^{'})$ have equal hyperbolic length.

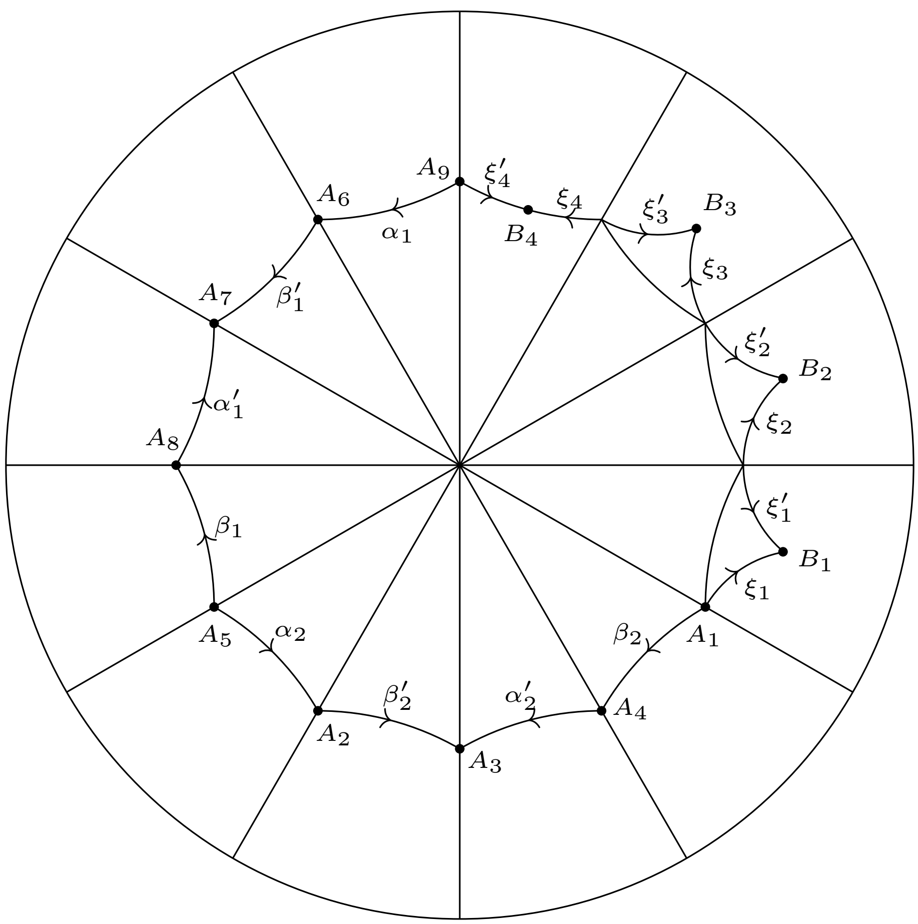


Figure 5.3.1: Construction of a fundamental region for a Fuchsian group with signature $(2 : p_1, p_2, p_3, 2)$.

If $t \to 0$, then $\text{area}_{\mathbb{H}}(Q(t)) \to 0$. By Theorem 1.7.4, as $t \to 1$,

$$\text{area}_{\mathbb{H}}(Q(t)) \to [(4g+2r)-2]\pi - \sum_{i=1}^{r} \frac{2\pi}{p_i} = 2\pi \left[(2g-1) + \sum_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)\right].$$

Since $\text{area}_{\mathbb{H}}(Q(t))$ is continuous, by the intermediate value theorem, for some $t_0 \in (0, 1)$,

$$\text{area}_{\mathbb{H}}(Q(t_0)) = 2\pi \left[(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)\right].$$

Since the hyperbolic lengths of the pairs are equal, there exist S_i, T_j, and G_k ($i, j = 1, 2, \ldots, g$ and $k = 1, 2, \ldots, r$) such that

$$S_i(\alpha_i^{'}) = \alpha_i,\ T_j(\beta_j^{'}) = \beta_j, \text{ and } G_k(\xi_k^{'}) = \xi_k.$$

Now, we compute the congruence classes of vertices. To begin with, let A_1 be the vertex where β_g begins. This vertex is congruent (via T_g^{-1}) to the vertex where β_g' begins, say A_2, which is same as where α_g ends. The vertex A_2 is congruent (via S_g^{-1}) to the vertex where α_g' ends, say A_3, which is same as where β_g' ends. The vertex A_3 is congruent (via T_g) to the vertex where β_g ends, say A_4, which is same as where α_g' begins. The vertex A_4 is congruent (via S_g) to the vertex where β_{g-1} begins. We continue this process until all the vertices form a congruent set.

Thus, we get

$$G_1 G_2 \ldots G_r S_1 T_1 S_1^{-1} T_1^{-1} \ldots S_g T_g S_g^{-1} T_g^{-1}(A_1) = A_1.$$

The remaining r vertices, say $B_1, B_2, \ldots, B_r$, form r congruent sets, each containing only one element. Since

$$\text{area}_{\mathbb{H}}(Q(t_0)) = 2\pi \left[(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)\right],$$

by the Gauss-Bonnet formula, the sum of angles at $A_1, \ldots, A_{4g+r}$ equal to 2π.

Let $\mathcal{S}$ be the group generated by

$$\langle S_i, T_j, G_k \mid i, j = 1,\ 2, \ldots, g \text{ and } k = 1, 2, \ldots, r\rangle.$$

It can be seen that the images of $Q(t_0)$ under $\mathcal{S}$ cover $\mathbb{D}$ without any overlaps, and thus $Q(t_0)$ is a fundamental region for $\mathcal{S}$. Therefore, the $\mathcal{S}$-orbit of each point is discrete, and so $\mathcal{S}$ acts properly discontinuously on $\mathbb{D}$. When taken to the upper half-plane, $\mathcal{S}$ becomes a Fuchsian group.

The quotient space $\mathbb{D}/\mathcal{S}$ is decomposed into $r + 1$ vertices, $2g + r$ edges, and one simply connected face. The Euler formula gives

$$2 - 2h = (r + 1) - (2g + r) = 2 - 2g,$$

where h is the genus of $\mathbb{D}/\mathcal{S}$. Therefore, $h = g$. Thus, there are r elliptic cycles, $\{B_1\}, \ldots, \{B_r\}$, with the order of stabilizer $p_1, p_2, \ldots, p_r$, respectively. Hence, $\mathcal{S}$ has signature $(g; p_1, p_2, \ldots, p_r)$. □

Assuming that a Fuchsian group $\mathcal{S}$ has r conjugacy classes of maximal elliptic cyclic subgroups of orders $p_1, \ldots, p_r$, s conjugacy classes of maximal parabolic cyclic subgroups, and associated quotient orbifold of genus g, we define the signature of $\mathcal{S}$ by the tuple $(g; p_1, \ldots, p_r; s)$. The hyperbolic area of the associated quotient orbifold can be calculated using this signature. When the expression for the area is positive, a Fuchsian group $\mathcal{S}$ with the given signature exists. In this case, s of the isosceles triangles must have vertices on $\partial\mathbb{D}$ with the internal angle zero. The algebraic structure of the group $\mathcal{S}$ is determined by its signature, and has the following presentation:

$$\begin{aligned}&\langle S_1,\ T_1,\ S_2,\ T_2, \ldots, S_g,\ T_g,\ G_1, \ldots,\ G_r, P_1, \ldots, P_s \mid\ G_1^{m_1} = \ldots = G^{m_r} \\ &= P_1 P_2 \ldots P_s\, G_1\, G_2 \ldots\ G_r\, S_1\, T_1\, S_1^{-1}\, T_1^{-1} \ldots S_g\, T_g\, S_g^{-1}\, T_g^{-1} = \mathrm{Id}\rangle,\end{aligned}$$

where A_i, T_j, G_k are as defined above, and P_t $(t = 1, \ldots, s)$ are the orientation-preserving isometries that fix vertices on $\partial\mathbb{D}$.

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