Commutativity of periodic mapping classes and their representations as words in Dehn twists

A thesis submitted in partial fulfilment of the requirements

for the award of the degree of

DOCTOR OF PHILOSOPHY

by

NEERAJ KUMAR DHANWANI

1510403

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL

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DEDICATION

I want to dedicate this thesis to my family.

ABSTRACT

Let $\mathrm{Mod}(S_g)$ be the mapping class group of the closed orientable surface S_g of genus $g \geq 2$. In the first part of the thesis, we develop the theory leading to necessary and sufficient conditions under which two finite-order mapping classes have representatives in their respective conjugacy classes that commute in $\mathrm{Mod}(S_g)$. As an application, we show that any finite-order mapping class, whose corresponding orbifold is not a sphere, has a conjugate that is liftable under a finite cyclic cover. Furthermore, we show that any torsion element in the centralizer of an irreducible finite order mapping class is of order at most 2. We also state equivalent conditions for the primitivity of a torsion element of $\mathrm{Mod}(S_g)$.

In the second part, we develop various methods (i.e. algorithms) for expressing a periodic mapping class in $\mathrm{Mod}(S_g)$ as a product of Dehn twists, up to conjugacy. To begin with, we derive a generalization of the star relation in $\mathrm{Mod}(S_g^3)$, for $g \geq 2$. The methods we derive are based the geometric realizations of torsion elements in $\mathrm{Mod}(S_g)$, the generalized star relation, and the symplectic representations of periodic mapping classes. By applying our methods, we provide an algorithm to write certain roots of Dehn twists in $\mathrm{Mod}(S_g)$ as words in Dehn twists. As the another application, we show that the irreducible periodic mapping classes of orders $4g$ and $4g+2$ in $\mathrm{Mod}(S_g)$, for $g \geq 2$, have representatives in their respective conjugacy classes whose product is pseudo-Anosov.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Let $S = S^b_{g,p}$ be the orientable surface of genus g with b boundary components and p punctures (or marked points). Let $\text{Homeo}^+(S)$ be the group of orientation-preserving homeomorphisms on S that restrict to identity on ∂S and preserve the set of punctures in S. Then the *mapping class group* $\text{Mod}(S)$ of S is defined to be group $\pi_0(\text{Homeo}^+(S))$. The Nielsen-Thurston classification [39] of surface diffeomorphisms states that:

Theorem 1.1.1 (Nielsen-Thurston classification)**.** *Any $F \in \text{Mod}(S)$ is represented by an $\mathcal{F} \in \text{Homeo}^+(S)$ such that at least one of the following holds.*

(a) $\mathcal{F}$ is periodic (i.e. $\mathcal{F}$ is of finite order).

(b) $\mathcal{F}$ is reducible (i.e $\mathcal{F}$ preserves a multicurve in S).

(c) $\mathcal{F}$ is pseudo-Anosov (i.e $\mathcal{F}$ preserves a transverse pair of foliations in S).

In this thesis, we will mostly be concerned with the closed orientable surface S_g (i.e. $b = p = 0$) and its mapping class group $\text{Mod}(S_g)$. In particular, we address the following questions pertaining to periodic mapping classes in $\text{Mod}(S_g)$, for $g \geq 2$.

Question 1. When do two periodic elements in $\text{Mod}(S_g)$ have conjugates that commute in $\text{Mod}(S_g)$?

Question 2. Can one develop methods for representing an arbitrary periodic element $F \in \text{Mod}(S_g)$ as a word $\mathcal{W}(F)$ in the standard generators of $\text{Mod}(S_g)$ (i.e. Dehn twists), up to conjugacy?

Before motivating our first question, we will provide a brief historical overview of the theory of group actions on surfaces that is relevant to our context. Viewing S_g as a Riemann surface, let $\text{Aut}(S_g)$ denote the group of automorphisms of S_g. In 1893, Hurwitz [15] showed that if a finite group G imbeds into $\mathrm{Aut}(S_g)$, then $|G| \leq 84(g-1)$. Around the same time, Wiman [41] proved that when G is cyclic, $|G| \leq 4g+2$ and that the bound is attainable for $g \geq 2$. In 1965, Harvey [12] applied the theory of Fuchsian groups [17] to classify cyclic actions on hyperbolic surfaces, and as a consequence, reproved the result of Wiman. In the following year, Maclachlan [22] showed that $4g+4$ is a realizable bound for $|G|$ when G is abelian. Recently, in 2007, Broughton-Wootton [4] derived a method for enumerating the conjugacy classes of abelian subgroups of $\mathrm{Mod}(S_g)$.

For $g \geq 2$, the Nielsen realization theorem [17, 29] says that given a finite subgroup $H < \mathrm{Mod}(S_g)$, there exists a hyperbolic structure on S_g that realizes H as a group of isometries. When H is cyclic, more recently, Parsad-Rajeevsarathy-Sanki [30] described an inductive way to construct hyperbolic structures on S_g that realize H as group of isometries, thereby providing explicit solutions to the Nielsen realization problem. Our first problem is primarily motivated by the extension of this work to the case when H is a two-generator finite abelian subgroup of $\mathrm{Mod}(S_g)$. In order to achieve this, we must first attain a better geometric understanding of two-generator abelian actions on S_g. In this connection, it is important to note that the (finite) groups generated by different (pairs of) conjugates of the same pair of finite-order mapping classes in $\mathrm{Mod}(S_g)$ may represent inequivalent actions on S_g. For example, consider the six involutions in $\mathrm{Mod}(S_8)$ shown in Figure 1.1 below, where each involution is realized as a π-rotation about an axis (passing through the origin) under a suitable embedding $S_8 \hookrightarrow \mathbb{R}^3$. Though all of these involutions

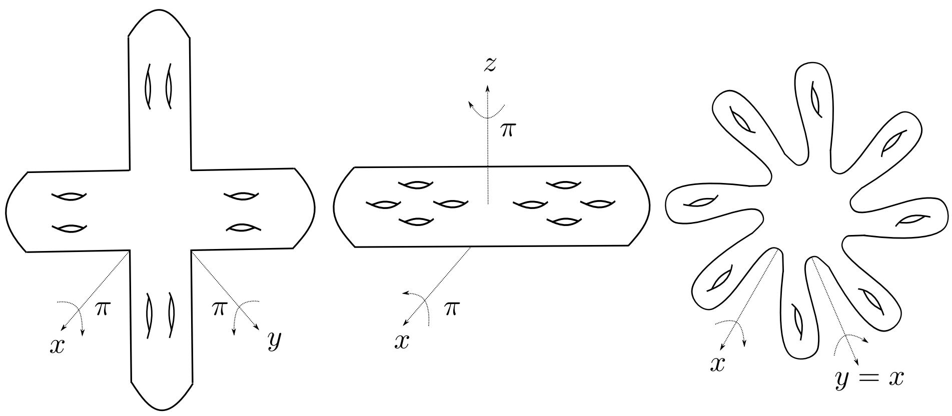

Figure 1.1: Six conjugate involutions in $\mathrm{Mod}(S_8)$

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are conjugate in $\mathrm{Mod}(S_g)$, note that each of the two pairs of involutions indicated in the first two subfigures clearly generate distinct subgroups of $\mathrm{Mod}(S_8)$ isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, while the pair of involutions appearing in the third subfigure can be shown to generate a subgroup isomorphic to D_8. Thus, even though finite abelian groups and their conjugacy classes in $\text{Mod}(S_g)$ have been widely studied [4, 12, 22], the solution to our problem yields two benefits.

(a) Provides a geometric understanding of two-generator finite abelian actions on S_g.

(b) Provides explicit solutions to the Nielsen realization problem for two-generator finite abelian subgroups of $\text{Mod}(S_g)$.

In order to motivate our second question, we first define a *Dehn twist* in $\text{Mod}(S_g)$, and briefly discuss its importance in the evolution of the theory of mapping class groups. A *Dehn twist* on an annulus $A \cong S^1 \times [0,1]$ is defined by the map

$$\phi : A \to A : (\theta, t) \overset{\phi}{\mapsto} (2\pi t + \theta, t).$$

Using this definition, we can define a Dehn twist T_α about a simple closed curve α in S_g. Consider a closed (regular) annular neighborhood N of α, and a homeomorphism $\eta : N \to A$. Then a *Dehn twist* T_α along the curve α is defined by

$$T_\alpha(x) := \begin{cases} \eta^{-1} \circ \phi \circ \eta(x), & \text{if } x \in N, \text{ and} \\ x, & \text{if } x \notin N. \end{cases}$$

Dehn twists are named after the German mathematician Max Dehn, who proved [7] that a finite set of Dehn twists in S_g generates $\text{Mod}(S_g)$. Subsequently, in 1964, Lickorish [20] proved that $\text{Mod}(S_g)$ is generated by $3g-1$ Dehn twists about nonseperating curves. This result was further improved by Humphries [14] in 1978, who also showed that $\text{Mod}(S_g)$ is generated by a minimal generating set comprising $2g+1$ Dehn twists about non-separating curves. Thus, it is a natural question to ask whether we can derive methods for representing an arbitrary periodic $F \in \text{Mod}(S_g)$ as a word $\mathcal{W}(F)$ in Dehn twists, up to conjugacy. In their seminal paper [3], Birman-Hilden derived an expression for $\mathcal{W}(F)$ when is of (largest possible) order $4g+2$ in $\text{Mod}(S_g)$, and consequently for F^{2g+1}, the hyperelliptic involution. This problem was solved for involutions in $\text{Mod}(S_2)$ by Matsumoto [26], which was later generalized to $g > 2$ by Korkmaz [18]. Using techniques in algebraic geometry, Hirose [13] derived expressions for $\mathcal{W}(F)$ for every periodic $F \in$

$\text{Mod}(S_g)$, for $2 \leq g \leq 4$. However, the specialized techniques of Hirose do not easily generalize for $g \geq 5$. In [30], a method was described to decompose an arbitrary periodic mapping class $F \in \text{Mod}(S_g)$ into irreducible components. We use this decomposition to develop various methods in this thesis for deriving $\mathcal{W}(F)$, for an arbitrary periodic mapping class $F \in \text{Mod}(S_g)$. The Dehn twists appearing in $\mathcal{W}(F)$ will depend on the nature of the periodic element F.

1.2 Layout of thesis

This thesis is divided into four chapters. In this first chapter, we have motivated the two problems that have been pursued from a conceptual and historical perspective. In Chapter 2, we provide several results about periodic mapping classes that are relevant to the theory that we will develop in Chapters 3 - 4. In Section 2.1, we introduce some essential notions from the theory of group actions of surfaces [16, 21]. In Section 2.2, we define a tuple of integers called a *data set* that encodes the conjugacy class of a periodic mapping class in $\text{Mod}(S_g)$. This language of data sets will be extensively in subsequent sections. In Section 2.2.1, we describe the method developed in [30] to decompose an arbitrary periodic mapping class into irreducible components. As an application of this decomposition, in Section 2.2.2, we give a procedure (from [30]) for obtaining the symplectic representation of a periodic mapping class up to conjugacy.

In Chapter 3, we provide a complete solution to the Question 1 (from Section 1.1). The research work detailed in this chapter is now part of a published manuscript [8]. In Section 3.1, we derive several properties of the automorphisms induced by periodic maps on quotient orbifolds. These properties together with theory of group actions on surfaces [12, 16, 21] and Thurston's orbifold theory [38] will provide the key ingredients for the proof of the main theorem in Section 3.2.10 that gives equivalent conditions under which two periodic mapping classes *weakly commute* (i.e have commuting conjugates) in $\text{Mod}(S_g)$. We provide several applications of the main result in Section 3.3. In Subsection 3.3.1, we give necessary and sufficient conditions for two involutions in $\text{Mod}(S_g)$ to weakly commute. In Subsection 3.3.2 , we derive conditions for the weak commutativity of irreducible periodics with other periodic mapping classes, and in Subsection 3.3.3, we derive similar conditions for periodic maps that generate free actions on S_g. In Subsection 3.3.4 we give a complete characterization of primitive periodic mapping classes, and in Subsection 3.3.5, we determine conditions under which a finite order mapping class can weakly commute with root of a Dehn twist [27, 32]. Finally, for a given weak conjugacy class of a two-generator finite abelian group, in Subsection 3.3.6, we provide an algorithm for determining the conjugacy classes of its generators. We indicate how this algorithm, along with theory developed in [30], leads to a procedure for determining the explicit hyperbolic structures that realize a two-generator abelian subgroup as a group of isometries. We conclude this chapter by providing some non-trivial geometric realizations of some of these subgroups. It is worth mentioning here that for $g = 3, 4$, Kuribayashi-Kuribayashi [19] have given a complete classification of the conjugacy classes of finite subgroups in $\mathrm{GL}(g, \mathbb{C})$ that arise as images under faithful representations of finite subgroups of $\mathrm{Mod}(S_g)$.

In Chapter 4, we answer Question 2 (from Section 1.1) in the affirmative by providing several methods to represent an arbitrary periodic mapping class $F \in \mathrm{Mod}(S_g)$ as a word $\mathcal{W}(F)$ in Dehn twists (up to conjugacy), based on the nature of F. In Section 4.1, we recall some known relations involving Dehn twists in $\mathrm{Mod}(S)$ that we will use to develop some of our methods in subsequent sections. Using these results, in Section 4.2, we represent the periodic elements in $\mathrm{Mod}(S_1)$ (up to conjugacy) as words in Dehn twists. In Section 4.3, we provide a method for deriving $\mathcal{W}(F)$ when F is *rotational* (i.e realizable a rotation of S_g). In Section 4.4, we use the well known chain relation in $\mathrm{Mod}(S)$ to develop a method (that we call the *chain method*) for representing a large family of periodic mapping classes (that we will call *chain-realizable* periodics) as words in Dehn twists. As an immediate application of the chain method, we represent the torison elements in $\mathrm{Mod}(S_2)$ (up to conjugacy) as words in Dehn twists. In Section 4.5, we generalize the star relation in $\mathrm{Mod}(S_1^3)$ to $\mathrm{Mod}(S_g^3)$, for $g \geq 2$, and then use this to describe a method (that we call the *star method*) for representing a even larger family of periodic mapping classes (that encompasses *chain-realizable* periodics) that we call *star-realizable* periodics, as words in Dehn twists. For an $F \in \mathrm{Mod}(S_g)$ that is neither rotational nor star-realizable, in Section 4.6, we apply results stated in Section 2.2.2, to formulate a method for deriving $\mathcal{W}(F)$. In Section 4.7, we provide two more applications of our methods. In Subsection 4.7.2, we give an algorithm to write certain roots of Dehn twists as words in Dehn twists. By applying the star and symplectic methods, in Subsection 4.7.1, we obtain representations for the torison elements in $\mathrm{Mod}(S_3)$ (up to conjugacy) as words in Dehn twists. Finally in Subsection 4.7.3, we show that the periodic mapping classes of order $4g$ and $4g + 2$ in $\text{Mod}(S_g)$, for $g \geq 2$, have representatives in their respective conjugacy classes whose product (in $\text{Mod}(S_g)$) is pseudo-Anosov.

CHAPTER 2

PERIODIC MAPPING CLASSES

2.1 Group actions on surfaces

A *Fuchsian group* [16, 21] Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$. If $\mathbb{H}/\Gamma$ is a compact orbifold, then Γ has a presentation of the form

$$\left\langle \alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \ldots, \xi_\ell \,|\, \xi_1^{n_1}, \ldots, \xi_\ell^{n_\ell}, \prod_{i=1}^{\ell} \xi_i \prod_{i=1}^{g_0} [\alpha_i, \beta_i] \right\rangle.$$

We represent Γ by a tuple $(g_0; n_1, n_2, \ldots, n_\ell)$ which is called its *signature*, and we write

$$\Gamma(g_0; n_1, n_2, \ldots, n_\ell) := \Gamma.$$

Let $\mathrm{Homeo}^+(S_g)$ denote the group of orientation-preserving homeomorphisms on S_g. Given a finite group $H < \mathrm{Homeo}^+(S_g)$, a faithful properly discontinuous H-action on S_g induces a branched covering

$$S_g \to \mathcal{O}_H := S_g/H,$$

which has ℓ branched points (or cone points) $x_1, \ldots, x_\ell$ on the quotient orbifold $\mathcal{O}_H \approx S_{g_0}$ ($\approx$ denotes "homeomorphic to") of orders $n_1, \ldots, n_\ell$, respectively. Thus, $\mathcal{O}_H$ has a signature given by

$$\Gamma(\mathcal{O}_H) := (g_0; n_1, n_2, \ldots, n_\ell),$$

and its orbifold fundamental group is given by

$$\pi_1^{\mathrm{orb}}(\mathcal{O}_H) := \Gamma(g_0; n_1, n_2, \ldots, n_\ell).$$

From orbifold covering space theory, the orbifold covering map $S_g \to \mathcal{O}_H$ corresponds to an exact sequence

$$1 \to \pi_1(S_g) \to \pi_1^{\text{orb}}(\mathcal{O}_H) \xrightarrow{\phi_H} H \to 1. \tag{2.1}$$

The epimorphism ϕ_H is classically known as the *surface kernel.* This leads us to the following result due to Harvey [12].

Lemma 2.1.1. *A finite group H acts faithfully on S_g with $\Gamma(\mathcal{O}_H) = (g_0; n_1, \ldots, n_\ell)$ if and only if it satisfies the following two conditions:*

(i) $\dfrac{2g-2}{|H|} = 2g_0 - 2 + \displaystyle\sum_{i=1}^{\ell}\left(1 - \frac{1}{n_i}\right)$, *and*

(ii) there exists a surjective homomorphism $\phi_H : \pi_1^{\text{orb}}(\mathcal{O}_H) \to H$ that preserves the orders of all torsion elements of $\pi_1^{\text{orb}}(\mathcal{O}_H)$.

2.2 Cyclic actions on surfaces

For $g \geq 1$, let $H = \langle F \rangle$ be a finite cyclic subgroup of $\text{Mod}(S_g)$ of order n. By the Nielsen realization theorem [29], we may also regard H as a finite cyclic subgroup of $\text{Homeo}^+(S_g)$ generated by an $\mathcal{F}$ of order n. We call $\mathcal{F}$ a *standard representative* of the mapping class F. We refer to both F and the group it generates, interchangeably, as a $\mathbb{Z}_n$*-action on* S_g. Moreover, F corresponds to an orbifold $\mathcal{O}_H \approx S_g/H \approx S_{g_0}$ called the *corresponding orbifold*, where for each i, the cone point x_i lifts to an orbit of size n/n_i on S_g. The local rotation induced by $\mathcal{F}$ around the points in the orbit is given by $2\pi c_i^{-1}/n_i$, where $c_i c_i^{-1} \equiv 1 \pmod{n_i}$. We denote a typical point in $\mathcal{O}_H$ by $[x]$, where x is a lift of $[x]$ under the branched cover $S_g \to \mathcal{O}_H$. We see that each cone point $[x] \in \mathcal{O}_H$ corresponds to a unique pair in the multiset $\{(c_1, n_1), \ldots, (c_\ell, n_\ell)\}$, we denote by (c_x, n_x). So, we define

$$\mathcal{P}_{[x]} := \begin{cases} (c_x, n_x), & \text{if } [x] \text{ is a cone point of } \mathcal{O}_H, \text{ and} \\ (0, 1), & \text{otherwise.} \end{cases}$$

We will now define a tuple of non-negative integers that will combinatorially encode the conjugacy class of a $\mathbb{Z}_n$-action on S_g. We will also associate an additional quantity (denoted by r) with the data set in order to encode free $\mathbb{Z}_n$-actions, which are generated by rotations of S_g by $2\pi r/n$, where $\gcd(r, n) = 1$, see figure below,

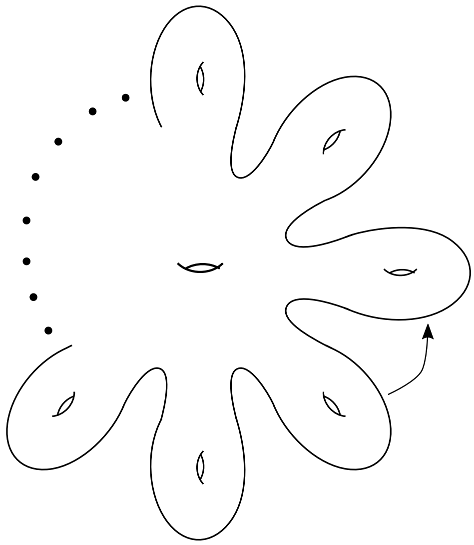


Figure 2.1: Free rotation of surface S_g by $2\pi/(g-1)$

Definition 2.2.1. A *data set of degree* n is a tuple

$$D = (n, g_0, r; (c_1, n_1), \ldots, (c_\ell, n_\ell)),$$

where $n \geq 2$, $g_0 \geq 0$, and $0 \leq r \leq n-1$ are integers, and each $c_i \in \mathbb{Z}_{n_i}^*$ such that:

(i) (a) $r > 0$ if and only if $\ell = 0$, and

(b) $\gcd(r, n) = 1$, whenever $r > 0$,

(ii) each $n_i \mid n$,

(iii) (a) $\mathrm{lcm}(n_1, \ldots \widehat{n_i}, \ldots, n_\ell) = \mathrm{lcm}(n_1, \ldots \widehat{n_j}, \ldots, n_\ell)$, for $1 \leq i \neq j \leq \ell$, and

(b) when $g_0 = 0$, $\mathrm{lcm}(n_1, \ldots \widehat{n_i}, \ldots, n_\ell) = n$, for all $1 \leq i \leq \ell$,

(iv) $\displaystyle\sum_{j=1}^{\ell} \frac{n}{n_j} c_j \equiv 0 \pmod{n}$.

The number g determined by the Riemann-Hurwitz equation

$$\frac{2-2g}{n} = 2 - 2g_0 + \sum_{j=1}^{\ell} \left(\frac{1}{n_j} - 1 \right)$$

is called the *genus* of the data set, denoted by $g(D)$.

Thus, the Nielsen-Kerckhoff theorem also implies that the canonical projection $\mathrm{Homeo}^+(S_g) \to \mathrm{Mod}(S_g)$ induces a bijective correspondence between the conjugacy classes of finite-order maps in $\mathrm{Homeo}^+(S_g)$ and the conjugacy classes of finite-order mapping classes in $\mathrm{Mod}(S_g)$. So, we have the following lemma, which is a consequence of [34, Theorem 3.8] and the results in [12].

Lemma 2.2.2. *For $g \geq 1$ and $n \geq 2$, data sets of degree n and genus g correspond to conjugacy classes of $\mathbb{Z}_n$-actions on S_g.*

Note that r will be non-zero if and only if D represents a free rotation of S_g by $2\pi r/n$ (i.e. a rotation of S_g through an axis by $2\pi r/n$ that does not have any fixed points). We will avoid writing r in the notation of a data set, whenever $r = 0$. Throughout this thesis, we will use data sets to denote the conjugacy classes of cyclic actions on S_g. Given a finite-order mapping class F, we denote the data set associated with its conjugacy class by D_F. Further, for convenience of notation, we also write the data set D as

$$D = (n, g_0, r; ((d_1, m_1), \alpha_1), \ldots, ((d_k, m_k), \alpha_k)),$$

where (d_i, m_i) are the distinct pairs in the multiset $S = \{(c_1, n_1), \ldots, (c_\ell, n_\ell)\}$, and the α_i denote the multiplicity of the pair (d_i, m_i) in S, again for simplicity of notation we don't write α_i whenever it's equal to 1.

Let $F \in \mathrm{Mod}(S_g)$ be of order n. Then F is said to be *rotational* if $\mathcal{F}$ is a rotation of the S_g through an axis by $2\pi r/n$, where $\gcd(r, n) = 1$. It is apparent that $\mathcal{F}$ is either has no fixed points, or $2k$ fixed points which are induced at the points of intersection of the axis of rotation with S_g. Moreover, these fixed points will form k pairs of points (x_i, x_i'), for $1 \leq i \leq k$, such that the sum of the angles of rotation induced by $\mathcal{F}$ around x_i and x_i' add up to 0 modulo 2π. Consequently, we have the following:

Proposition 2.2.3. *Let $F \in \mathrm{Mod}(S_g)$ be a rotational mapping class of order n.*

(i) When $\mathcal{F}$ is a non-free rotation, then D_F has the form

$$(n, g_0; \underbrace{(s, n), (n-s, n), \ldots, (s, n), (n-s, n)}_{k\,pairs}),$$

for integers $k \geq 1$ and $0 < s \leq n-1$ with $\gcd(s, n) = 1$, and $k = 1$, if $n > 2$.

(ii) When $\mathcal{F}$ is a free rotation, then D_F has the form

$$(n, \frac{g-1}{n} + 1, r;).$$

We say F is of *Type 1* if $\Gamma(\mathcal{O}_F)$ has the form $(g_0; n_1, n_2, n)$, and F is said to be of *Type 2* if F is neither rotational nor of Type 1. Gilman [11] showed that a periodic mapping class $F \in \mathrm{Mod}(S_g)$ is irreducible if and only if $\mathcal{O}_F$ is a sphere with three cone points. Thus, F is an irreducible Type 1 mapping class if and only if $\Gamma(\mathcal{O}_F)$ has the form $(0; n_1, n_2, n)$.

2.2.1 Decomposing periodic maps into irreducibles

In [2, 30], a method was described to decompose an arbitrary non-rotational periodic element $F \in \mathrm{Mod}(S_g)$, for $g \geq 2$, into irreducible Type 1 components, which are realized as rotations of certain unique hyperbolic polygons with side-pairings.

Theorem 2.2.4. *For $g \geq 2$, consider an irreducible Type 1 action $F \in \mathrm{Mod}(S_g)$ with*

$$D_F = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n)).$$

Then F can be realized explicitly as the rotation by $\theta_F = 2\pi c_3^{-1}/n$ of a hyperbolic polygon $\mathcal{P}_F$ with a suitable side-pairing $W(\mathcal{P}_F)$, where $\mathcal{P}_F$ is a hyperbolic $k(F)$-gon with

$$k(F) := \begin{cases} 2n, & \text{if } n_1, n_2 \neq 2, \text{ and} \\ n, & \text{otherwise,} \end{cases}$$

and for $0 \leq m \leq n-1$,

$$W(\mathcal{P}_F) = \begin{cases} \prod_{i=1}^{n} a_{2i-1}a_{2i} \text{ with } a_{2m+1}^{-1} \sim a_{2z}, & \text{if } k(F) = 2n, \text{ and} \\ \prod_{i=1}^{n} a_i \text{ with } a_{m+1}^{-1} \sim a_z, & \text{otherwise,} \end{cases}$$

where $z \equiv m + qj \pmod{n}$ with $q = (n/n_2)c_3^{-1}$ and $j = n_2 - c_2$.

Example 2.2.5. Consider the order 8 mapping class $F \in \mathrm{Mod}(S_2)$ with

$$D_F = (8, 0; (1, 2), (3, 8), (1, 8)),$$

which is realized as a $\theta = 2\pi/8$ rotation of a hyperbolic 8-gon $\mathcal{P}_F$ with side-pairing $W(P_F) = a_1a_2a_3a_4a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}$.

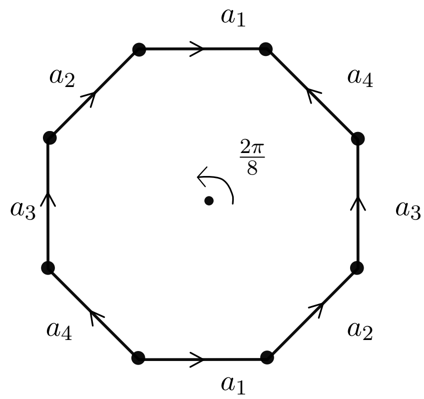


Figure 2.2: A realization of a $\mathbb{Z}_8$-action on S_2

Consequently, the process yielded explicit solutions to the Nielsen realization problem [17, 29] for the cyclic case. Further, it was shown that the process of decomposition can be reversed by piecing together the irreducible Type 1 components (described in Theorem 2.2.4) using the methods that we will now describe in Constructions 2.2.6 and 2.2.8.

Construction 2.2.6 (k-compatibility)**.** For $i = 1, 2$, let $F_i \in \text{Mod}(S_{g_i})$ be of order n. Suppose that the actions of $\langle \mathcal{F}_i \rangle$ on the S_{g_i} induces a pair of orbits O_i such that $|O_1| = |O_2|$ and the rotation angles induced by the $\langle \mathcal{F}_i \rangle$-action around points in the O_i add up to 0 (mod 2π). Then we remove (cyclically permuted) $\langle \mathcal{F}_i \rangle$-invariant disks around points in the O_i and then attach k-annuli A_i connecting the resulting boundary components, to obtain an $F \in \text{Mod}(S_g)$ of order n, where $g(F) := g = g_1 + g_2 + k - 1$. This method of constructing F is called a *k-compatibility*, and we say that F *is realizable as a k-compatible pair* (F_1, F_2) *of genus* $g(F)$. Further, we denote $A(F) := \sqcup_{i=1}^{k} A_i$. A typical 1-compatibility between irreducible Type 1 maps is illustrated in Figure 2.3 below. (For a visualization of a k-compatibility for $k \geq 2$, see Figure 2.5).

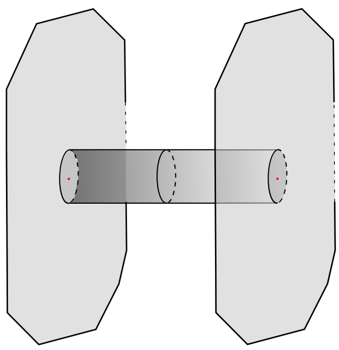


Figure 2.3: A 1-compatibility of a pair of irreducible Type 1 maps.

Let $\Sigma_i(F) := \overline{S_{g_i} \setminus A(F)}$. Then by construction, the maps $\mathcal{F}|_{\Sigma_i(F)}$ and $\mathcal{F}|_{A(F)}$ commute with each other.

If in the construction above, the orbits O_i are induced by a single action on a surface S_g, then the method is called a *self k-compatibility*, wherein the resultant action is on S_{g+k}.

Generalizing the ideas in Construction 2.2.6, we have the following.

Definition 2.2.7. Let $F \in \text{Mod}(S_g)$ be of order n. We say F is a *linear s-tuple* $(F_1, F_2, \ldots F_s)$ *of degree n and genus g* if for $1 \leq i \leq s$, there exists $F_i \in \text{Mod}(S_{g_i})$ of order n satisfying the following conditions.

(i) $F_{i,i+1} := (F_i, F_{i+1})$ is realizable as a k_i-compatible pair of genus $g(F_{i,i+1})$, for $1 \leq i \leq s-1$.

(ii) Let

$$\Sigma_i(F) := \begin{cases} \overline{S_{g_i} \setminus A(F_{i,i+1})}, & \text{if } i = 1, \\ \overline{S_{g_i} \setminus A(F_{i-1,i})}, & \text{if } i = s, \text{ and} \\ \overline{S_{g_i} \setminus (A(F_{i-1,i}) \sqcup A(F_{i,i+1}))}, & \text{for } 2 \leq i \leq s-1. \end{cases}$$

Then $\mathcal{F}|_{\Sigma_i(F)} = \mathcal{F}_i|_{\Sigma_i(F)}$, for $1 \leq i \leq s$.

(iii) $S_g = \sqcup_{i=1}^s \Sigma_i(F) \sqcup_{i=1}^{s-1} A(F_{i,i+1})$, where

$$g = \sum_{i=1}^{s} g_i + \sum_{i=1}^{s-1}(k_i - 1).$$

Given a linear s-tuple $F = (F_1, F_2, \ldots, F_s)$ as in Definition 2.2.7, we denote $g(F) := g$, and further, we fix the following notation that for $1 \leq i < j-1 < s$, we denote

$$F_{i,j} := (F_i, F_{i+1}, \ldots, F_j) \text{ and } \Sigma_{i,j}(F) := \cup_{k=i}^{j} \Sigma_k(F).$$

Construction 2.2.8 (Permutation additions and deletions)**.** The *addition of a g'-permutation component* to a periodic map $\mathcal{F}$ is a process that involves the removal of (cyclically permuted) invariant disks around points in an orbit of size n and then pasting n copies of $S_{g'}^1$ (i.e. $S_{g'}$ with one boundary component) to the resultant boundary components. This realizes an action on $S_{g+ng'}$ with the same fixed point and orbit data as $\mathcal{F}$. A visualization of a permutation addition to an irreducible Type 1 map is shown in Figure 2.4 below.

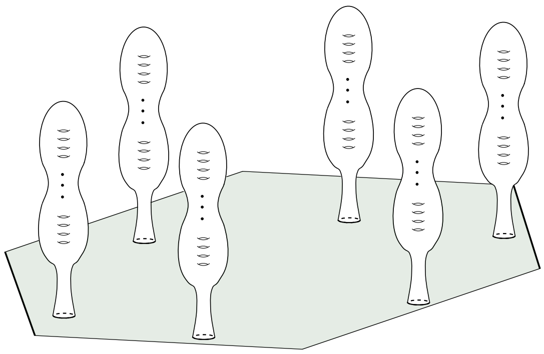


Figure 2.4: Addition of a g'-permutation component to an irreducible Type 1 map.

The reversal of this process, wherein a g'-permutation is removed from $\mathcal{F}$ (when possible), is called the *deletion of g'-permutation component.*

The upshot of the discussion above is the following:

Theorem 2.2.9. *For $g \geq 2$, an arbitrary non-rotational periodic mapping class in* $\mathrm{Mod}(S_g)$ *can be constructed through finitely many k-compatibilities, permutation additions, and permutation deletions on irreducible Type 1 mapping classes.*

Definition 2.2.10. Given integers $s > 0$, $u, v, w \geq 0$, an *admissible* (s, u, v, w)*-tuple* $\mathcal{T}$ is a tuple of integers of the form

$$\mathcal{T} = [((i_1, j_1), k_1), \ldots, ((i_u, j_u), k_u); (i'_1, g'_1), \ldots, (i'_v, g'_v); \\ ((i''_1, j''_1), g''_1), \ldots, ((i''_w, j''_w), g''_w)],$$

where for each q, $1 \leq i_q < j_q \leq s$, $k_q \geq 1$, $1 \leq i'_q \leq s$, $1 \leq i''_q < j''_q \leq s$, and $g'_q, g''_q > 0$.

Definition 2.2.11. Given a linear s-tuple $(F_1, \ldots, F_s)$ of degree n and genus g as in Definition 2.2.7 and an admissible (s, u, v, w)-tuple $\mathcal{T}$ as in Definition 2.2.10, we construct a *compatible* $(F, \mathcal{T})$*-tuple* $F_{\mathcal{T}}$ of degree n and genus $g(F_{\mathcal{T}})$ through the constructions in the following sequence of steps.

Step 1. If $u = 0$, then we skip this step. Otherwise, for $1 \leq q \leq u$, we perform a self k_q-compatibility in $\mathcal{F}_{i_q,j_q}$, if $\mathcal{F}_{i_q,j_q}$ admits such a compatibility.

Step 2. If $v = 0$, then we skip this step. Otherwise, for $1 \leq q \leq v$, we perform a g'_q-permutation addition on the $\mathcal{F}_{i'_q}$.

Step 3. If $w = 0$, then we skip this step. Otherwise, for $1 \leq q \leq w$, we perform a g''_q-permutation deletion on the $\mathcal{F}_{i''_q,j''_q}$, if $\mathcal{F}_{i''_q,j''_q}$ admits such a deletion.

Note that a compatible $(F, \mathcal{T})$-tuple, where $\mathcal{T}$ is an admissible $(s, 0, 0, 0)$-tuple simply refers to the linear s-tuple F. With this notation in place, Theorem 2.2.9 can be now restated as follows.

Theorem 2.2.12. *Given an arbitrary non-rotational periodic mapping class* $G \in \mathrm{Mod}(S_g)$*, for* $g \geq 2$*, there exists a linear* s*-tuple* $F \in \mathrm{Mod}(S_g)$ *of irreducible Type 1 actions, and an admissible* (s, u, v, w)*-tuple* $\mathcal{T}$ *of integers such that* $G = F_{\mathcal{T}}$*.*

2.2.2 Symplectic representations of periodic mapping classes

For $g \geq 1$, let $\Psi : \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g; \mathbb{Z})$ be the surjective representation afforded by the action of $\mathrm{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z})$. In this subsection, we will state some results from [30, Section 4] that are relevant to this thesis.

Let $F \in \mathrm{Mod}(S_g)$ be an irreducible Type 1 action that is realized by the rotation of a hyperbolic polygon $\mathcal{P}_F$ with a boundary word $W(\mathcal{P}_F)$ (when read counterclockwise) as in Theorem 2.2.4. An application of the *handle normalization algorithm* detailed in [35, Section 3.4] shows that $W(\mathcal{P}_F)$ is of the form $QaRbSa^{-1}Tb^{-1}U$, for some words Q, R, S, T, U (possibly empty), and letters a, b, and we have the following.

Proposition 2.2.13. *Let* $W(\mathcal{P}_F) = QaRbSa^{-1}Tb^{-1}U$*. Suppose that* $\mathcal{P}'$ *is the polygon with boundary word* $W(\mathcal{P}') = QTSRUxyx^{-1}y^{-1}$ *obtained by applying the handle normalization algorithm once to* $\mathcal{P}_F$*. Then* x *and* y *are homotopically equivalent to* $QTb^{-1}U$ *and* $U^{-1}R^{-1}a^{-1}Tb^{-1}U$*, respectively.*

We fix the following notation.

(a) We denote by $\mathcal{N}^i(\mathcal{P}_F)$, the polygon obtained from $\mathcal{P}_F$ after i successive applications of the normalization procedure described in Proposition 2.2.13.

(b) We denote by $L(\mathcal{P}_F)$, the set of distinct letters in $W(\mathcal{P}_F)$.

(c) We denote by $\mathcal{B}(\mathcal{P}_F)$, the set of standard generators of $H_1(S_g; \mathbb{Z})$ expressed in terms of elements in $L(\mathcal{P}_F)$.

Let $W = W(\mathcal{P}_F)$ and $W' = W(\mathcal{P}')$ be as in Proposition 2.2.13. Then the map

$$\mathcal{B}(\mathcal{P}') \to \mathcal{B}(\mathcal{P}_F) \,:\, x \mapsto QTb^{-1}U,\, y \mapsto U^{-1}R^{-1}a^{-1}Tb^{-1}U,\, z \mapsto z,$$

for all $z \in \mathcal{B}(\mathcal{P}') \setminus \{x, y\}$, uniquely determines an isomorphism on $H_1(S_g;\mathbb{Z})$, which we denote by $f_{\mathcal{P}',\mathcal{P}_F}$, which leads us to the following lemma.

Lemma 2.2.14. *Let $F \in \mathrm{Mod}(S_g)$ be an irreducible Type 1 action that is realized by the rotation of a hyperbolic polygon $\mathcal{P}_F$ as in Theorem 2.2.4. Then*

$$W(\mathcal{N}^g(\mathcal{P}_F)) = \prod_{i=1}^{g}[x_i, y_i],$$

and the mapping

$$f_{\mathcal{P}_F} = \prod_{i=1}^{g-1} f_{\mathcal{N}^i(\mathcal{P}_F),\mathcal{N}^{i-1}(\mathcal{P}_F)}$$

defines an isomorphism of the homology group $H_1(S_g;\mathbb{Z})$ such that

$$\mathcal{B}(\mathcal{P}_F) \xmapsto{f_{\mathcal{P}_F}} \mathcal{B}(\mathcal{P}_F).$$

For an isomorphism $\varphi : H_1(S_g;\mathbb{Z}) \to H_1(S_g;\mathbb{Z})$, let M_φ denote the matrix of φ with respect to the standard homology generators. The following theorem describes the structure of $\Psi(F)$, up to conjugacy.

Theorem 2.2.15. *Let $F \in \mathrm{Mod}(S_g)$ be an irreducible Type 1 with $D_F = ((n, 0; (c_1, n_1), (c_2, n_2), (c_3, n))$. Then up to conjugacy, $\Psi(F) = M_\varphi$, where $\varphi = f_{\mathcal{P}_F}^{-1}\phi_{\mathcal{P}_F} f_{\mathcal{P}_F}$, with $f_{\mathcal{P}_F}$ as in Lemma 2.2.14, and $\mathcal{B}(\mathcal{P}_F) \xmapsto{\phi_{\mathcal{P}_F}} \mathcal{B}(\mathcal{P}_F)$ is induced by $a_i \mapsto a_j$, where*

$$j \equiv \begin{cases} i + 2c_3^{-1} \pmod{2n}, & \text{if } n_1, n_2 \neq 2, \text{ and} \\ i + c_3^{-1} \pmod{n}, & \text{otherwise.} \end{cases}$$

We conclude this subsection with the following remark.

Remark 2.2.16. By Theorem 2.2.9, an arbitrary non-rotational periodic mapping class $F \in \mathrm{Mod}(S_g)$ can be decomposed into irreducible Type 1 mapping classes. This decomposition induces a decomposition of $\Psi(F)$ (up to conjugacy) into a block-diagonal matrix, where each diagonal block is of one of the following types.

(i) The image under Ψ of an irreducible Type 1 component (of F) as described in Theorem 2.2.15.

(ii) Let F' be a component of F resulting from a k-compatibility (or a self k-compatibility), and let S_g^b denote the surface of genus g with b boundary components. Then there exists a subsurface S (of S_g) homeomorphic to S_{k-1}^2 (shown in Figure 2.5 below) in which $\mathcal{F}'$ cyclically permutes the disjoint union of the k annuli $A_F \subset S$ involved in the construction.

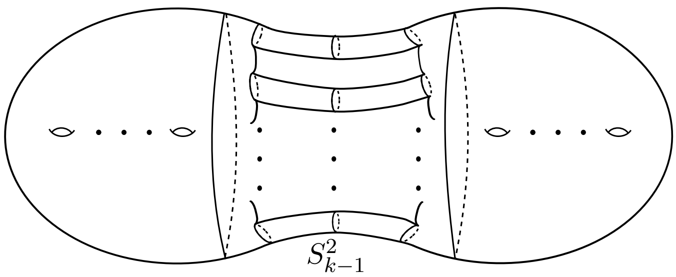


Figure 2.5: The subsurface $S \approx S_{k-1}^2$.

The diagonal block is obtained from the well-defined action of such an F' on $H_1(S, \mathbb{Z})$.

(iii) The image under Ψ under a permutation component of F, which permutes n subsurfaces of S_g homeomorphic to some $S_{g'}^1$ as in Construction 2.2.8.

Note that as the blocks of type (ii) and (iii) are simple permutation blocks, one can obtain a complete description of $\Psi(F)$ (up to conjugacy).

CHAPTER 3

COMMUTING CONJUGATES OF PERIODIC MAPPING CLASSES

In this chapter, we will apply the concepts introduced in Chapter 2 to provide a complete answer to Question 1 posed in Section 1.1.

3.1 Induced automorphisms on quotient orbifolds

Consider a finite group $H < \text{Homeo}^+(S_g)$, and a subgroup $H' \lhd H$. Then it is known [40] that the actions of H and H' on S_g together induce an action of H/H' on $\mathcal{O}_{H'}(= S_g/H')$. In this section, we analyze this induced action for the case when H is a two-generator finite abelian group, and H' is one of its cyclic factor subgroups. We will derive several properties of these induced actions, which will form the core of the theory developed in this thesis.

Definition 3.1.1. Let $H < \text{Homeo}^+(S_g)$ be a finite cyclic group. We say a $\bar{\mathcal{F}} \in \text{Homeo}^+(\mathcal{O}_H)$ is an *automorphism of* $\mathcal{O}_H$ if for $[x], [y] \in \mathcal{O}_H$, we have $\mathcal{P}_{[x]} = \mathcal{P}_{[y]}$, whenever $\bar{\mathcal{F}}([x]) = [y]$.

We denote the group of automorphisms of $\mathcal{O}_H$ by $\text{Aut}(\mathcal{O}_H)$. We derive three technical lemmas, which give necessary conditions under which a given orbifold automorphism is induced by a finite-order map. These lemmas will be used extensively in subsequent sections.

Lemma 3.1.2. *Let $\mathcal{G}, \mathcal{F} \in \text{Homeo}^+(S_g)$ be commuting maps of order m, n, respectively, and let $H = \langle \mathcal{F} \rangle$. Then:*

(i) $\mathcal{G}$ induces a $\bar{\mathcal{G}} \in \text{Homeo}^+(\mathcal{O}_H)$ such that

$$\mathcal{O}_H/\langle \bar{\mathcal{G}} \rangle = S_g/\langle \mathcal{F}, \mathcal{G} \rangle,$$

(ii) $|\bar{\mathcal{G}}|$ divides $|\mathcal{G}|$, and

(iii) $|\bar{\mathcal{G}}| < m$ *if and only if* $\mathcal{F}^l = \mathcal{G}^k$, *for some* $0 < l < n$ *and* $0 < k < m$.

Proof. Given $[x] \in S_g/\langle \mathcal{F} \rangle$, we define $\bar{\mathcal{G}}([x]) := [\mathcal{G}(x)]$. The assertion in (i) now follows immediately from this definition. Moreover, (ii) follows from the fact that

$$\bar{\mathcal{G}}^m([x]) = [\mathcal{G}^m(x)] = [x], \text{ for } [x] \in S_g/\langle \mathcal{F} \rangle.$$

To prove (iii), we first assume that $t := |\bar{\mathcal{G}}| < m$. Then

$$\bar{\mathcal{G}}^t([x]) = [x] \Leftrightarrow [\mathcal{G}^t(x)] = [x],$$

for all $[x] \in \mathcal{O}_H$. Thus, for each $[x] \in \mathcal{O}_H$, there exists $1 \leq l_x \leq n$ such that $\mathcal{G}^t \mathcal{F}^{l_x}(y) = y$, for all $y \in S_g$ in the preimage of $[x]$ under the branched cover $S_g \to \mathcal{O}_H$. Suppose we assume on the contrary that $\mathcal{F}^l \neq \mathcal{G}^k$, for $1 \leq l < n$ and $1 \leq k < m$. Then, since $t < m$, for each l_x, $\mathcal{G}^t \mathcal{F}^{l_x}$ is a non-trivial homeomorphism. This would imply that every point of S_g is fixed by some element of the abelian group $\langle \mathcal{F}, \mathcal{G} \rangle$ of order mn, which is impossible (as the action of $\langle \mathcal{F}, \mathcal{G} \rangle$ on S_g is properly discontinuous). The converse follows directly from the definition of $\bar{\mathcal{G}}$.

□

We call the map $\bar{\mathcal{G}}$ in Lemma 3.1.2 the *induced map on* $\mathcal{O}_{\langle \mathcal{F} \rangle}$ *by* $\mathcal{G}$. For an action of a group H on a set X, we denote the stabilizer of a point $x \in X$ by $\mathrm{Stab}_H(x)$. We will also need the following well-known result [28, Proposition 3.1] from the theory of finite group actions on surfaces.

Lemma 3.1.3. *Let* $H < \mathrm{Homeo}^+(S_g)$ *be finite. Then* $\mathrm{Stab}_H(x)$ *is a cyclic group, for every* $x \in S_g$.

Lemma 3.1.4. *Let* $\mathcal{F}, \mathcal{G} \in \mathrm{Homeo}^+(S_g)$ *be of orders* n, m, *respectively, and let* $\bar{\mathcal{F}} \in \mathrm{Homeo}^+(\mathcal{O}_{\langle \mathcal{G} \rangle})$ *be induced by* $\mathcal{F}$ *as in Lemma 3.1.2. Suppose that* $\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F}$, *and* $\mathcal{F}^p \neq \mathcal{G}^q$, *for any* $1 \leq p < n$ *and* $1 \leq q < m$. *If for some* $x \in S_g$, $\mathcal{G}^k(x) = x$ *and* $\bar{\mathcal{F}}^l([x]) = [x]$, *for some* $1 \leq k < m$ *and* $1 \leq l < n$, *then*

$$|\bar{\mathcal{F}}^l| = ba,$$

where $\gcd(b, |\mathcal{G}^k|) = 1$ *and* $a \mid \frac{m}{|\mathcal{G}^k|}$.

Proof. Let $|\bar{\mathcal{F}}^l| = t$, $\gcd(t, \frac{m}{|\mathcal{G}^k|}) = a$, and $b = t/a$. Since $\bar{\mathcal{F}}^l([x]) = [x]$, there exists $\mathcal{G}' \in \langle \mathcal{G} \rangle$ such that $\mathcal{G}'\mathcal{F}^l(x) = x$ and $|\mathcal{G}'\mathcal{F}^l| = t'$. Moreover, $\mathrm{Stab}_{\langle \mathcal{F},\mathcal{G}\rangle}(x)$ contains $\mathcal{G}'\mathcal{F}^l$ and $\mathcal{G}^k$. Suppose we assume on the contrary that $\gcd(b, |\mathcal{G}^k|) = \alpha \neq 1$. Then $|(\mathcal{G}^k)^{\frac{|\mathcal{G}^k|}{\alpha}}| = |(\mathcal{G}'\mathcal{F}^l)^{\frac{t'}{\alpha}}| = \alpha$, which is impossible, as a finite cyclic group has a unique subgroup of a given order dividing its order. □

Lemma 3.1.5. *Let $\mathcal{G}, \mathcal{F} \in \mathrm{Homeo}^+(S_g)$ be commuting homeomorphisms of orders m, n, respectively. Let $\bar{\mathcal{F}}$ be the induced map on $S_g/\langle \mathcal{G} \rangle$ as in Lemma 3.1.2. Then:*

(i) For $[x], [y] \in \mathcal{O}_{\langle \mathcal{G} \rangle}$, if $\bar{\mathcal{F}}([x]) = ([y])$, then $\mathcal{P}_x = \mathcal{P}_y$.

(ii) Consider the action of $\langle \bar{\mathcal{F}} \rangle$ on $\mathcal{O}_{\langle \mathcal{G} \rangle}$ and the action of $\bar{\mathcal{G}}$ on $\mathcal{O}_{\langle \mathcal{F} \rangle} \approx S_{g_1}$. For each orbit O of size $|\bar{\mathcal{F}}|$, there exists a point $[x(O)] \in \mathcal{O}_{\bar{\mathcal{G}}}$ and a $[y] \in O$ such that $\mathcal{P}_{[x(O)]} = \mathcal{P}_{[y]}$.

(iii) Let $\mathcal{F}$ have β fixed points in S_g. If $\bar{\beta}$ denotes the number of fixed points of $\bar{\mathcal{F}}$, then

$$\left\lceil \frac{\beta}{m} \right\rceil \leq \bar{\beta} \leq \left\lfloor \frac{(m-1)(2g-2+2n)}{m(n-1)} \right\rfloor + \left\lceil \frac{\beta}{m} \right\rceil.$$

Proof. (i) Suppose that $\bar{\mathcal{F}}([x]) = [y]$. Then there exists $x', y' \in S_g$ in the pre-images of $[x], [y]$ (under the branched cover) such that $\mathcal{F}(x') = y'$. Then

$$\mathcal{G}^{m/n_x}(y') = \mathcal{G}^{m/n_x}(\mathcal{F}(x')) = \mathcal{F}(\mathcal{G}^{m/n_x}(x')) = \mathcal{F}(x') = y',$$

where $\mathcal{P}_{[x]} = (c_x, n_x)$ (We had fixed this notation in the discussion preceding Definition 2.2.1). By a similar argument, we can show that $\mathcal{G}^{m/n_y}(x') = x'$, and so it follows that $n_x = n_y$.

To show that $c_x = c_y$, we first consider the case when $n_x = n_y = m$, then $c_x = c_y$. Without loss of generality, we assume that $c_y = 1$. Now, there exists an $\mathcal{G}$-invariant disk D_2 around y that $\mathcal{G}$ rotates by $2\pi/m$, and there exists a $\mathcal{G}$-invariant disk D_1 around x that $\mathcal{G}$ rotates by $2\pi c_x^{-1}/m$. So, we must have $\mathcal{F}\mathcal{G}\mathcal{F}^{-1} = \mathcal{G}^{c_x^{-1}}$, which is only possible when $c_x = 1$, as $\mathcal{F}$ and $\mathcal{G}$ commute. For the case when $n_x = n_y < m$, we may apply the same arguments as above to the subgroup $\langle \mathcal{F}, \mathcal{G}^{n/n_x} \rangle$ (of $\langle \mathcal{F}, \mathcal{G} \rangle$) to prove the required assertion.

(ii) Suppose that $\mathcal{G}$ has n fixed points $\{x_1, \ldots, x_n\}$ in S_g that form an orbit under the action of $\mathcal{F}$ on S_g. Then clearly $\{[x_1], \ldots, [x_n]\}$ forms an orbit of size $|\bar{\mathcal{F}}|$ under the action of $\langle \bar{\mathcal{F}} \rangle$ on $\mathcal{O}_{\langle \mathcal{G} \rangle}$. Moreover, $[x] = \{x_1, \ldots, x_n\}$ is a point in $\mathcal{O}_{\langle \mathcal{F} \rangle} \approx S_{g_1}$, which is fixed under the action of $\bar{\mathcal{G}}$. Thus, the assertion follows.

(iii) If $\mathcal{F}(x) = x$, then by definition, $\bar{\mathcal{F}}([x]) = [x]$, and so we have $\mathcal{F}(\mathcal{G}^i(x)) = \mathcal{G}^i(x)$, for each i. If $\mathcal{F}$ has β fixed points, then there exist at least $\frac{\beta}{m}$ distinct orbits which contain points fixed by $\mathcal{F}$. Hence, the lower bound follows.

To show the upper bound, we observe that if $\bar{\mathcal{F}}([x]) = [x]$, then by definition, there exists $0 \leq i \leq m-1$ such that $\mathcal{G}^i\mathcal{F}(x) = x$. When $i \neq 0$, by a direct application of the Riemann-Hurwitz equation, it follows that $\left\lfloor \frac{(2g-2+2n)}{(n-1)} \right\rfloor$ is the maximum number cone points of order n in $\mathcal{O}_{\langle \mathcal{G}^i \mathcal{F} \rangle}$, which completes the argument.

□

The necessary conditions that appear in lemmas above, under which a given orbifold automorphism is induced, are summarized in the following two definitions.

Definition 3.1.6. Let $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$ be of orders n and m respectively, and let $H = \langle \mathcal{G} \rangle$. We say a map $\bar{\mathcal{F}} \in \text{Aut}(\mathcal{O}_H)$ satisfies the *induced map property (IMP) with respect to* $(\mathcal{F}, \mathcal{G})$, if the following conditions hold.

(i) For $[x], [y] \in \mathcal{O}_H$, if $\bar{\mathcal{F}}([x]) = ([y])$, we have $\mathcal{P}_x = \mathcal{P}_y$.

(ii) Consider the action of $\langle \bar{\mathcal{F}} \rangle$ on $\mathcal{O}_{\langle \mathcal{G} \rangle}$ and the action of $\bar{\mathcal{G}}$ on $\mathcal{O}_{\langle \mathcal{F} \rangle} \approx S_{g_1}$. For each orbit O of size $|\bar{\mathcal{F}}|$, there exists a point $[x(O)] \in \mathcal{O}_{\bar{\mathcal{G}}}$ and a $[y] \in O$ such that $\mathcal{P}_{[x(O)]} = \mathcal{P}_{[y]}$.

(iii) Let $\mathcal{F}$ have β fixed points in S_g. If $\bar{\beta}$ denotes the number of fixed points of $\bar{\mathcal{F}}$, then

$$\left\lceil \frac{\beta}{m} \right\rceil \leq \bar{\beta} \leq \left\lfloor \frac{(m-1)(2g-2+2n)}{m(n-1)} \right\rfloor + \left\lceil \frac{\beta}{m} \right\rceil .$$

(iv) If $[x]$ is a cone point of order n' in $\mathcal{O}_H$, then $\bar{\mathcal{F}}^l([x]) = [x]$, only if $\mid \bar{\mathcal{F}}^l \mid = ba$, where $\gcd(b, n') = 1$ and $a \mid \frac{m}{n'}$.

Definition 3.1.7. Let $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$ be finite-order maps with $D_F = (n, g_1, r_1; ((c_1, n_1), \alpha_1), \ldots, ((c_r, n_r), \alpha_r))$ and $D_G = (m, g_2, r_2; ((d_1, m_1), \beta_1), \ldots, ((d_k, m_k), \beta_k))$, where $m \mid n$. Then $(\mathcal{G}, \mathcal{F})$ are said to form an *essential pair* if the following three conditions hold.

(i) There exists an $\mathcal{F}' \in \text{Homeo}^+(S_{g_2})$ with $D_{\mathcal{F}'} = (n, g_0, r_1^o; (c_1^o, n_1^o), \ldots, (c_s^o, n_s^o))$ on S_{g_2} which induces an $\bar{\mathcal{F}} \in \text{Aut}(O_{\langle \mathcal{G} \rangle})$ that satisfies the IMP with respect to $(\mathcal{F}, \mathcal{G})$.

(ii) There exists a $\mathcal{G}' \in \text{Homeo}^+(S_{g_1})$ with $D_{\mathcal{G}'} = (m, g_0, r_2^o; (d_1^o, m_1^o), \ldots, (d_t^o, m_t^o))$, which induces a $\bar{\mathcal{G}} \in \text{Aut}(\mathcal{O}_{\langle \mathcal{F} \rangle})$ that satisfies the IMP with respect to $(\mathcal{G}, \mathcal{F})$.

(iii) $\Gamma(\mathcal{O}_{\langle \mathcal{G} \rangle}/\langle \bar{\mathcal{F}} \rangle) = \Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}/\langle \bar{\mathcal{G}} \rangle)$.

The number mn (written as $m \cdot n$) is called the *order* of the essential pair $(\mathcal{G}, \mathcal{F})$.

Example 3.1.8. Let $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_7)$ with $D_{\mathcal{F}} = D_{\mathcal{G}} = (6, 2, 1;)$. Then $(\mathcal{G}, \mathcal{F})$ is an essential pair of order $6 \cdot 6$, as $\mathcal{F}, \mathcal{G}$ induce $\bar{\mathcal{F}}, \bar{\mathcal{G}} \in \text{Homeo}^+(S_2)$ (resp.) with $D_{\bar{\mathcal{F}}} = D_{\bar{\mathcal{G}}} = (6, 0; ((1, 2), 2), (1, 3), (2, 3))$, and $\Gamma(\mathcal{O}_{\langle \mathcal{G} \rangle}/\langle \bar{\mathcal{F}} \rangle) = \Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}/\langle \bar{\mathcal{G}} \rangle) = (0; 2, 2, 3, 3)$.

Given a quotient orbifold $\mathcal{O}_H$, where $H = \langle \mathcal{F} \rangle$, we now state a set of necessary conditions (as we will show later in Theorem 3.2.10) for a given $\bar{\mathcal{G}} \in \text{Aut}(\mathcal{O}_H)$ to be induced by a finite-order map $\mathcal{G}$ such that $\langle \mathcal{G}, \mathcal{F} \rangle$ forms a two-generator abelian group.

Definition 3.1.9. For finite-order maps $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$, let $(\mathcal{G}, \mathcal{F})$ form an essential pair of order $m \cdot n$ as in Definition 3.1.7. Then $(\mathcal{G}, \mathcal{F})$ is said to be a *weakly abelian pair of order* $m \cdot n$ if the following conditions hold.

(i) If $\Gamma(\mathcal{O}_{\langle \mathcal{G} \rangle}/\langle \bar{\mathcal{F}} \rangle) = \Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}/\langle \bar{\mathcal{G}} \rangle) = (g_0; m_1'n_1', \ldots, m_l'n_l')$ such that for each i, $m_i'n_i' \neq 1$ and $m_i'n_i' \mid n$.

(ii) If $g_0 = 0$ in condition (i), then there exists a sub-multiset $A = \{n_{11}, \ldots, n_{l1}\}$ of the multiset $B = \{m_1'n_1', \ldots, m_l'n_l'\}$ such that $\text{lcm}(\widehat{A}) = \text{lcm}(\{n_{11}, \ldots, \widehat{n_{i1}}, \ldots, n_{l1}\}) = n$ and $m \mid \text{lcm}(B \setminus A)$.

(iii) (a) Denoting $\text{lcm}(\{m_k'n_k' : m_k' \neq 1\}) = B_1$, if $\displaystyle\sum_{n_i' \neq 1} \frac{n}{\gcd(n, n_i'm_i')} c_i \equiv -\delta_2 \pmod{n}$, where $m_i' \in \{1, m_1^o, \ldots, m_t^o\}$ and $n_i' \in \{1, n_1, \ldots, n_r\}$, then $\frac{n}{B_1} | \delta_2$.

(b) Denoting $\text{lcm}(\{m_l'n_l' : n_l' \neq 1\}) = \bar{B}_2$, and $\gcd(\bar{B}_2, m) = B_2$, if $\displaystyle\sum_{m_i' \neq 1} \frac{m}{\gcd(m, m_i'n_i')} d_i \equiv -\delta_1 \pmod{m}$, where $m_i' \in \{1, m_1, \ldots, m_k\}$ and $n_i' \in \{1, n_1^o, \ldots, n_s^o\}$, then $\frac{m}{B_2} | \delta_1$.

Example 3.1.10. Let $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_2)$ with $D_F = (6, 0; ((1, 6), 2), (2, 3))$, $D_G = (2, 0; ((1, 2), 6))$, respectively. Then $(\mathcal{G}, \mathcal{F})$ is an essential pair of order $2 \cdot 6$, with $D_{\bar{\mathcal{F}}} =$

$(6,0;(1,6),(5,6))$ and $D_{\bar{\mathcal{G}}} = (2,0;((1,2),2))$, where

$$\Gamma(\mathcal{O}_{\langle \mathcal{G} \rangle}/\langle \bar{\mathcal{F}} \rangle) = \Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}/\langle \bar{\mathcal{G}} \rangle) = (0;2,6,6).$$

It is easy to check that $(\mathcal{G},\mathcal{F})$ is also a weak abelian pair of order $2 \cdot 6$.

Note that the $\mathcal{F}$ and $\mathcal{G}$ in Example 3.1.8 do not form a weakly abelian pair $(\mathcal{G},\mathcal{F})$, as they do not satisfy condition (ii) of Definition 3.1.9.

In order to improve the clarity of exposition, we will divide the proof of our main result into four subcases, of which the first two cases (that will form bulk of our proof) assume the following condition on the quotient orbifolds (of the cyclic factor subgroups).

Definition 3.1.11. Let $H < \text{Homeo}^+(S_g)$ be a finite cyclic group, and let $\Gamma(\mathcal{O}_H) = (g_0; n_1, \dots, n_\ell)$. We say the action of H on S_g satisfies the *lcm condition* if

$$\text{lcm}(\{n_1, \dots, n_\ell\}) = |H|.$$

We conclude this section with another lemma that will be used in one of the subcases of our main result.

Lemma 3.1.12. *Let $\mathcal{F},\mathcal{G} \in \text{Homeo}^+(S_g)$ be of orders n and m, respectively. If $\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F}$ and $S_g/\langle \mathcal{F},\mathcal{G}\rangle \approx S_0$, then there exists an $\mathcal{F}' \in \langle \mathcal{F},\mathcal{G}\rangle$ of order n such that the action of $\langle \mathcal{F}' \rangle$ on S_g satisfies the lcm condition.*

Proof. Consider the map $\bar{\mathcal{F}} \in \text{Aut}(\mathcal{O}_{\langle \mathcal{G} \rangle})$ induced by $\mathcal{F}$. Since $\mathcal{O}_{\langle \mathcal{G} \rangle}/\langle \bar{\mathcal{F}} \rangle = S_g/\langle \mathcal{F},\mathcal{G}\rangle$ (in view of Lemma 3.1.2 (i)) the action of $\bar{\mathcal{F}}$ on $\mathcal{O}_{\langle \mathcal{G} \rangle}$ satisfies the lcm condition. Let $D_{\bar{\mathcal{F}}} = (n, 0; (c_1', n_1'), \dots, (c_s', n_s'))$. Consider a minimal subset $\{n_{11}, \dots, n_{1l}\}$ of the multiset $\{n_1', n_2', \dots, n_s'\}$ with the property $\text{lcm}(\{n_{11}, \dots, n_{1l}\}) = n$. Now, for each n_{1i}, there exists l_i such that $\mathcal{G}^{l_i}\mathcal{F}^{\frac{nc_{1i}}{n_{1i}}}(x_i) = x_i$, for some $x_i \in S_g$. It is apparent that $|\mathcal{G}^{l_i}\mathcal{F}^{\frac{nc_{1i}}{n_{1i}}}| \geq n_{1i}$. For each $1 \leq i \leq l$, we choose an appropriate power of $\mathcal{G}^{l_i}\mathcal{F}^{\frac{nc_{1i}}{n_{1i}}}$ that we denote by $\mathcal{F}_i'$, so that $\gcd(|\mathcal{F}_i'|, |\mathcal{F}_j'|) = 1$, when $i \neq j$, and $\text{lcm}(\{|\mathcal{F}_1'|, \dots, |\mathcal{F}_l'|\} = n$. Thus, the assertion follows by choosing $\mathcal{F}' = \mathcal{F}_1'\mathcal{F}_2' \dots \mathcal{F}_l'$. □

3.2 Main theorem

By a *two-generator finite abelian action of order mn* (written as $m \cdot n$), we mean a tuple $(H, (\mathcal{G}, \mathcal{F}))$, where $m \mid n$, $H < \text{Homeo}^+(S_g)$, and

$$H = \langle \mathcal{G}, \mathcal{F} \,|\, \mathcal{G}^m = \mathcal{F}^n = 1,\, [\mathcal{F}, \mathcal{G}] = 1 \rangle.$$

Definition 3.2.1. Two finite abelian actions $(H_1, (\mathcal{G}_1, \mathcal{F}_1))$ and $(H_2, (\mathcal{G}_2, \mathcal{F}_2))$ of order $m \cdot n$ are said to be *weakly conjugate* if there exists an isomorphism, $\psi : \pi_1^{\text{orb}}(\mathcal{O}_{H_1}) \to \pi_1^{\text{orb}}(\mathcal{O}_{H_2})$ and an isomorphism $\chi : H_1 \to H_2$ such that the following conditions hold.

(i) $\chi((\mathcal{G}_1, \mathcal{F}_1)) = (\mathcal{G}_2, \mathcal{F}_2)$.

(ii) For $i = 1, 2$, let $\phi_{H_i} : \pi_1^{\text{orb}}(\mathcal{O}_{H_i}) \to H_i$ be the surface kernel (appearing in the exact sequence (2.1) in Section 2.1). Then $(\chi \circ \phi_{H_1})(g) = (\phi_{H_2} \circ \psi)(g)$, whenever $g \in \pi_1^{\text{orb}}(\mathcal{O}_{H_1})$ is of finite order.

(iii) The pair $(\mathcal{G}_1, \mathcal{F}_1)$ is conjugate (component-wise) to the pair $(\mathcal{G}_2, \mathcal{F}_2)$ in $\text{Homeo}^+(S_g)$.

The notion of weak conjugacy induces an equivalence relation on the two-generator finite abelian subgroups of $\text{Homeo}^+(S_g)$, and we will call the equivalence classes as *weak conjugacy classes*.

Remark 3.2.2. It is important to note that while the notion of weak conjugacy induces an equivalence relation on the set of all triples of the form $(H, (\mathcal{G}, \mathcal{F}))$, (unlike conjugacy,) it does not define an equivalence relation on the set of finite abelian subgroups of $\text{Homeo}^+(S_g)$ in the traditional sense.

From Remark 3.2.2, it is apparent that weak-conjugacy and conjugacy are fundamentally different notions. In fact, it is possible for two finite non-conjugate abelian subgroups of $\text{Mod}(S_g)$ to be weakly conjugate.

Example 3.2.3. In [4, Example 5.1], Broughton-Wootton have shown that there exists 2 conjugacy classes of a free $\mathbb{Z}_5 \oplus \mathbb{Z}_5$-action on S_{26}. But from Definition 3.2.1, it is apparent that there exists only one weakly conjugacy class of this action.

We will now define an abstract tuple of integers that encode, as we will see shortly in Proposition 3.2.5, the weak conjugacy class of a two-generator finite abelian action.

Definition 3.2.4. An *abelian data set* of *degree* $m \cdot n$ and *genus* g is a tuple

$$(m \cdot n, g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{r1}, n_{r1}), (c_{r2}, n_{r2}), n_r]),$$

where $m, n \geq 2$, $g_0 \geq 0$, and $g \geq 2$ are integers satisfying the following conditions:

(a) $m \mid n$,

(b) $\dfrac{2g-2}{mn} = 2g_0 - 2 + \displaystyle\sum_{i=1}^{r} \left(1 - \frac{1}{n_i}\right)$,

(c) $\mathrm{lcm}(n_1, \ldots, n_r) = \mathrm{lcm}(n_1, \ldots, \widehat{n}_k, \ldots, n_r) = N$, and if $g_0 = 0$, then $N = n$,

(d) for each i, $n_{i1}|m$, $n_{i2}|n$, and $\mathrm{lcm}(n_{i1}, n_{i2}) = n_i$,

(e) for each i, j, either $\gcd(c_{ij}, n_{ij}) = 1$, or $c_{ij} = 0$, and $c_{ij} = 0$, if and only if $n_{ij} = 1$,

(f) $\displaystyle\sum_{i=1}^{r} \frac{m}{n_{i1}} c_{i1} \equiv 0 \pmod m$ and $\displaystyle\sum_{i=1}^{r} \frac{n}{n_{i2}} c_{i2} \equiv 0 \pmod n$, and

(g) when $g_0 = 0$, there exists $(\ell_1, \ldots, \ell_r), (k_1, \ldots, k_r) \in \mathbb{Z}^r$ such that

 (i) $\displaystyle\sum_{i=1}^{r} \frac{m}{n_{i1}} c_{i1} \ell_i \equiv 0 \pmod m$ and $\displaystyle\sum_{i=1}^{r} \frac{n}{n_{i2}} c_{i2} \ell_i \equiv 1 \pmod n$, and

 (ii) $\displaystyle\sum_{i=1}^{r} \frac{m}{n_{i1}} c_{i1} k_i \equiv 1 \pmod m$ and $\displaystyle\sum_{i=1}^{r} \frac{n}{n_{i2}} c_{i2} k_i \equiv 0 \pmod n$.

Proposition 3.2.5. *For $m, n, g \geq 2$ and $m \mid n$, abelian data sets of degree $m \cdot n$ and genus g correspond to the weak conjugacy classes of $\mathbb{Z}_m \oplus \mathbb{Z}_n$-actions on S_g.*

Proof. Let $H = \mathbb{Z}_m \oplus \mathbb{Z}_n$, $\Gamma(\mathcal{O}_H) = (g_0; n_1, \ldots, n_r)$, and $\Gamma(g_0; n_1, \ldots, n_r) = \Gamma$. Let D be an abelian data set of degree $m \cdot n$ and genus g as in Definition 3.2.4. By Lemma 2.1.1, it suffices to show there exists a surjective map $\phi : \pi_1^{\mathrm{orb}}(\mathcal{O}_H) \to H$ that preserves the order of torsion elements. Let the presentations of Γ and $\mathbb{Z}_m \oplus \mathbb{Z}_n$ be given by

$$\langle \alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \ldots, \xi_r \mid \xi_1^{n_1} = \cdots = \xi_r^{n_r} = \prod_{i=1}^{r} \xi_i \prod_{i=1}^{g_0} [\alpha_i, \beta_i] = 1 \rangle \text{ and}$$

$$\mathbb{Z}_m \oplus \mathbb{Z}_n = \langle x, y \mid x^m = y^n = [x, y] = 1 \rangle, \text{ respectively.}$$

First, we show the result for the case when $g_0 = 0$. We consider the map

$$\xi_i \to x^{\frac{m}{n_{i1}} c_{i1}} y^{\frac{n}{n_{i2}} c_{i2}}, \text{ for } 1 \leq i \leq r.$$

Since $|x^{\frac{m}{n_{i1}}c_{i1}}| = n_{i1}$ and $|y^{\frac{n}{n_{i2}}c_{i2}}| = n_{i2}$, condition (d) implies that ϕ is an order-preserving map. Moreover, condition (f) implies that ϕ satisfies the long relation $\prod_{i=1}^{r} \xi_i = 1$. In order to show that ϕ is surjective, we establish that $\phi(\Gamma)$ generates the group $\mathbb{Z}_m \oplus \mathbb{Z}_n$. But condition (g) ensures that $\{\phi(\xi_i) : 1 \leq i \leq r\}$ generates $\mathbb{Z}_m \oplus \mathbb{Z}_n$, and hence it follows that D determines a $\mathbb{Z}_m \oplus \mathbb{Z}_n$-action on S_g. When $g_0 > 0$, $\pi_1^{\text{orb}}(\mathcal{O}_H)$ also has hyperbolic generators (i.e. the α_i and the β_i), which can be mapped surjectively to the generators of $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

Conversely, suppose that there is a $\mathbb{Z}_m \oplus \mathbb{Z}_n$-action H on S_g such that $\mathcal{O}_H$ had genus g_0. Then by Theorem 2.1.1, there exists a surjective homomorphism

$$\phi : \Gamma \to \mathbb{Z}_m \oplus \mathbb{Z}_n : \xi_i \mapsto x^{\frac{m}{n_{i1}}c_{i1}} y^{\frac{n}{n_{i2}}c_{i2}}, \text{ for } 1 \leq i \leq r,$$

that is order-preserving on the torsion elements. This yields an abelian data set of degree $m \cdot n$ and genus g as in Definition 3.2.4, and the result follows. □

Example 3.2.6. The weak conjugacy classes of the abelian actions illustrated in the first two subfigures of Figure 1.1 (in Section 1.2) are represented by the abelian data sets

$$(2 \cdot 2, 2; [(0,1),(1,2),2],[(1,2),(0,1),2],[(1,2),(1,2),2]) \text{ and}$$
$$(2 \cdot 2, 1; [(0,1),(1,2),2],[(1,2),(0,1),2],[(1,2),(1,2),2]_5),$$

where the suffix 5 in the second data set denotes the multiplicity of the subtuple $[(1,2),(1,2),2]$. We will discuss such actions in more detail in Section 3.3.

Definition 3.2.7. Two elements of a group H are said to *weakly commute* if there exists representatives in their respective conjugacy classes (in H) that commute.

For a group H, if $g, h \in H$ weakly commute, then we denote it by $[\![g,h]\!] = 1$. It is clear from Definition 3.2.7 that if $[\![g,h]\!] \neq 1$, then g and h cannot commute in H.

Remark 3.2.8. It follows immediately from Definition 3.2.7 and the Nielsen-Kerckhoff theorem that given $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$ of finite-order, $[\![\mathcal{F},\mathcal{G}]\!] = 1$ if and only if the mapping classes $F, G \in \text{Mod}(S_g)$ they represent satisfy $[\![F,G]\!] = 1$.

The proof of the main theorem we will also require the following elementary number-theoretic lemma.

Lemma 3.2.9. *Let* $\delta \in \mathbb{Z}_n$*, and* $k_1, \ldots, k_r$ *are positive integers such that* $\mathrm{lcm}(\{k_1, \ldots, k_r\}) = \beta$ *and* $\beta \mid n$*. If* $\frac{n}{\beta}|\delta$*, then there exists* $\delta_1, \ldots, \delta_r \in \mathbb{Z}_n$ *such that* $\frac{n}{k_i}|\delta_i$ *and* $\sum_{i=1}^r \delta_i \equiv \delta \pmod{n}$*.*

Proof. Since $\mathrm{lcm}(\{k_1, \ldots, k_r\}) = \beta$, we have $\gcd(\{\frac{n}{k_1}, \ldots, \frac{n}{k_r}\})|\frac{n}{\beta}$. Denoting $\gcd(\{\frac{n}{k_1}, \ldots, \frac{n}{k_r}\}) = c$, we see that there exists integers c_i such that $c = \sum_{i=1}^r c_i \frac{n}{k_i}$. For some integer t, if $\delta = ct$, where $c = \gcd(\{\frac{n}{k_1}, \ldots, \frac{n}{k_r}\})$, then $\delta = \sum_{i=1}^r tc_i \frac{n}{k_i}$. Taking $\delta_i = tc_i \frac{n}{k_i}$, the assertion follows. □

We are now in a position of prove the main result of this chapter, which essentially asserts that a pair of finite-order homeomorphisms commute if and only if they have conjugates that form a weakly abelian pair. In view of Proposition 3.2.5, the notion of abelian data set will be used extensively in our proof.

Theorem 3.2.10 (Main Theorem)**.** *Let* $F, G \in \mathrm{Mod}(S_g)$ *be of finite order. Then* $[\![F, G]\!] = 1$ *and their commuting conjugates form a two-generator abelian group, if and only if* $(\mathcal{G}, \mathcal{F})$ *is a weakly abelian pair of order* $|G| \cdot |F|$*.*

Proof. Let $|F| = n$ and $|G| = m$, where $m \mid n$, and let $H = \langle \mathcal{F} \rangle$. Let $D_F = (n, g_1, r_1; ((c_1, n_1), \alpha_1), \ldots, ((c_r, n_r), \alpha_r))$ and $D_G = (m, g_2, r_2; ((d_1, m_1), \beta_1), \ldots, ((d_k, m_k), \beta_k))$, respectively. First, we assume that $[\![F, G]\!] = 1$, and show that $(\mathcal{G}, \mathcal{F})$ form a weakly abelian pair of order $m \cdot n$. Without loss of generality, we may assume that F and G commute in $\mathrm{Mod}(S_g)$. Further, by the Nielsen-Kerckhoff theorem, we may assume that $\mathcal{F}$ and $\mathcal{G}$ commute in $\mathrm{Homeo}^+(S_g)$. Then by Lemmas 3.1.2 and 3.1.5, it follows that $(\mathcal{G}, \mathcal{F})$ forms an essential pair of order $m \cdot n$. It remains to show that $(\mathcal{G}, \mathcal{F})$ is a weakly abelian pair as in Definition 3.1.9. Condition (i) in this definition is a consequence of Lemma 3.1.5, while condition (ii) is a direct consequence of condition (g) of Definition 3.2.4. To show condition (iii), it suffices to consider the case when

$$D_F = (n, g_1; ((c_1, n_1), m), \ldots, ((c_r, n_r), m)),$$

as all other cases follow from similar arguments. First, we assume that $\mathcal{G}$ induces a $\bar{\mathcal{G}} \in \mathrm{Aut}(\mathcal{O}_H)$ which does not fix any cone point of $\mathcal{O}_H$. Let $\Gamma(\mathcal{O}_H/\langle\bar{\mathcal{G}}\rangle) = (g_0; n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+l})$, and let $\phi : \Gamma(\mathcal{O}_H/\langle\bar{\mathcal{G}}\rangle) \to \langle \mathcal{F}, \mathcal{G} \rangle$ be the surface kernel. Following the notation in the proof of Proposition 3.2.5, we map $\xi_i \overset{\phi}{\mapsto} F^{\frac{n}{n_i}c_i}$, for $1 \leq i \leq r$. The relation

$\prod_{i=1}^{r+l} \xi_i = (\prod_{j=1}^{g_0}[\alpha_j, \beta_j])^{-1}$ in the presentation of $\pi_1^{\text{orb}}(\mathcal{O}_H/\langle \bar{G} \rangle)$ would now imply that $\prod_{i=1}^{r+l} \phi(\xi_i) = 1$. If $\sum_{i=1}^{r} \frac{n}{n_i} c_i \equiv 0 \pmod n$, then condition (iii) holds trivially. On the other hand, if $\sum_{i=1}^{r} \frac{n}{n_i} c_i \not\equiv 0 \pmod n$, since $\prod_{i=1}^{r+l} \phi(\xi_i) = 1$, we have that $\prod_{i=1}^{r} \phi(\xi_i) = (\prod_{i=r+1}^{r+l} \phi(\xi_i))^{-1}$. Consequently, n/B divides $\sum_{i=1}^{r} \frac{n}{n_i} c_i$, where $B = \text{lcm}(n_{r+1}, \cdots, n_{r+l})$. Thus, it follows that condition (iii) is necessary.

Conversely, suppose that $(\mathcal{G}, \mathcal{F})$ forms a weakly abelian pair of degree $m \cdot n$ as in Definition 3.1.9. By Remark 3.2.8, it suffices to show that our assumption yields an abelian data set as desired. We now break our argument into four cases.

Case 1: Let $\text{lcm}(\{n_1, \ldots, n_r\}) = n$. We further assume that $m_i' n_i' = B_1$, where $m_i' \neq 1$, for some i. We may assume, without loss of generality, that $i = 1$. Then we show that the tuple

$$(m * n, g_0; \left[(d_1', m_1'), \left(\frac{\alpha c_1' + \delta}{\alpha \kappa}, \frac{m_1' n_1'}{\kappa}\right), m_1' n_1'\right],$$
$$\left[(d_2', m_2'), \left(\frac{c_2'}{\kappa_2}, \frac{m_2' n_2'}{\kappa_2}\right), m_2' n_2'\right], \ldots, \left[(d_l', m_l'), \left(\frac{c_l'}{\kappa_l}, \frac{m_l' n_l'}{\kappa_l}\right), m_l' n_l'\right]),$$

where $\gcd(c_j', m_j') = \kappa_j$, $\kappa = \gcd(c_1 + \frac{\delta}{\alpha}, m_1' n_1')$, $\alpha = \frac{n}{m_{t_1}' n_{t_1}'}$, $d_{t_i}' = 0$, if $m_i' \notin \{m_1^o, m_o^2, \ldots, m_t^o\}$, and $c_i' = 0$, if $n_i' \notin \{n_1, n_2, \ldots, n_r\}$, forms an abelian data set. Since $[\![F, G]\!] = 1$, $(\mathcal{G}, \mathcal{F})$ forms an essential pair, from which conditions (a) - (c) of Definition 3.2.4 follow. Moreover, for each i, we have $\gcd(d_i', m_i') = 1$ and $\gcd\left(\frac{c_i'}{\kappa_i}, \frac{m_i' n_i'}{\kappa_i}\right) = 1$, and by our choice of κ_i, we have $\text{lcm}(m_i', \frac{m_i' n_i'}{\kappa_i}) = m_i' n_i'$, from which conditions (d) - (e) of Definition 3.2.4 follow. Furthermore, our choice of c_i' and δ_2 ensures that

$$\sum_{i=1}^{l} \frac{n}{m_i' n_i'} c_i' + \delta_2 \equiv 0 \pmod n \text{ and } \sum_{i=1}^{l} \frac{m}{m_i'} d_i' \equiv 0 \pmod m,$$

which yields condition (f) of Definition 3.2.4. It now remains to show condition (g), for the case when $g_0 = 0$. Following the notation used in the proof of Theorem 3.2.5, we show that the generators y, x (of $\mathbb{Z}_m \oplus \mathbb{Z}_n$) can be expressed as products of elements in the set $\{\phi(\xi_i) : 1 \leq i \leq l\}$. Consider the set $S = \{\phi(\xi_i)^{m_i} : 1 \leq i \leq l\}$. Then by our choice of the map ϕ, each element of S equals some power of x, and $|\phi(\xi_i)^{m_i}| = n_i$. Since $\text{lcm}(n_1, \ldots, n_l) = n$, we have $\langle S \rangle = \langle x \rangle$. Now consider the set $T = \{\phi(\xi_r) : \phi(\xi_r) = y^a x^b,\ a \neq 0\}$. Since $(\mathcal{G}, \mathcal{F})$ is an essential pair, yx^t is a product of elements in T, and the assertion follows.

Now suppose that $\mathrm{lcm}(\{m'_k n'_k : m'_k \neq 1\}) = B_1$, where no $m'_k n'_k$ equals B_1. Without loss of generality, we may assume that $\mathrm{lcm}(\{m'_k n'_k : m'_k \neq 1 \text{ and } 1 \leq k \leq p\}) = B_1$. Then by Lemma 3.2.9, there exists δ'_i, for $1 \leq i \leq p$, such that $\sum_{i=1}^{p} \delta'_i \equiv \delta_2 \pmod{n}$. For each δ'_i, we choose $\alpha_i = \frac{n}{m'_i n'_i}$ and consider the tuple

$$(m * n, g_0; \left[(d'_1, m'_1), \left(\frac{\alpha_1 c'_1 + \delta'_1}{\alpha_1 \xi'_1}, \frac{m'_1 n'_1}{\xi'_1}\right), m'_1 n'_1\right], \dots, \left[\left(d'_p, m'_p\right), \left(\frac{\alpha_p c'_p + \delta'_p}{\alpha_p \xi'_p}, \frac{m'_p n'_p}{\xi'_p}\right), m'_p n'_p\right],$$
$$\left[\left(d'_{p+1}, m'_{p+1}\right), \left(\frac{c'_{p+1}}{\xi_{p+1}}, \frac{m'_{p+1} n'_{p+1}}{\xi_{p+1}}\right), m'_{p+1} n'_{p+1}\right], \dots, \left[(d'_l, m'_l), \left(\frac{c'_l}{\xi_l}, \frac{m'_l n'_l}{\xi_l}\right), m'_l n'_l\right]),$$

where $\xi'_j = \gcd(\{c_j + \frac{\delta'_j}{\alpha_j}, m'_j n'_j : 1 \leq j \leq p\})$ and $\gcd(c'_i, m'_i) = \xi_i$, for $p+1 \leq i \leq l$. As before, this tuple will satisfy all the conditions of an abelian data set.

Case 2: Let $\mathrm{lcm}(\{m_1, \dots, m_k\}) = m$ and $\mathrm{lcm}(\{n_1, \dots, n_r\}) < n$. By an argument analogous to Case 1, we obtain a representation $\phi : \Gamma \to \mathbb{Z}_m \oplus \mathbb{Z}_n$ such that the generators y, x (of $\mathbb{Z}_m \oplus \mathbb{Z}_n$) can be expressed as products of elements in the set $\{\phi(\xi_i) : 1 \leq i \leq l\}$. Consider the set $S = \{\phi(\xi_i)^{n_i} : 1 \leq i \leq l\}$. Then by our choice of ϕ and Proposition 3.1.4, it follows that each element of S equals some power of y and $|\phi(\xi_i)^{n_i}| = m_i$. Since $\mathrm{lcm}(m_1, \dots, m_l) = m$, we have $\langle S \rangle = \langle y \rangle$. Now consider the set $T = \{\phi(\xi_r) : \phi(\xi_r) = y^a x^b,\ b \neq 0\}$. As (G, F) forms an essential pair, xy^t is a product of elements in T, and the assertion follows.

Case 3: Let $\mathrm{lcm}(\{m_1, \dots, m_k\}) < m$, $\mathrm{lcm}(\{n_1, \dots, n_r\}) < n$, and $g_0 > 1$. Then the abelian data set and the representation ϕ from Case 1 also works for this case.

Case 4: Let $\mathrm{lcm}(\{m_1, \dots, m_k\}) < m$, $\mathrm{lcm}(\{n_1, \dots, n_r\}) < n$, and $g_0 = 0$. Then by Lemma 3.1.12, it follows that there exists an $\mathcal{F}' \in \langle \mathcal{F}, \mathcal{G} \rangle$ such that $|\mathcal{F}'| = n$ and $D_{F'} = (g'_0; (c_1, n_1), \dots, (c_r, n_r))$ satisfies $\mathrm{lcm}(\{n_1, \dots, n_r\}) = n$. Since (G, F) is a weakly abelian pair, so is $(\mathcal{G}, \mathcal{F}')$, and hence this case reduces to Case 1. □

3.3 Applications

In this section, we derive several applications of the theory developed earlier in the chapter.

3.3.1 Weak commutativity of involutions

It is well known that the conjugacy class of an involution $F \in \text{Mod}(S_g)$ is represented by $D_F = (2, g_0; ((1,2), k))$, where $k = 2(g - 2g_0 + 1)$, if F is a non-free action on S_g, and $D_F = (2, (g+1)/2, 1;)$, otherwise. In this section, we will derive conditions under which two involutions in $\text{Mod}(S_g)$ will weakly commute.

Corollary 3.3.1. *Let $F, G \in \text{Mod}(S_g)$ be involutions such that*

$$D_F = (2, g_0', r'; ((1,2), 2k')) \text{ and } D_G = (2, g_0'', r''; ((1,2), 2k'')),$$

respectively. Then $[\![F, G]\!] = 1$ if and only if, the following conditions hold.

(a) There exists $\bar{\mathcal{G}} \in \text{Homeo}^+(S_{g_0'})$ with $D_{\bar{\mathcal{G}}} = (2, g_0, r_1; ((1,2), 2s''))$ such that $g + k'' + 1 \geq 2s'' \geq k''$.

(b) There exists $\bar{\mathcal{F}} \in \text{Homeo}^+(S_{g_0''})$ with $D_{\bar{\mathcal{F}}} = (2, g_0, r_2; ((1,2), 2s'))$ such that $g + k' + 1 \geq 2s' \geq k'$.

Proof. It suffices to show that conditions (a) - (b) mentioned above hold true if and only if $(\mathcal{G}, \mathcal{F})$ is a weakly abelian pair. If $(\mathcal{G}, \mathcal{F})$ is a weakly abelian pair, then it is also an essential pair, and so a direct computation shows that shows that conditions (i) and (ii) in the definition of an essential pair are equivalent to the conditions (a) and (b) above, respectively. Conversely, it is easy to see that conditions (a) - (b) imply that $(\mathcal{G}, \mathcal{F})$ is an essential pair. It remains to show that conditions (i) - (iii) of Definition 3.1.9 hold true. A simple application of the Riemann-Hurwitz equation to the four data sets that appear in the statement above leads to a system of (four) linear equations, which can be simplified to yield the condition:

$$2s' - k' = 2s'' - k'',$$

from which (i)-(ii) follow. When g is odd, $4 \mid \sum_{i=1}^{r} \alpha_i \frac{n}{n_i} c_i$, and so each δ_i appearing in (iii) is 0. If g is even, then as no involution generates a free action, we have $B_i = 2$. Thus, condition (iii) is satisfied, and the assertion follows. □

Let the conjugacy classes $D_F = (2, g_0', r'; ((1,2), 2k'))$ and $D_G = (2, g_0'', r''; ((1,2), 2k''))$, be represented by involutions $\mathcal{F}$ and $\mathcal{G}$, which commute. Then, by Corollary 3.3.1, we have $D_{FG} = (2, g_0, r'''; ((1, 2), 2k))$, where $k = 2s' - k' = 2s'' - k''$. Using this idea, one can obtain a geometric realization of a Klein 4-subgroup K_4 of $\mathrm{Mod}(S_g)$ by obtaining an embedding of $\iota : S_g \hookrightarrow \mathbb{R}^3$ that is symmetric about origin such that $\iota(S_g)$ intersects, the x-axis at $2k'$ points, the y-axis at $2k''$ points, and the z-axis at $2k$ points. It is now apparent that under this embedding the non-trivial elements of K_4 are realized as π-rotations about the three coordinates axes. This property is illustrated in the following example.

Example 3.3.2. Consider $F, G \in \mathrm{Mod}(S_7)$ whose conjugacy classes are given by $D_F = (2, 4, 1;)$, $D_G = (2, 3; ((1, 2), 4))$, respectively. By the preceding discussion, there exist three possible choices for the conjugacy class of FG, namely:

(a) $D_{FG} = (2, 4, 1;)$

(b) $D_{FG} = (2, 2; ((1, 2), 8))$

(c) $D_{FG} = (2, 0; ((1, 2), 16))$

The realization of the group $\{1, F, G, FG\}$ in each case is given in Figures 3.1-3.3 below.

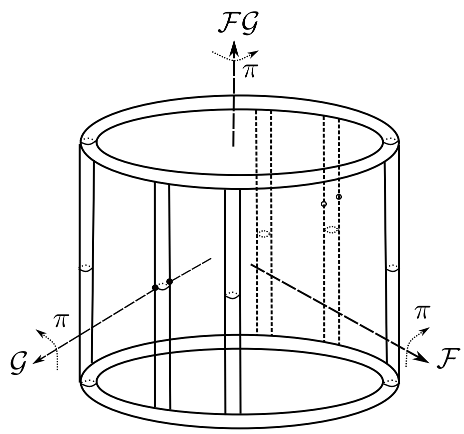


Figure 3.1: Case (a)

In fact, all Klein 4-subgroups of $\mathrm{Mod}(S_g)$ can be realized in an analogous manner.

3.3.2 Finite abelian groups with irreducible finite-order mapping classes

We say a $\mathbb{Z}_n$-action is *irreducible* if it is irreducible as a mapping class. By a result of Gilman [11], this is equivalent to requiring that the corresponding orbifold of the action

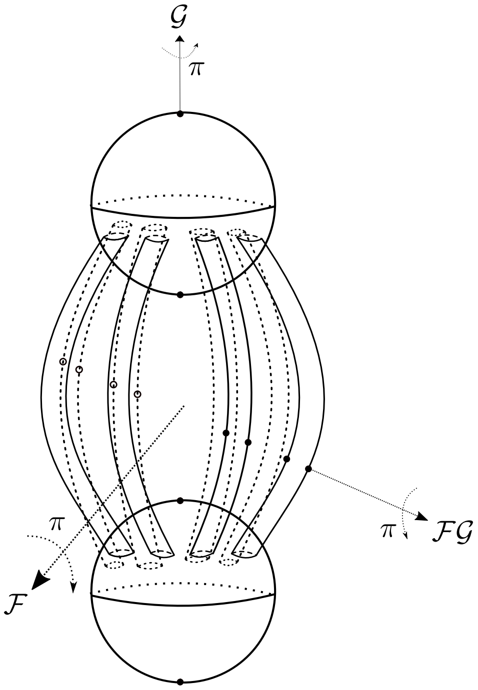


Figure 3.2: Case (b)

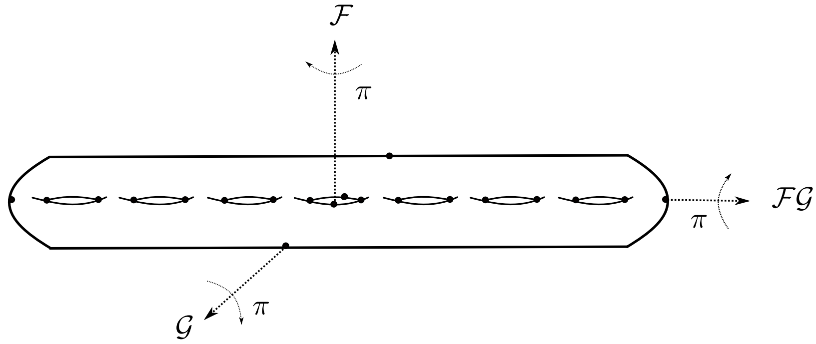


Figure 3.3: Case (c)

is a sphere with 3 cone points. Following the nomenclature in [2] and [30], a $\mathbb{Z}_n$-action on S_g is said to be *rotational* if it can be realized as a rotation about an axis under a suitable embedding of $S_g \hookrightarrow \mathbb{R}^3$. A non-rotational action is said of be of *Type 1* if its quotient orbifold has signature $(g_0; n_1, n_2, n)$, otherwise, it is called a *Type 2* action. The following corollary characterizes the weak commutativity of Type 2 actions with finite-order maps.

Corollary 3.3.3. *There exists no finite non-cyclic abelian subgroup of* $\text{Mod}(S_g)$ *that contains an irreducible Type 2 action.*

Proof. By Remark 3.2.8, it suffices to show that an irreducible Type 2 $\mathbb{Z}_n$-action F does not commute with any other Z_m-action. Since F is a Type 2 action, we have $\Gamma(\mathcal{O}_{\langle F\rangle}) = (0; n_1, n_2, n_3)$, where $n_i \neq n_j$ and $n_i < n$, for $1 \leq i \neq j \leq 3$. In view of Theorem 3.2.10, if some $G \in \text{Mod}(S_g)$ such that $[\![F, G]\!] = 1$, then there exists $\bar{\mathcal{G}} : \mathcal{O}_{\langle \mathcal{F}\rangle} \to \mathcal{O}_{\langle \mathcal{F}\rangle}$ which satisfies the IMP with respect to $(\mathcal{G}, \mathcal{F})$. This would imply that $\bar{\mathcal{G}}$ fixes all three cone points of $\mathcal{O}_{\langle \mathcal{F} \rangle}$. This is impossible, as any non-trivial automorphism on a sphere can fix at most two points, and the assertion follows. □

We now give a similar characterization for the weak commutativity of Type 1 actions .

Corollary 3.3.4. *Suppose that there exists a finite non-cyclic abelian subgroup H of* $\mathrm{Mod}(S_g)$ *that contains an irreducible Type 1 action. Then $H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2g+2}$.*

Proof. Let F be an irreducible Type 1 action with $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}) = (0; n_1, n_2, n_3)$. Since F is of Type 1, there exists at least one n_i (say n_1) such that $n_1 = n$, and the following cases arise.

Case 1: $n_2 \neq n_3$ and $n_2, n_3 < n$. By an argument analogous to the one used in the proof of Corollary 3.3.3, it follows that $\mathcal{F}$ does not commute with any other finite-order element of $\mathrm{Mod}(S_g)$.

Case 2 : $n_i = n$, for $1 \leq i \leq 3$. Then by the Riemann-Hurwitz equation, we have that $n = 2g + 1$. By applying a result of Maclachlan [22] that bounds the order of a finite abelian subgroup of $\mathrm{Mod}(S_g)$ by $4g + 4$, it follows that only an involution can commute with $\mathcal{F}$. When such an involution $\mathcal{G}$ does commute with $\mathcal{F}$, it follows immediately that $\langle \mathcal{F}, \mathcal{G} \rangle \cong \mathbb{Z}_{4g+2}$.

Case 3: $n_1 = n_2 = n \neq n_3$. Once again, by similar arguments as above, we can conclude that F cannot commute with any other finite-order $G \in \mathrm{Mod}(S_g)$ with $|G| \geq 3$. When $\mathcal{F}$ commutes with an involution $\mathcal{G}$, the induced map $\bar{\mathcal{G}} \in \mathrm{Aut}(\mathcal{O}_{\langle \mathcal{F} \rangle})$ fixes the cone point of order n_3 in $\mathcal{O}_{\langle \mathcal{F} \rangle}$ and permutes the remaining 2 cone points. Consequently, we have $\langle \mathcal{F}, \mathcal{G} \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_n$. By the Riemann-Hurwitz equation, it follows that $n \geq 2g + 1$, and hence $n = 2g + 2$, as $2n \leq 4g + 4$. □

3.3.3 Weak commutativity with free cyclic actions

Any non-trivial finite m-sheeted cover of S_g, for $g \geq 2$ with cyclic group of deck transformation, has the form $p : S_{m(g-1)+1} \to S_g$, where p is a covering map. Let $\mathrm{LMod}_p(S_g)$ be the subgroup of $\mathrm{Mod}(S_g)$ of all mapping classes that have representatives that lift to homeomorphisms of $S_{m(g-1)+1}$ under p. It is natural question to ask whether a given $F \in \mathrm{Mod}(S_g)$ of finite-order will have a conjugate F' such that $F' \in \mathrm{LMod}_p(S_g)$. In this subsection, we answer this question for certain types of finite-order maps. We begin by determining when certain types of free cyclic actions weakly commute with other cyclic actions.

Corollary 3.3.5. *Let $F, G \in \text{Mod}(S_g)$ with $D_F = (n, g_1, r;)$ and $D_G = (m, g_0, r'; ((d_1, m_1), \beta_1), \ldots, ((d_k, m_k), \beta_k))$, respectively. Suppose that F induces a free action on $\mathcal{O}_{\langle G \rangle}$. Then $[\![F, G]\!] = 1$ if and only if:*

(a) $\beta_j = 0 \pmod n$, for $1 \leq j \leq k$,

(b) $n | (g_0 - 1)$, and

(c) $\sum_{i=1}^{k} \frac{\beta_i}{n} \frac{m}{m_i} d_i \equiv 0 \pmod m$.

Proof. We show that conditions (a) - (c) are sufficient, as it follows directly from Theorem 3.2.10 that they are necessary. By conditions (a) - (b) of our hypothesis, it follows that there exists a free action on S_{g_0}, which induces an $\bar{\mathcal{F}} \in \text{Aut}(\mathcal{O}_{\langle \mathcal{G} \rangle})$. The Riemann-Hurwitz equation and Lemma 2.1.1 imply that there exists a $\bar{\mathcal{G}} \in \text{Aut}(\mathcal{O}_{\langle \mathcal{F} \rangle})$ with

$$D_{\bar{G}} = (m, \frac{(g_0 - 1)}{n} + 1, r''; ((d_1, m_1), \frac{\beta_1}{n}), \ldots, ((d_k, m_k), \frac{\beta_k}{n})).$$

Hence, it follows that $(\mathcal{G}, \mathcal{F})$ forms an essential pair, and the fact that they form an weakly abelian pair follows directly from condition (c) of our hypothesis and by the hypothesis that F is a free action. $\square$

In the following result, we show that a finite-order mapping class whose corresponding orbifold has positive genus has a conjugate that is liftable under a finite-sheeted cover of S_g.

Corollary 3.3.6. *Consider an $F \in \text{Mod}(S_g)$ of finite-order such that $\mathcal{O}_{\langle F \rangle}$ has positive genus. Let $p : S_{m(g-1)+1} \to S_g$ be an m-sheeted cover whose deck transformation group is given by $\langle G \rangle \cong \mathbb{Z}_m$. Then there exists a conjugate F' of F such that $F' \in \text{LMod}_p(S_g)$.*

Proof. Let $D_F = (n, g_0, r_1; (c_1, n_1), \ldots, (c_r, n_r))$, and consider $\widetilde{F} \in \text{Mod}(S_{m(g-1)+1})$ with $D_{\widetilde{F}} = (n, m(g_0 - 1) + 1, \bar{r}; ((c_1, n_1), m), \ldots, ((c_r, n_r), m))$. Then by Corollary 3.3.5, we have that $[\![G, \widetilde{F}]\!] = 1$. Without loss of generality, we may assume that $\mathcal{G}$ and $\widetilde{\mathcal{F}}$ commute in $\text{Homeo}^+(S_g)$. By the IMP, it now follows that $\widetilde{\mathcal{F}}$ induces $\mathcal{F}' \in \text{Homeo}^+(S_g)$, whose mapping class F' is conjugate to F in $\text{Mod}(S_g)$. $\square$

In the following corollary, we provide conditions under which certain finite-order mapping classes whose corresponding orbifolds are spheres have conjugates that lift under a finite cover of S_g.

Corollary 3.3.7. *Let* $F \in \mathrm{Mod}(S_g)$ *with* $D_F = (n, 0; (c_1, n_1), \ldots, (c_r, n_r))$*. Let* $p : S_{m(g-1)+1} \to S_g$ *be an* m*-sheeted cover whose deck transformation group is given by* $\langle G \rangle \cong \mathbb{Z}_m$*. Then there exists a conjugate* F'' *of* F *such that* $F'' \in \mathrm{LMod}_p(S_g)$*, if the following conditions hold.*

(i) $m \mid n_1$ *and* $m \mid n_2$*.*

(ii) For $k = 1, 2$*, there exists residue classes* c_k' *modulo* (n_k/m) *such that* $\gcd(c_k', n_k/m) = 1$ *and the tuple*

$$D = (n, 0; (c_1', n_1/m), (c_2', n_2/m), ((c_3, n_3), m), \ldots, ((c_r, n_r), m))$$

forms a data set.

Proof. Consider an $F' \in \mathrm{Mod}(S_{m(g-1)+1})$ with $D_{F'} = D$. It is straightforward to check that $(\mathcal{G}, \mathcal{F}')$ forms a weakly abelian pair. Thus, by Theorem 3.2.10, we have that $[\![F', G]\!] = 1$. So, $\mathcal{F}'$ induces $\mathcal{F}'' \in \mathrm{Homeo}^+(S_g)$ whose mapping class is conjugate to F. □

3.3.4 Primitivity of finite-order mapping classes

Let H be group. We say an $x \in H$ has a *root of degree* n if there exists $y \in H$ such that $y^n = x$ and $|y| = n|x|$. If $g \in H$ has no root of any degree greater than one, then g is said to be *primitive* in H. It is known [41] that the order of a finite cyclic subgroup of $\mathrm{Mod}(S_g)$ is bounded above by $4g + 2$. This would imply that no finite-order mapping class with order $> 2g + 1$ can have a nontrivial root. The following proposition gives conditions under which an arbitrary finite-order mapping class can have a root.

Proposition 3.3.8. *Let* $F \in \mathrm{Mod}(S_g)$ *with* $D_F = (n, g_0, r_1; (c_1, n_1), \ldots, (c_r, n_r))$*, and let* $H = \langle F \rangle$*. Then* F *has a root of degree* m *if and only if there exists a* $\mathcal{G} \in \mathrm{Homeo}^+(S_{g_0})$ *with* $D_G = (m, g', r'; (d_1, m_1), \ldots, (d_k, m_k))$*, which induces a* $\bar{\mathcal{G}} \in Aut(\mathcal{O}_H)$ *that satisfies the following conditions.*

(i) $\Gamma(\mathcal{O}_H/\langle\bar{\mathcal{G}}\rangle) = (g'; n_1', \ldots, n_l')$, *where for each* i, n_i' *belongs to the following union of multisets*

$$\{n_1, \ldots, n_r\} \cup \{m_i | \gcd(m_i, n) = 1\} \cup \{n_j m_i | \gcd(m_i, n) = 1\} \cup \{nm_j\}.$$

(ii) There exists an $F' \in \text{Mod}(S_g)$ *with* $D_{F'} = (mn, g', r''; (c_1', n_1'), \ldots, (c_l', n_l'))$ *such that for each* i,

$$c_i' \equiv \begin{cases} c_j, & \text{if } n_i' = n_j, \text{ and} \\ c_j \pmod{n_j}, & \text{if } n_i' = n_j m_i. \end{cases}$$

Proof. First, we note that the conjugacy of $(F')^m$ is represented by D_F. Thus, we have that $(F')^m$ and F are conjugate. So, we can find a conjugate of F', say F'', such that $(F'')^m = F$. Hence, F'' is a root of F of order m.

Conversely, suppose that F has a root F' of order m. Since $\mathcal{F}'$ commutes with $(\mathcal{F}')^m$, the map $\bar{\mathcal{G}}([x]) := [\mathcal{F}'(x)]$ defines an automorphism of $\mathcal{O}_H$, where $H = \langle(\mathcal{F}')^m\rangle$. Furthermore, we have that

$$\Gamma(\mathcal{O}_H/\langle\bar{\mathcal{G}}\rangle) = \Gamma(S_g/\langle\mathcal{F}'\rangle) = (g'; t_1, t_2, \ldots, t_l).$$

Note that,

$$t_i \in \{n_1, \ldots, n_r\} \cup \{m_1, \ldots, m_k\} \cup \{n_i m_j | 1 \leq i \leq r, 1 \leq j \leq k\}.$$

So, it remains to prove if $t_i = m_j$, then $\gcd(m_j, n) = 1$, and if $t_i = m_j n_k$ then either $n_k = n$ or $\gcd(m_j, n) = 1$. However, this follows directly from the structures of $D_{F'}$ and D_F. □

A consequence of this proposition is the following corollary, which pertains to the roots of a mapping class of order $g - 1$.

Corollary 3.3.9. *Let* $F \in \text{Mod}(S_g)$ *be represented by the generator of a free cyclic action on* S_g.

(i) If F *has a nontrivial root* F', *then* $\mathcal{O}_{\langle F'\rangle} \not\approx S_0$, *and*

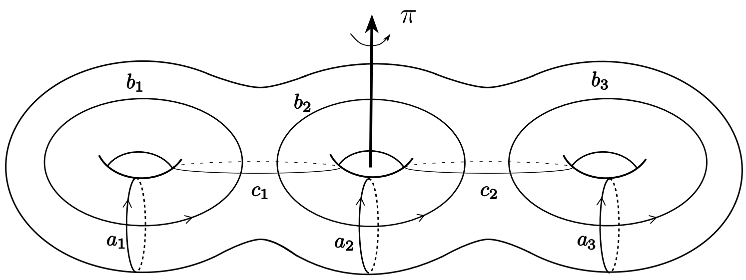


Figure 3.4: Realization of free involution in S_3

(ii) If $|F| = g-1$, then F has a root F' if and only if g is even. In this case, F' is a root of degree 2.

Proof. (i) Suppose that $\mathcal{O}_{\langle \mathcal{F}' \rangle} \approx S_0$. Then, as discussed in the proof of Proposition 3.3.8, all its powers of prime order have a fixed point.

(ii) Let $H = \langle \mathcal{F} \rangle$. Since $|F| = g-1$, the Riemann-Hurwitz equation would imply that $\mathcal{O}_H \approx S_2$. Moreover, by Proposition 3.3.8, $\mathcal{F}'$ induces an $\bar{\mathcal{F}} \in \text{Aut}(\mathcal{O}_H)$ of order n. In view of (i), it is clear that $n \leq 2$, and further, by condition (i) of Proposition 3.3.8, this is only possible when $2 \nmid (g-1)$, that is, g is even. Conversely, if g is even, then it can be easily verified that $F' \in \text{Mod}(S_g)$ with $D_{F'} = (2g-2, 1; (1,2), (1,2))$ is a root of F of degree 2.

□

Another consequence concerns the primitivity of a rotation of S_g of order g.

Corollary 3.3.10. *If $6 \mid g$, then an $F \in \text{Mod}(S_g)$ with $D_F = (g, 1; (c, g), (g-c, g))$ is primitive.*

Proof. Since order of any action on S_1 is not coprime with 6 and every action on S_1 has at least three cone points, Condition (i) of Proposition 3.3.8 would be violated, from which our assertion follows. □

The following corollary characterizes the primitivity of involutions.

Corollary 3.3.11. *An involution $F \in \text{Mod}(S_g)$ is primitive if and only if $D_F = (2, 2, 1;)$.*

Proof. By Proposition 3.3.8, it follows that an F with $D_F = (2, 2, 1;)$ is primitive, whose realization is given in Figure 3.4. It remains to show that no other involution can be primitive. Let $F \in \text{Mod}(S_g)$ be an arbitrary involution with $D_F = (2, g_0, r; ((1,2), n))$. We divide our argument into the following cases.

Case 1: $n = 0$ and $g_0 \geq 3$. Then there exists a $G \in \text{Homeo}^+(S_{g_0})$ with $D_G = (g_0 - 1, 2, 1;)$ that satisfies condition (i) of Proposition 3.3.8. Further, the mapping class $F' \in \text{Mod}(S_g)$ with $D_{F'} = (2(g_0-1), 2, 1;)$ satisfies condition (ii). Thus, F' has a conjugate F'' such that $(F'')^{g_0-1} = F$.

Case 2: $n \neq 0$ and g_0 is even. Then there exists a $G \in \text{Homeo}^+(S_{g_0})$ with , $D_G = (2, \frac{g_0}{2}; ((1,2), 2))$ inducing $\bar{\mathcal{G}} \in \text{Aut}(\mathcal{O}_{\langle \mathcal{F} \rangle})$ such that

$$\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}/\langle \bar{\mathcal{G}} \rangle) = (\frac{g_0}{2}; 4, 4, \underbrace{2, \ldots, 2}_{\frac{n-2}{2} \text{ times}}).$$

Moreover, the $F' \in \text{Mod}(S_g)$ with

$$D_{F'} = (4, \frac{g_0}{2}; (1,4), (c,4), ((1,2), \frac{n-2}{2})), \text{ where } c = \begin{cases} 1, & \text{if } \frac{n-2}{2} \text{ is odd, and} \\ 3, & \text{if } \frac{n-2}{2} \text{ is even.} \end{cases}$$

Thus, by Proposition 3.3.8, F' has a conjugate F'' such that $(F'')^2 = F$.

Case 3: $n > 2$ and g_0 is odd. Then consider $\mathcal{G} \in \text{Homeo}^+(S_{g_0})$ with data set $D_G = (2, \frac{g_0-1}{2}; ((1,2), 4))$ such that $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}/\langle \bar{\mathcal{G}} \rangle) = (\frac{g_0-1}{2}; 4, 4, 4, 4, \underbrace{2, \ldots, 2}_{\frac{n-4}{2} \text{ times}})$. Furthermore, there exists $F' \in \text{Mod}(S_g)$ with

$$D_{F'} = (4, \frac{g_0-1}{2}; ((1,4), 3), (c,4), ((1,2), \frac{n-4}{2})), \text{ where } c = \begin{cases} 3, & \text{if } \frac{n-4}{2} \text{ is odd, and} \\ 1, & \text{if } \frac{n-4}{2} \text{ is even.} \end{cases}$$

Thus, as in the previous case, $F = (F'')^2$.

Case 4: $n = 2$ and g_0 is odd. Suppose that $g_0 \geq 2$. Then we take $\mathcal{G} \in \text{Homeo}^+(S_{g_0})$ with $D_G = (g_0, 1; (1, g_0), (g_0 - 1, g_0))$ so that $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}/\langle \bar{\mathcal{G}} \rangle) = (1; 2g_0, 2g_0)$. Further, by taking $F' \in \text{Mod}(S_g)$ with $D_{F'} = (2g_0, 1; (1, 2g_0), (2g_0 - 1, 2g_0))$, by Proposition 3.3.8, we obtain a conjugate F'' (of F') such that $(F'')^{(g_0)} = F$. When $g_0 = 1$, the mapping class $F' \in \text{Mod}(S_g)$ with data set $D_{F'} = (6, 0; (1,3), (5,6), (5,6))$ has a conjugate F'' such that $(F'')^3 = F$.

□

3.3.5 Weak commutativity of finite-order maps with roots of Dehn twists

Let c be a simple closed curve in S_g, for $g \geq 2$, and let $T_c \in \text{Mod}(S_g)$ denote the left-handed Dehn twist about c. A *root of* T_c *of degree* n is an $F \in \text{Mod}(S_g)$ such that $F^n = T_c$. Consider an $F \in \text{Mod}(S_g)$ that is either an order-n mapping class that preserves c, or a root of T_c of degree n. Then up to isotopy, we can assume that F has a representative $\mathcal{F}$ such that $\mathcal{F}(c) = c$, and that $\mathcal{F}$ preserves a closed annular neighborhood N of c. Let $\widehat{S_g(c)}$ denote the surface obtained by capping off the components of $\overline{S_g \setminus N}$. Then by the theory developed in [27, 30, 32], it follows that $\mathcal{F}$ induces an order-n map $\widehat{\mathcal{F}}_c \in \text{Homeo}^+(\widehat{S_g(c)})$ by coning. Throughout this subsection, we will denote the representative of an arbitrary mapping class $F \in \text{Mod}(S_g)$ by $\mathcal{F}$. The following remark describes the construction of a root of a T_c, when c is nonseparating.

Remark 3.3.12. When c is nonseparating, it is well known [23, 27] that (up to conjugacy) a root F of T_c of degree n in $\text{Mod}(S_g)$ determines a $\mathbb{Z}_n$-action $\widehat{\mathcal{F}}_c$ on S_{g-1}, which has two (distinguished) fixed points on $\widehat{S_g(c)}$, where it induces rotation angles add up to $2\pi/n \pmod{2\pi}$. (We will call such an action a *nonseparating root-realizing* $\mathbb{Z}_n$*-action.*) Conversely, consider a $\mathbb{Z}_n$-action on S_{g-1}, which has two (distinguished) fixed points, where it induces rotation angles that add up to $2\pi/n \pmod{2\pi}$. Then we can remove invariant disks around the fixed points and attach a 1-handle N with an $1/n^{th}$ twist connecting the resulting boundary components to obtain a root of Dehn twist about the non-separating curve in N.

Moreover, it was shown in [27, 32] that no root of T_c can switch the two sides of c.

Remark 3.3.13. Suppose that a $\mathbb{Z}_m$-action $G \in \text{Mod}(S_g)$ preserves a curve c. Then G induces an order-m map $\widehat{G}_c$ on $\widehat{S_g(c)}$. When c is non-separating, $\widehat{S_g(c)} \approx S_{g-1}$, and when c is separating, $\widehat{S_g(c)} \approx S_{g_1} \sqcup S_{g_2}$ (in symbols $S_g = S_{g_1} \#_c S_{g_2}$), where $g_1 + g_2 = g$. Let N be a closed annular neighborhood of c such that $\mathcal{G}(N) = N$. Then the two distinguished points P, Q that lie at the center of the capping disks (of the two boundary components of the surface $\overline{S_g \setminus N}$) are either fixed under the action of $\widehat{\mathcal{G}}_c$, or form an orbit of size 2. Conversely, given a $\mathbb{Z}_m$-action $\widehat{\mathcal{G}}_c$ on a surface ($\approx \widehat{S_g(c)}$) with two distinguished points P, Q, which are either fixed with locally induced rotation angles (around P and Q) adding up to 0 (mod 2π), or form a orbit of size 2, we may reverse the above process to obtain $\mathbb{Z}_m$-action on S_g. Note that by [30] P, Q is an orbit of size 2, only when $|\widehat{G}_c| = 2$.

This leads us to the following characterization of weak commutativity of finite-order maps with roots of Dehn twists about nonseparating curves.

Corollary 3.3.14. *Let $F \in \text{Mod}(S_g)$ be a root of T_c, where c is nonseparating, and $G \in \text{Mod}(S_g)$ be of finite order. Then $[\![F, G]\!] = 1$ if and only if $G(c) = c$ and $[\![\widehat{F}_c, \widehat{G}_c]\!] = 1$. In particular, if $\widehat{F}_c$ is primitive, then F and G cannot commute in $\text{Mod}(S_g)$.*

Proof. Suppose that $[\![F, G]\!] = 1$. Then up to conjugacy, we assume that F commutes with G, and so we have $T_c = GT_cG^{-1} = T_{G(c)}$. Hence, we may assume up to isotopy that $\mathcal{G}(c) = c$, and both $\mathcal{G}$ and $\mathcal{F}$ preserve the same annular neighborhood N of c. Thus, $\widehat{\mathcal{F}}_c$ and $\widehat{\mathcal{G}}_c$, which are induced by $\mathcal{F}$ and $\mathcal{G}$, respectively, must commute as maps on S_{g-1}, and so it follows that $[\![\widehat{\mathcal{F}}_c, \widehat{\mathcal{G}}_c]\!] = 1$.

Conversely, let us assume hypothesis of statement hold true. Then $\widehat{\mathcal{F}}_c$ and $\widehat{\mathcal{G}}_c$ share the same set of two distinguished points P and Q (as in Remark 3.3.12) that are either fixed or form an orbit of size 2, under their actions. By Remarks 3.3.12-3.3.13, we construct maps $\mathcal{F}$ and $\mathcal{G}$, which commute in $\text{Homeo}^+(S_g)$. Therefore, as mapping classes they satisfy $[\![F, G]\!] = 1$.

Let $H = \langle \widehat{\mathcal{F}}_c, \widehat{\mathcal{G}}_c \rangle$. To show the final part of the assertion, we first observe that $\text{Stab}_H(P) = H$, when $|G| > 2$. Since H is cyclic (by Lemma 3.1.3), it follows that $\widehat{\mathcal{F}}_c$ has a root of degree $|\mathcal{G}|$. Further, it was shown in [27] that any nontrivial root of T_c is of odd degree. So, when $|\mathcal{G}| = 2$, it is apparent that H is cyclic. Therefore, if $\widehat{F}_c$ is primitive, then F and G cannot commute in $\text{Mod}(S_g)$. □

Note that the conditions $\gcd(|\widehat{F}_c|, |\widehat{G}_c|) = 1$ and $|\widehat{F}_c||\widehat{G}_c| \leq (4(g-1)+2)$ determine an upper bound for $|G|$.

Remark 3.3.15. Let c is a separating curve in S_g so that $S_g = S_{g_1} \#_c S_{g_2}$. It is known [33] that (up to conjugacy) a root F of T_c of degree n corresponds to a pair $\widehat{F}_c = (\widehat{\mathcal{F}}_{1,c}, \widehat{\mathcal{F}}_{2,c})$ of finite order maps, where $\widehat{\mathcal{F}_{i,c}} \in \text{Homeo}^+(S_{g_i})$ with $|\widehat{\mathcal{F}_{i,c}}| = n_i$, for $i = 1, 2$, with distinguished fixed points $P_i \in S_{g_i}$ around which the locally induced rotational angles θ_i, which satisfy

$$\theta_1 + \theta_2 \equiv 2\pi/n \pmod{2\pi}, \text{ where } n = \text{lcm}(n_1, n_2).$$

Further, if $\mathcal{G}$ is a finite-order map with $\mathcal{G}(c) = c$ and $|\mathcal{G}| > 2$, then there is a decomposition of $\widehat{\mathcal{G}}_c$ into a pair of actions $(\widehat{\mathcal{G}}_{1,c}, \widehat{\mathcal{G}}_{2,c})$, where $\widehat{\mathcal{G}}_{i,c}$ is a $\mathbb{Z}_m$-action on S_{g_i}, for $i = 1, 2$. However, when $|\mathcal{G}| = 2$, $\widehat{\mathcal{G}}_c$ is either a single action on $S_g(c)$ that permutes the components S_{g_i} (in which case $g_1 = g_2$), or it decomposes into a pair of actions $(\widehat{\mathcal{G}}_{1,c}, \widehat{\mathcal{G}}_{2,c})$ as before.

The ideas in Remarks 3.3.13 and 3.3.15 lead to the following analog of Corollary 3.3.14 for the roots of Dehn twists about separating curves.

Corollary 3.3.16. *Let c is a separating curve in S_g so that $S_g = S_{g_1} \#_c S_{g_2}$. Let $F \in \text{Mod}(S_g)$ be a root of T_c so that $\widehat{F}_c = (\widehat{F}_{1,c}, \widehat{F}_{2,c})$. Then a $G \in \text{Mod}(S_g)$ of finite order satisfies $[\![F, G]\!] = 1$ if and only if:*

(i) $\mathcal{G}(c) = c$, *and*

(ii) either $\widehat{G}_c = (\widehat{G}_{1,c}, \widehat{G}_{2,c})$ and $[\![\widehat{F}_{i,c}, \widehat{G}_{i,c}]\!] = 1$, for $i = 1, 2$, or $\widehat{F}_{1,c}$ is conjugate with $\widehat{F}_{2,c}$.

3.3.6 Hyperbolic structures realizing abelian actions

In [2] and [30], a procedure to obtain the hyperbolic structures that realize cyclic subgroups of $\text{Mod}(S_g)$ as isometries was described. In this section, we use this procedure, and theory developed in Sections 3.1-3.2 to give an algorithm for obtaining the hyperbolic structures that realize a given two-generator finite abelian subgroup of $\text{Mod}(S_g)$ as an isometry group. Given a finite subgroup $H < \text{Mod}(S_g)$, let $\text{Fix}(H)$ denote the subspace of fixed points in the Teichmüller space $\text{Teich}(S_g)$ under the action of H. With this notation in place, we have the following elementary lemma.

Lemma 3.3.17. *Let $F, G \in \text{Mod}(S_g)$ be commuting finite-order mapping classes. Then*

$$\text{Fix}(\langle F, G\rangle) = \text{Fix}(\langle F\rangle) \cap \text{Fix}(\langle G\rangle).$$

Proof. Suppose that $x \in \text{Fix}(\langle F, G\rangle)$. Then $x \in \text{Fix}(\langle F\rangle)$ and $x \in \text{Fix}(\langle G\rangle)$, and so $x \in \text{Fix}(\langle F\rangle) \cap \text{Fix}(\langle G\rangle)$

Conversely, given $x \in \text{Fix}(\langle F\rangle) \cap \text{Fix}(\langle G\rangle)$, thus $F(x) = G(x) = x$ so $F^l G^k(x) = x$, for all l, k, which implies that $x \in \text{Fix}(\langle F, G\rangle)$. □

Given a weak conjugacy class of an abelian action $(H, (\mathcal{G}, \mathcal{F}))$ represented by

$$(m \cdot n, g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{r1}, n_{r1}), (c_{r2}, n_{r2}), n_r]),$$

we will now describe an algorithmic procedure for obtaining the conjugacy classes of its generators.

Step 1. It follows directly from our theory that the data sets

$$D_{\bar{G}} = (m, g_0; (c_{11}, n_{11}), \ldots, (c_{r1}, n_{r1})) \text{ and}$$
$$D_{\bar{F}} = (n, g_0; (c_{12}, n_{12}), \ldots, (c_{r2}, n_{r2}))$$

represent the conjugacy classes of the actions $\bar{\mathcal{G}}$ and $\bar{\mathcal{F}}$ induced on the orbifolds $\mathcal{O}_{H_1}$ and $\mathcal{O}_{H_2}$ by the actions of H_1 and H_2 on S_g, respectively.

Step 2. We now note that the orbifold signatures $\Gamma(\mathcal{O}_{H_i})$ have the form

$$\Gamma(\mathcal{O}_{H_1}) = (g(D_{\bar{G}}); (\underbrace{\frac{n_1}{n_{11}}, \ldots \frac{n_1}{n_{11}}}_{\frac{m}{n_{11}} \text{ times}}, \ldots, \underbrace{\frac{n_r}{n_{r1}}, \ldots \frac{n_r}{n_{r1}}}_{\frac{m}{n_{r1}} \text{ times}})) \text{ and}$$
$$\Gamma(\mathcal{O}_{H_2}) = (g(D_{\bar{F}}); (\underbrace{\frac{n_1}{n_{12}}, \ldots \frac{n_1}{n_{12}}}_{\frac{n}{n_{12}} \text{ times}}, \ldots, \underbrace{\frac{n_r}{n_{r2}}, \ldots \frac{n_r}{n_{r2}}}_{\frac{n}{n_{r2}} \text{ times}})),$$

with the understanding that if $n_i/n_{ij} = 1$, for some $1 \leq i \leq r$ and $j = 1, 2$, then we exclude it from the signatures.

Step 3. We choose conjugacy classes

$$D_F = (n, g_1; ((c_1, \frac{n_1}{n_{11}}), \frac{m}{n_{11}}), \ldots, ((c_r, \frac{n_r}{n_{r1}}), \frac{m}{n_{r1}})) \text{ and}$$
$$D_G = (m, g_2; ((d_1, \frac{n_1}{n_{12}}), \frac{n}{n_{12}}), \ldots, ((d_r, \frac{n_r}{n_{r2}}), \frac{n}{n_{r2}})),$$

where $c_i \equiv c_{i2} \frac{n_{i1}}{\gcd(n_{i1}, n_{i2})} \pmod{n_i/n_{i1}}$, $d_i \equiv c_{i1} \frac{n_{i2}}{\gcd(n_{i1}, n_{i2})} \pmod{n_i/n_{i2}}$, and the g_i are determined by Equation 2.2.1 in Definition 2.2.1.

Step 4. Finally, using Lemma 3.3.17, Theorem 2.2.4, and the subsequent discussion on the theory developed in [2, 30], we can obtain the hyperbolic structures that realize $\langle \mathcal{F}, \mathcal{G} \rangle$ as group of isometries.

In Table 3.1 at the end of this section, we give a complete classification of weak conjugacy classes of two-generator finite abelian subgroups of $\text{Mod}(S_3)$. Using the algorithm described above, in Figures 3.5-3.7 below, we provide geometric realizations of the weak conjugacy classes of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$-actions in S.Nos 7, 9 and 17 (of Table 3.1). The pairs of integers labeled in each figure are the pairs $\mathcal{P}_{[x]}$, which correspond to cone points $[x]$ in the quotient orbifold $\mathcal{O}_{\langle \mathcal{F} \rangle}$.

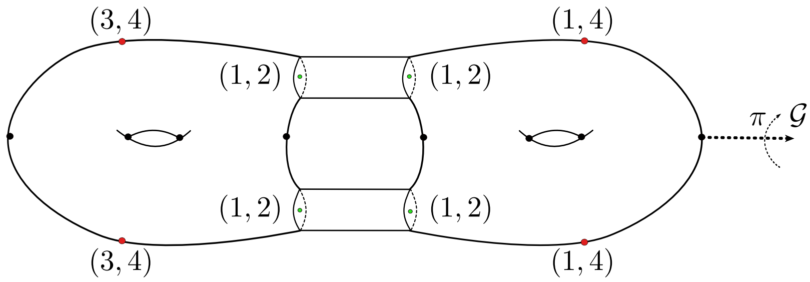


Figure 3.5: A realization of the action in S.No.7 of Table 3.1, with $D_G = (2, 0; ((1, 2), 8))$ and $D_F = (4, 0; ((1, 4), 2), ((3, 4), 2))$. D_F can be realized as a 2-compatibility between two actions F' and F'', where $D_{F'} = (4, 0; (1, 2), ((3, 4), 2))$ and $D_{F''} = (4, 0; (1, 2), ((1, 4), 2))$. Note that F' and F'' are realized rotations of the polygons $\mathcal{P}_{F'}$ and $\mathcal{P}_{F''}$ described in Theorem 2.2.4.

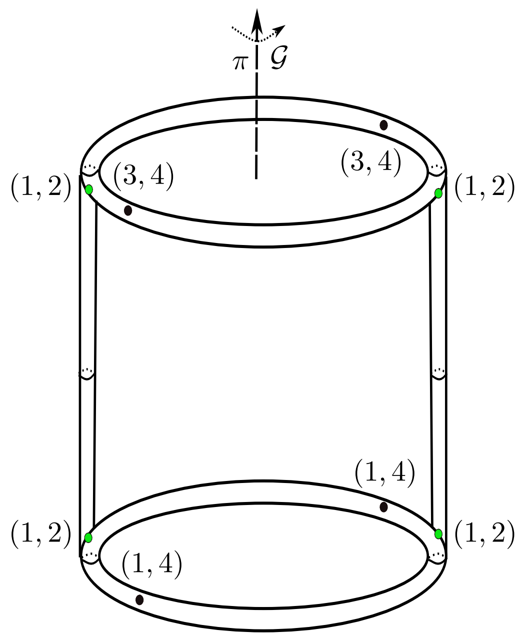


Figure 3.6: A realization of the action in S.No.17 of Table 3.1, with $D_G = (2, 2, 1;)$ and $D_F = (4, 0; ((1, 4), 2), ((3, 4), 2))$. Here, D_F can be realized as a 2-compatibility between two actions F' and F'' (realized as before), where $D_{F'} = (4, 0; (1, 2), ((3, 4), 2))$ and $D_{F''} = (4, 0; (1, 2), ((1, 4), 2))$.

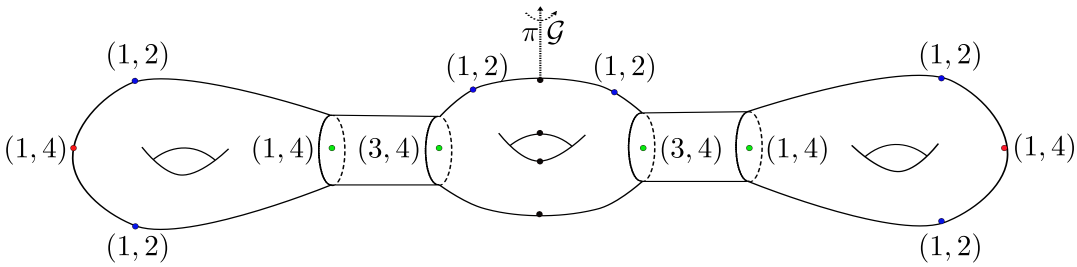


Figure 3.7: A realization of action in S.No.9 of Table 3.1, with $D_G = (2, 1; ((1, 2), 4))$ and $D_F = (4, 0; ((1, 2), 3), ((1, 4), 2))$. Here, D_F can be realized by 1-compatibilities of the two actions F' and F'' with F''', where $D_{F'} = (4, 0; (1, 2), ((1, 4), 2)), D_{F''} = (4, 0; (1, 2), ((1, 4), 2))$, and $D_{F'''} = (4, 0; (1, 2), ((3, 4), 2)$. Again F', F'' and F''' are irreducible Type 1 actions realized as rotations of polygons described in Theorem 2.2.4.

Note that the actions in S.Nos 19-26 in Table 3.1 have irreducible Type 1 cyclic actions as one of their generators. As the structures realizing such cyclic actions are unique, by Lemma 3.3.17, the abelian groups representing these weak conjugacy classes are realized as isometry groups by a unique structure.

S.No.	Abelian Data Set	Cyclic factors $[D_G; D_F]$
1	$(2\cdot 2, 1; [(1,2),(0,1),2]_2)^*$	$[(2,1;((1,2),4));(2,2,1;)]$
2	$(2\cdot 2, 1; [(1,2),(1,2),2]_2)$	$[(2,2,1;);(2,2,1;)]$
3	$(2\cdot 2, 0; [(1,2),(0,1),2]_2, [(0,1),(1,2),2]_2, [(1,2),(1,2),2]_2)$	$[(2,1;((1,2),4));(2,1;((1,2),4))]$
4	$(2\cdot 2, 0; [(1,2),(0,1),2]_4, [(0,1),(1,2),2]_2)$	$[(2,0;((1,2),8));(2,1;((1,2),4))]$
5	$(2\cdot 2, 0; [(1,2),(0,1),2]_4, [(1,2),(1,2),2]_2)$	$[(2,0;((1,2),8));(2,2,1;)]$
6	$(2\cdot 2, 0; [(1,2),(0,1),2]_2, [(1,2),(1,2),2]_4)$	$[(2,1;((1,2),4));(2,2,1;)]$
7	$(2\cdot 4, 0; [(1,2),(0,1),2]_2, [(0,1),(1,4),4], [(0,1),(3,4),4])$	$[(2,0;((1,2),8));(4,0;((1,4),2),((3,4),2))]$
8	$(2\cdot 4, 0; [(1,2),(0,1),2]_2, [(1,2),(1,4),4], [(1,2),(3,4),4])$	$[(2,0;((1,2),8));(4,1;((1,2),2))]$
9	$(2\cdot 4, 0; [(1,2),(0,1),2], [(0,1),(1,2),2], [(1,2),(1,4),4], [(0,1),(1,4),4])$	$[(2,1;((1,2),4));(4,0;((1,2),3),((1,4),2))]$
10	$(2\cdot 4, 0; [(1,2),(0,1),2], [(0,1),(1,2),2], [(1,2),(3,4),4], [(0,1),(3,4),4])$	$[(2,1;((1,2),4));(4,0;((1,2),3),((3,4),2))]$
11	$(2\cdot 4, 0; [(1,2),(0,1),2], [(1,2),(1,2),2], [(0,1),(1,4),4]_2)$	$[(2,1;((1,2),4));(4,0;((1,4),4))]$
12	$(2\cdot 4, 0; [(1,2),(0,1),2], [(1,2),(1,2),2], [(0,1),(3,4),4]_2)$	$[(2,1;((1,2),4));(4,0;((3,4),4))]$
13	$(2\cdot 4, 0; [(1,2),(0,1),2], [(1,2),(1,2),2], [(1,2),(1,4),4]_2)$	$[(2,1;((1,2),4));(4,1;((1,2),2))]$
14	$(2\cdot 4, 0; [(1,2),(0,1),2], [(1,2),(1,2),2], [(1,2),(3,4),4]_2)$	$[(2,1;((1,2),4));(4,1;((1,2),2))]$
15	$(2\cdot 4, 0; [(0,1),(1,2),2], [(1,2),(1,2),2], [(0,1),(1,4),4], [(1,2),(3,4),4])$	$[(2,2,1;);(4,0;((1,2),3),((1,4),2))]$
16	$(2\cdot 4, 0; [(0,1),(1,2),2], [(1,2),(1,2),2], [(0,1),(3,4),4], [(1,2),(1,4),4])$	$[(2,2,1;);(4,0;((1,2),3),((3,4),2))]$
17	$(2\cdot 4, 0; [(1,2),(1,2),2]_2, [(0,1),(1,4),4], [(0,1),(3,4),4])$	$[(2,2,1;);(4,0;((1,4),2),((3,4),2))]$
18	$(2\cdot 4, 0; [(1,2),(1,2),2]_2, [(1,2),(3,4),4], [(1,2),(1,4),4])$	$[(2,2,1;);(4,1;((1,2),2))]$
19	$(2\cdot 8, 0; [(1,2),(0,1),2], [(0,1),(1,8),8], [(1,2),(7,8),8])$	$[(2,0;((1,2),8));(8,0;(3,4),((1,8),2))]$
20	$(2\cdot 8, 0; [(1,2),(0,1),2], [(0,1),(3,8),8], [(1,2),(5,8),8])$	$[(2,0;((1,2),8));(8,0;(1,4),((3,8),2))]$
21	$(2\cdot 8, 0; [(1,2),(0,1),2], [(0,1),(5,8),8], [(1,2),(3,8),8])$	$[(2,0;((1,2),8));(8,0;(3,4),((5,8),2))]$
22	$(2\cdot 8, 0; [(1,2),(0,1),2], [(0,1),(7,8),8], [(1,2),(1,8),8])$	$[(2,0;((1,2),8));(8,0;(1,4),((7,8),2))]$
23	$(2\cdot 8, 0; [(1,2),(1,2),2], [(0,1),(1,8),8], [(1,2),(3,8),8])$	$[(2,2,1;);(8,0;(3,4),((1,8),2))]$
24	$(2\cdot 8, 0; [(1,2),(1,2),2], [(0,1),(3,8),8], [(1,2),(1,8),8])$	$[(2,2,1;);(8,0;(1,4),((3,8),2))]$
25	$(2\cdot 8, 0; [(1,2),(1,2),2], [(0,1),(5,8),8], [(1,2),(7,8),8])$	$[(2,2,1;);(8,0;(3,4),((5,8),2))]$
26	$(2\cdot 8, 0; [(1,2),(1,2),2], [(0,1),(7,8),8], [(1,2),(5,8),8])$	$[(2,2,1;);(8,0;(1,4),((7,8),2))]$
27	$(4\cdot 4, 0; [(1,4),(0,1),4], [(3,4),(3,4),4], [(0,1),(1,4),4])$	$[(4,0;((1,4),4));(4,0;((1,4),4))]$
28	$(4\cdot 4, 0; [(1,4),(0,1),4], [(3,4),(1,4),4], [(0,1),(3,4),4])$	$[(4,0;((1,4),4));(4,0;((3,4),4))]$
29	$(4\cdot 4, 0; [(1,4),(0,1),4], [(1,4),(1,4),4], [(1,2),(3,4),4])$	$[(4,0;((1,4),4));(4,1;((1,2),2))]$
30	$(4\cdot 4, 0; [(1,4),(0,1),4], [(1,4),(3,4),4], [(1,2),(1,4),4])$	$[(4,0;((1,4),4));(4,1;((1,2),2))]$
31	$(4\cdot 4, 0; [(3,4),(0,1),4], [(0,1),(3,4),4], [(1,4),(1,4),4])$	$[(4,0;((3,4),4));(4,0;((3,4),4))]$
32	$(4\cdot 4, 0; [(3,4),(0,1),4], [(1,2),(1,4),4], [(3,4),(3,4),4])$	$[(4,0;((3,4),4));(4,1;((1,2),2))]$
33	$(4\cdot 4, 0; [(3,4),(0,1),4], [(3,4),(1,4),4], [(1,2),(3,4),4])$	$[(4,0;((3,4),4));(4,1;((1,2),2))]$
34	$(4\cdot 4, 0; [(1,2),(1,4),4], [(3,4),(1,4),4], [(3,4),(1,2),4])$	$[(4,1;((1,2),2));(4,1;((1,2),2))]$
35	$(4\cdot 4, 0; [(1,2),(3,4),4], [(3,4),(1,2),4], [(3,4),(3,4),4])$	$[(4,1;((1,2),2));(4,1;((1,2),2))]$
36	$(4\cdot 4, 0; [(1,4),(1,4),4], [(1,4),(1,2),4], [(1,2),(1,4),4])$	$[(4,1;((1,2),2));(4,1;((1,2),2))]$

Table 3.1: The weak conjugacy classes of two-generator finite abelian subgroups of $\mathrm{Mod}(S_3)$. (*The suffix refers to the multiplicity of the tuple in the abelian data set.)

CHAPTER 4

REPRESENTING PERIODIC MAPPING CLASSES AS WORDS IN DEHN TWISTS

In this chapter, depending on the nature of periodic mapping class $F \in \text{Mod}(S_g)$, we will use the concepts introduced in Chapter 2 and Section 4.1 to develop various methods for writing F as a word $\mathcal{W}(F)$ in Dehn twists (up to conjugacy), Consequently, we answer Question 2 (from Section 1.1) in the affirmative.

4.1 Relations involving Dehn twists

Let $i(c,d)$ denote the geometric intersection number of simple closed curves c and d in S_g. A collection $\mathcal{C} = \{c_1, \ldots, c_k\}$ of simple closed curves in S_g is said to form a *chain* if $i(c_i, c_{i+1}) = 1$, for $1 \leq i \leq k-1$, and $i(c_i, c_j) = 0$, if $|i-j| > 1$. We state the following basic fact [9, Section 1.2] about chains.

Lemma 4.1.1. *For $g \geq 1$, there is a unique chain in S_g of size $2g$, up to homeomorphism. Moreover, when $g > 1$, there is a unique chain in S_g of size $2g+1$, up to homeomorphism.*

A closed regular neighborhood of the union of curves in $\mathcal{C}$ is a subsurface S of S_g that has one or two boundary components, depending on whether k is even or odd. Let the isotopy classes of ∂S be represented by the curves d (resp. d_1, d_2) when k is even (resp. odd). Let T_c denote the left-handed Dehn twist about a simple closed curve c in S_g. We will make extensive use of the well known chain relation involving Dehn twists.

Proposition 4.1.2 (Chain relation)**.** *Let $\mathcal{C} = \{c_1, \ldots, c_k\}$ be a chain in S_g. Then:*

$$(T_{c_1} T_{c_2} \ldots T_{c_k})^{2k+2} = T_d, \quad \textit{when } k \textit{ is even, and}$$

$$(T_{c_1} T_{c_2} \ldots T_{c_k})^{k+1} = T_{d_1} T_{d_2}, \quad \textit{when } k \textit{ is odd.}$$

Equivalently, we have

$$\begin{aligned}(T_{c_1}^2 T_{c_2} \dots T_{c_k})^{2k} &= T_d, && \textit{when } k \textit{ is even, and} \\ (T_{c_1}^2 T_{c_2} \dots T_{c_k})^{k} &= T_{d_1} T_{d_2}, && \textit{when } k \textit{ is odd.}\end{aligned}$$

In each case of Proposition 4.1.2, we will denote the word enclosed within the parenthesis on the left hand side by $W_{\mathcal{C}}$. Note that for all $1 < i < k$, $W_{\mathcal{C}}(c_i) = c_{i+1}$. We will also use the following relations also known as the *hyperelliptic relations* (see [9, Chapter 5]) in $\text{Mod}(S_g)$, for $g \geq 2$.

Proposition 4.1.3. *For $g \geq 2$, let $\{c_1, \dots, c_{2g+1}\}$ be a chain in S_g. Then:*

$$\begin{aligned}(T_{c_{2g+1}} \dots T_{c_1} T_{c_1} \dots T_{c_{2g+1}})^2 &= 1, \textit{ and} \\ \left[T_{c_{2g+1}} \dots T_{c_1} T_{c_1} \dots T_{c_{2g+1}}, T_{c_{2g+1}}\right] &= 1,\end{aligned}$$

where $T_{c_{2g+1}} \dots T_{c_1} T_{c_1} \dots T_{c_{2g+1}}$ represents the conjugacy class of the standard hyperelliptic involution in $\text{Mod}(S_g)$ *encoded by* $(2, 0; ((1, 2), 2g + 2))$.

Let $c_1, c_2, c_3, d_1, d_2, d_3$, and b be the curves in S_1^3, as indicated in Figure 4.1 below. We

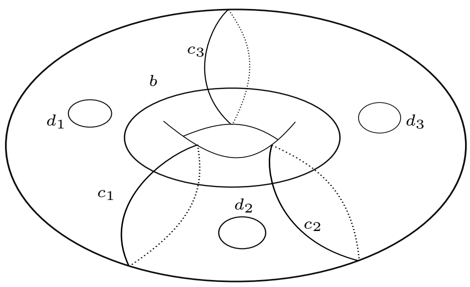


Figure 4.1: The curves involved in the star relation in $\text{Mod}(S_1^3)$.

will use the following relation in $\text{Mod}(S_1^3)$ due to Gervais [10], also known as the *star relation*, to develop a method for writing periodic mapping classes of order 3 as words in Dehn twists.

Proposition 4.1.4 (Star relation)**.** *Let $c_1, c_2, c_3, d_1, d_2, d_3$, and b be the curves in S_1^3, as indicated in Figure 4.1. Then:*

$$(T_{c_1}T_{c_2}T_{c_3}T_b)^3 = T_{d_1}T_{d_2}T_{d_3}.$$

In Section 4.5, we will derive a generalization of this relation, which we will apply to develop a method for obtaining the word representations for a larger family of periodic maps. The final result we state in this subsection pertains to the Burkhardt *handle swap* map [5, 24], which swaps the "i^{th} handle" in S_g (for $g \geq 2$) with the "$(i+1)^{st}$ handle".

Proposition 4.1.5. *For $g \geq 2$, let $a_1, b_1, \ldots, a_g, b_g$ be the curves that represent the standard generators of $H_1(S_g, \mathbb{Z})$. Then for $1 \leq i \leq g-1$, the i^{th} handle swap map is given by*

$$H_{i+1,i} := (T_{a_{i+1}}T_{b_{i+1}}T_{x_i}T_{a_i}T_{b_i})^3,$$

where x_i is a carefully chosen simple closed curve that represents the homology class $a_{i+1} + b_i$.

4.2 Periodic maps on the torus as words in Dehn twists

Since $\mathrm{Mod}(S_1) \cong \mathrm{SL}(2,\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, any nontrivial periodic element in $\mathrm{Mod}(S_1)$ is of order 2, 3, 4, or 6. Moreover, since $\{a, b\}$ (as indicated in Figure 4.2 below) is a chain in S_1, it follows by Proposition 4.1.2 that T_aT_b is of order 6, and $T_a^2T_b$ is of order 4 in $\mathrm{Mod}(S_1)$. Note that T_aT_b (resp. $T_a^2T_b$) is represented by a rotation of a regular hexagon (resp. square) with opposite sides identified, by $2\pi/6$ (resp. $\pi/2$).

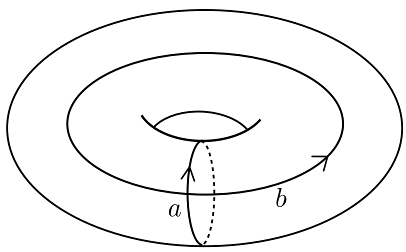


Figure 4.2: A chain in the torus.

Taking the powers of these maps, we obtain a word $\mathcal{W}(F)$ (in Dehn twists) representing the conjugacy class of each periodic element $F \in \mathrm{Mod}(S_1)$.

$\lvert F \rvert$	D_F	$\mathcal{W}(F)$
6	$(6, 0; (1, 2), (1, 3), (1, 6))$	$T_a T_b$
6	$(6, 0; (1, 2), (2, 3), (5, 6))$	$(T_a T_b)^5$
4	$(4, 0; (1, 2), (1, 4), (1, 4))$	$T_a^2 T_b$
4	$(4, 0; (1, 2), (3, 4), (3, 4))$	$(T_a^2 T_b)^3$
3	$(3, 0; (1, 3), (1, 3), (1, 3))$	$(T_a T_b)^2$
3	$(3, 0; (2, 3), (2, 3), (2, 3))$	$(T_a T_b)^4$
2	$(2, 0; ((1, 2), 4))$	$(T_a T_b)^3$

Table 4.1: Words (in Dehn twists) representing the conjugacy classes of periodic elements in $\mathrm{Mod}(S_1)$.

4.3 Rotational mapping classes as words in Dehn twists

In this section, we will provide a method for writing rotational mapping classes as products of Dehn twists. The key idea is to write given rotational mapping class as a product of two involutions, whose representations (as words) will be discussed in the following subsection.

4.3.1 Non-free involutions as words in Dehn twists

By Proposition 2.2.3, given an arbitrary involution $F \in \mathrm{Mod}(S_g)$, D_F has one of the following forms:

$$(2, g_0; ((1, 2), 2k)) \text{ or } (2, (g + 1)/2, 1;),$$

depending on whether $\mathcal{F}$ is non-free or free. First, we consider the cases $g = 1, 2$, where there are three possible conjugacy classes of involutions.

(a) The hyperelliptic involution in $\mathrm{Mod}(S_1)$: $(2, 0; ((1, 2), 4))$.

(b) The hyperelliptic involution in $\mathrm{Mod}(S_2)$: $(2, 0; ((1, 2), 6))$.

(c) The rotation of S_2 with two fixed points: $(2, 1; ((1, 2), 2))$.

The word representation for the involution in (a) was featured in Table 4.1, while the word for (b), the hyperelliptic involution, is known from Proposition 4.1.3. Since (c) swaps the two genera of S_2, it is the map $H_{2,1}$ from Propostion 4.1.5. We will collectively call these involutions the *fundamental involutions*. We will show that an arbitrary involution can be obtained by piecing together the fundamental involutions via 1-compatibilities. We will require the following lemmas, which are simple consequences of Proposition 4.1.2.

Lemma 4.3.1. *Let $\overline{H_{2,1}}$ be the restriction of $H_{2,1}$ on S_2^2. Then,*

$$\overline{H_{2,1}}^2 = T_{d_1} T_{d_2},$$

where d_1, d_2 are the boundary curves of S_2^2 as shown in Figure 4.3 below.

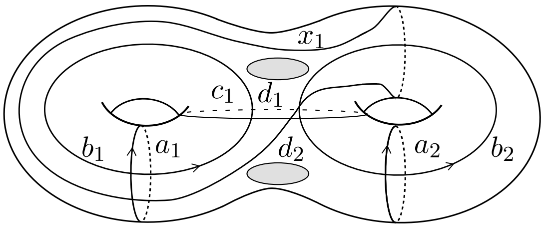


Figure 4.3: The boundary curves d_1, d_2 in S_2^2.

Lemma 4.3.2. *Let a, b, a', d_1, d_2 be the curves in S_1^2 as indicated in the Figure 4.4 below. Then*

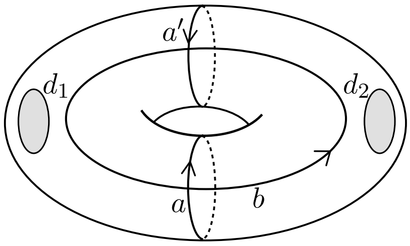


Figure 4.4: The curves a, b, a', d_1, d_2 in S_1^2.

$$(T_a T_b T_{a'})^4 = T_{d_1} T_{d_2}.$$

We will now provide an algorithm for writing involutions as words in the Dehn twists.

Algorithm 4.3.3. *Let $F \in \mathrm{Mod}(S_g)$ be a non-free involution with $D_F = ((2, g_0; ((1, 2), 2k))$ (by virtue of Lemma 2.2.3).*

Step 1. If $k = 1$, then:

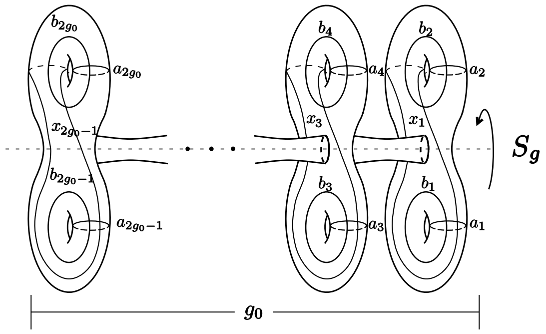


Figure 4.5: Decomposition of $\mathcal{F}$ for $k = 1$ into fundamental involutions.

Step 1a. We decompose $\mathcal{F}$ into fundamental involutions as shown in the Figure 4.5.

Step 1b. We set

$$\mathcal{W}(F) = \prod_{i=1}^{g_0} (H_{2i,2i-1})^{(-1)^{i-1}}.$$

Step 2. If $k > 1$, then:

Step 2a. We decompose $\mathcal{F}$ into fundamental involutions as shown in the Figure 4.6.

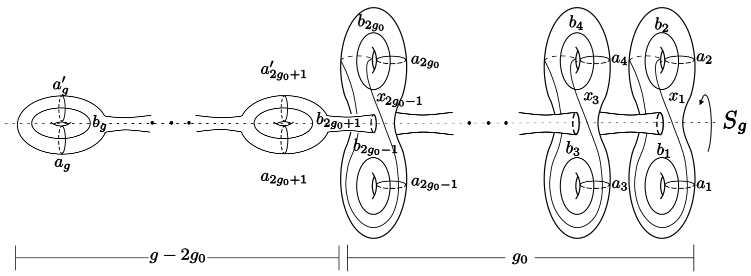


Figure 4.6: Decomposition of $\mathcal{F}$ for $k > 1$ into fundamental involutions.

Step 2b. We set

$$\mathcal{W}(F) = \prod_{i=1}^{g_0} (H_{2i,2i-1})^{(-1)^{i-1}} \prod_{j=2g_0+1}^{g} (T_{a_j} T_{b_j} T_{a'_j})^{2(-1)^{j+g_0-1}},$$

Step 3. By Proposition 4.1.5 and Lemmas 4.3.1-4.3.2, $\mathcal{W}(F)$ is the desired representation of F as a word in Dehn twists, up to conjugacy.

We apply above algorithm to get the word (upto conjugacy) of involutions in $\mathrm{Mod}(S_4)$.

Example 4.3.4. Up to conjugacy there are three distinct involutions F_i, $1 \le i \le 3$, in $\mathrm{Mod}(S_4)$ with

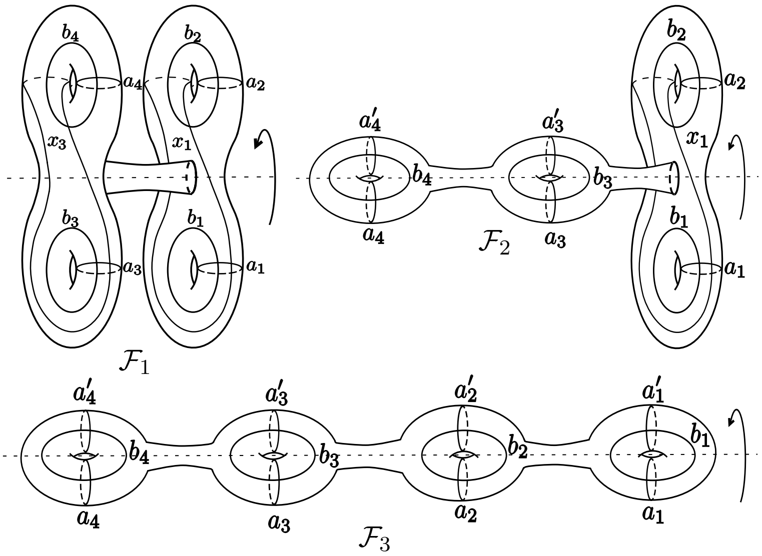


Figure 4.7: Realization of involutions in $\text{Mod}(S_4)$

(i) $D_{F_1} = (2, 2; ((1, 2), 2))$

(ii) $D_{F_2} = (2, 1; ((1, 2), 6))$

(iii) $D_{F_3} = (2, 0; ((1, 2), 10))$

The realizations of these involutions are given in Figure 4.7. By applying Algorithm 4.3.3, we can compute $\mathcal{W}(F_i)$.

(i) $\mathcal{W}(F_1) = (T_{a_2}T_{b_2}T_{x_1}T_{a_1}T_{b_1})^3(T_{a_4}T_{b_4}T_{x_3}T_{a_3}T_{b_3})^{-3}$

(ii) $\mathcal{W}(F_2) = (T_{a_2}T_{b_2}T_{x_1}T_{a_1}T_{b_1})^3(T_{a_3}T_{b_3}T_{a_3'})^{-2}(T_{a_4}T_{b_4}T_{a_4'})^2$

(iii) $\mathcal{W}(F_3) = (T_{a_1}T_{b_1}T_{a_1'})^2(T_{a_2}T_{b_2}T_{a_2'})^{-2}(T_{a_3}T_{b_3}T_{a_3'})^2(T_{a_4}T_{b_4}T_{a_4'})^{-2}$

It may be noted that in $\mathcal{W}(F_2)$, a_4 and a_4' are the isotopic curves. Similarly, in $\mathcal{W}(F_3)$, a_1 is isotopic to a_1' and a_4 is isotopic to a_4'.

A simple computation reveals that in general, Algorithm 4.3.3 will express an arbitrary non-free involution (up to conjugacy) as a word in $3g - 2 + \lfloor g/2 \rfloor$ Dehn twists about non-separating curves. However, it is important to note each application of Algorithm 4.3.5 may involve up to $3g - 2 + \binom{g}{2}$ (distinct) Dehn twists about nonseparating curves.

4.3.2 Surface rotations as words in Dehn twists

For $g \geq 2$, any rotation of S_g (that is free or non-free) of order $n \geq 2$ can be written as a product of two involutions, as illustrated in Figure 4.8 below. This leads to the following method for writing surface rotations as words in Dehn twists.

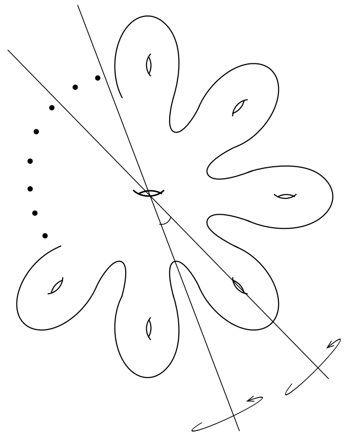


Figure 4.8: A surface rotation as a product of two involutions.

Algorithm 4.3.5. *Let $F \in \text{Mod}(S_g)$ be realized either as a rotation $\mathcal{F}$ of order $n > 2$ or as a free involution. Then by Proposition 2.2.3, D_F has the form*

$$(n, g_0; \underbrace{(s,n),(n-s,n),\ldots,(s,n),(n-s,n)}_{k\,pairs}) \text{ or } (n, \frac{g-1}{n}+1, r;),$$

depending on whether $\mathcal{F}$ is free rotation or not.

Step 1. Consider an embedding of S_g in $\mathbb{R}^3$, as indicated in Figure 4.8, with the understanding that there is a "genus in the middle", only when $\mathcal{F}$ is free.

Step 2. For $i = 1, 2$, let the reflection along the axis X_i (as shown in the figure) be Θ_i, where Θ_i is a non-free involution determined by Algorithm 4.3.3. We set $R_g = \Theta_1 \cdot \Theta_2$.

Step 3. If $\mathcal{F}$ is free, then we set $\mathcal{W}(F) = R_g^{(g-1)r/n}$, else we set $\mathcal{W}(F) = R_g^{\frac{gs^{-1}}{n}}$.

Step 4. $\mathcal{W}(F)$ is the desired representation of F as a word in Dehn twists, up to conjugacy.

4.4 Chain method

In this section, we provide a method by which one can write certain periodic mapping classes as words in Dehn twists by repeated application of the chain relation. The key idea is to decompose certain periodic mapping classes into components that are realized as powers of irreducible Type 1 actions that are representable (as words) using the chain relation. Let $F \in \text{Mod}(S_g)$ be an irreducible Type 1 mapping class. Let $F \in \text{Mod}(S_g)$ be of order n, and let $(1, n)$ be a pair in D_F representing a fixed point of the $\langle \mathcal{F} \rangle$-action on S_g. Now consider the mapping class F^m, for some integer $1 \leq m \leq |F|$. Then in D_{F^m}, there exists a pair (c', n') (representing a fixed point of the $\langle \mathcal{F}^m \rangle$-action on S_g) that originated from the pair $(1, n)$ such that $n' = |F^m| = n/\gcd(m, n)$ and $(c')^{-1} \equiv m/\gcd(m, n) \pmod{n'}$. We will denote this pair (c', n') in D_{F^m} by $(1, n)_{m,F}$.

Definition 4.4.1. Let $F \in \text{Mod}(S_g)$ be realizable as a linear s-tuple $(F_1, \ldots, F_s)$ of degree n and genus g as in Definition 2.2.7. Then F is said to be *chain-realizable* if F admits a realization as a linear s-tuple $(F_1, \ldots, F_s)$ of genus g such that the following conditions hold.

(i) For $1 \leq i \leq s$, there exists an irreducible Type 1 mapping class $\tilde{F}_i \in \text{Mod}(S_{g_i})$, a filling chain $\mathcal{C}(\tilde{F}_i)$ in S_{g_i}, and an $m_i \geq 1$ such that F_i is conjugate to $(W_{\mathcal{C}(\tilde{F}_i)})^{m_i}$. Then:

  (a) For each i, $D_{\tilde{F}_i}$ has one of the following forms on S_{g_i}

   1. $(2g_i + 1, 0; (2g_i - 1, 2g_i + 1), (1, 2g_i + 1), (1, 2g_i + 1))$
   2. $(2g_i + 2, 0; (g_i, g_i + 1), (1, 2g_i + 2), (1, 2g_i + 2))$,
   3. $(4g_i, 0; (1, 2), (1, 4g_i), (2g_i - 1, 4g_i))$,
   4. $(4g_i + 2, 0; (1, 2), (g_i, 2g_i + 1), (1, 4g_i + 2))$

  (b) For $1 \leq i \leq s - 1$, $k_i = 1$, and for each pair (F_i, F_{i+1}), the 1-compatibility is across a pair of fixed points represented by pairs of the form (c_i, n) (in D_{F_i}) and $(n - c_i, n)$ (in $D_{F_{i+1}}$), where $(c_i, n) = (1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}$ and $(n - c_i, n) = (1, |\tilde{F}_{i+1}|)_{m_{i+1}, \tilde{F}_{i+1}}$.

Definition 4.4.2. A periodic mapping class $G \in \text{Mod}(S_g)$ is said to be *chain-realizable* if there exists a chain-realizable linear s-tuple $F \in \text{Mod}(S_g)$ and a nonzero integer q such that $G = F^q$.

Given $c \in \mathbb{Z}_n^{\times}$, we will fix the following notation.

(a) $c^+ = c(+1) := \{d \in \mathbb{Z} : cd \equiv 1 \pmod{n}\} \cap [0, n]$.

(b) $c^- = c(-1) := \{d \in \mathbb{Z} : cd \equiv 1 \pmod{n}\} \cap [-n, 0]$.

Lemma 4.4.3. *Let $F \in \text{Mod}(S_g)$ be realizable as a chain-realizable s-tuple of degree n and genus g as in Definition 4.4.1. For all i, let $\mathcal{W}(F_i) = (W_{\mathcal{C}(\tilde{F}_i)})^{\beta_i \bar{c}_i}$, where $\bar{c}_i = c_i((-1)^{i+1})$ and $\beta_i = \frac{|\tilde{F}_i|}{|F_i|}$. Then:*

$$\mathcal{W}(F) = \prod_{i=1}^{s} \mathcal{W}(F_i)$$

is conjugate to F.

Proof. By Proposition 4.1.2, for each i, $((W_{\mathcal{C}(\tilde{F}_i)})^{\beta_i \bar{c}_i})^{|F_i|}$ equals either $(T_{d_1} T_{d_2})^{\bar{c}_i}$ or $(T_{d_1})^{\bar{c}_i}$, depending upon whether $|\mathcal{C}(\tilde{F}_i)|$ is odd or even. Thus, $\bar{c}_i$ measures the amount of twisting along the boundary of a closed neighborhood of the chain $\mathcal{C}(\tilde{F}_i)$. Thus, by Construction 2.2.6 and Definition 4.4.1, we have that $\mathcal{W}(F)$ is conjugate of F. Finally, since $\mathcal{W}(F_i)$ commutes with $\mathcal{W}(F_j)$ for all $1 \leq i, j \leq s$, we have

$$(\mathcal{W}(F))^n = (\prod_{i=1}^{s} \mathcal{W}(F_i))^n = \prod_{i=1}^{s} \mathcal{W}(F_i)^n = \prod_{i=1}^{s} (T_{d_{i1}} T_{d_{i2}})^{\bar{c}_i} = 1,$$

where d_{i2} is taken to be the trivial curve when $i = 1, s$. □

We will now provide an algorithm for representing a chain-realizable periodic mapping classes as words in Dehn twists.

Algorithm 4.4.4 (Chain method)**.** *Let $G \in \text{Mod}(S_g)$ be a chain-realizable periodic mapping class.*

Step 1. Write $G = F^q$, where F is a compatible chain-realizable s-tuple $(F_1, \ldots, F_s)$ of degree n and genus g as in Definition 4.4.1.

Step 2. Set $\mathcal{W}(F_i) = (W_{\mathcal{C}(\tilde{F}_i)})^{\beta_i \bar{c}_i}$, where $\bar{c}_i = c_i((-1)^{i+1})$, and set

$$\mathcal{W}(F) = \prod_{i=1}^{s} \mathcal{W}(F_i).$$

Step 3. By Lemma 4.4.3, $\mathcal{W}(G) = \mathcal{W}(F)^q$ is the desired representation of G as a word in Dehn twists, up to conjugacy.

Example 4.4.5. For $i = 1, 2$, consider the order 6 mapping classes $F_i \in \text{Mod}(S_1)$ with $D_{F_1} = (6, 0; (1, 2), (1, 3), (1, 6))$ and $D_{F_2} = (6, 0; (1, 2), (2, 3), (5, 6))$. The $\mathcal{F}_i$ admit a 1-compatibility along a pair of compatible fixed points that correspond to the pairs $(1, 6)$ and $(5, 6)$ in the D_{F_i} where the induced rotation angles are $2\pi/6$ and $10\pi/6$, respectively.

This 1-compatibility yields an $F = (F_1, F_2) \in \text{Mod}(S_2)$ with $D_F = (6, 0; (1, 2), (1, 2), (1, 3), (2, 3))$. If $\mathcal{C}(F_1) = \{a_1, b_1\}$ and $\mathcal{C}(F_2) = \{a_2, b_2\}$, then by Table 4.1 and Algorithm 4.4.4, F is represented up to conjugacy by the word

$$\mathcal{W}(F) = (T_{a_1}T_{b_1})(T_{a_2}T_{b_2})^{-1}.$$

4.4.1 Periodic maps on S_2 as words in Dehn twists

Let a_1, b_1, c_1, a_2, b_2, and x_1 be curves in S_2, as indicated in Figure 4.9 below.

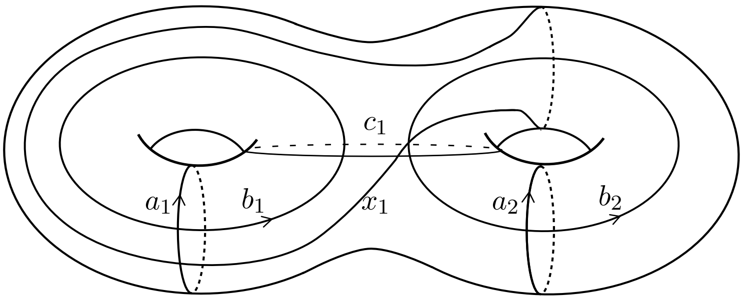


Figure 4.9: Curves a_1, b_1, c_1, a_2, b_2, and x_1 in S_2.

Using Algorithms 4.3.3 and 4.4.4, in Table 4.2 below, we provide a word $\mathcal{W}(F)$ (in Dehn twists) representing the conjugacy class of each periodic element $F \in \text{Mod}(S_2)$.

4.5 Generalized star method

In this section, we first derive a generalization of Proposition 4.1.4 for $g \geq 2$. Using this result, we will develop a method to represent a much larger family of periodic mapping classes as words in Dehn twists, as compared with the chain method. As we will see, this family will also encompass the family of periodics described in Definition 4.4.1. Let

$$a_1', a_1, \ldots a_g, b_1, \ldots, b_g, c_1, \ldots, c_{g-1}, d_1, d_2, \text{ and } d_3$$

be the isotopy classes of the simple closed curves in S_g^3, as shown in Figure 4.10 below. Note that the curves a_1 and a_1' are isotopic in the surface ($\approx S_g^2$) obtained by capping off the boundary curve d_2. Further, we consider the surface S_g^2 obtained by capping off the boundary curve d_3. We have the following generalization of the star relation, which is due to Salter [37] (for three boundary components) and Matsumoto [25] (for two boundary components). However, we provide an alternative proof of this result using the Alexander

$\lvert F\rvert$	D_F	$\mathcal{W}(F)$	Algorithm
10	$(10, 0; (1, 2), (2, 5), (1, 10))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2})$	4.4.4
10	$(10, 0; (1, 2), (1, 5), (3, 10))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2})^7$	4.4.4
10	$(10, 0; (1, 2), (4, 5), (7, 10))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2})^3$	4.4.4
10	$(10, 0; (1, 2), (3, 5), (9, 10))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2})^9$	4.4.4
8	$(8, 0; (1, 2), (1, 8), (3, 8))$	$T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}$	4.4.4
8	$(8, 0; (1, 2), (5, 8), (7, 8))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2})^5$	4.4.4
6	$(6, 0; ((1, 2), 2), (1, 3), (2, 3))$	$(T_{a_1}T_{b_1})(T_{a_2}T_{b_2})^{-1}$	4.4.4
6	$(6, 0; (2, 3), (1, 6), (1, 6))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{a_2})$	4.4.4
6	$(6, 0; (1, 3), (5, 6), (5, 6))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{a_2})^5$	4.4.4
5	$(5, 0; ((1, 5), 2), (3, 5))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{a_2})$	4.4.4
5	$(5, 0; ((2, 5), 2), (1, 5))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{a_2})^3$	4.4.4
5	$(5, 0; ((3, 5), 2), (4, 5))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{a_2})^2$	4.4.4
5	$(5, 0; ((4, 5), 2), (2, 5))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{a_2})^4$	4.4.4
4	$(4, 0; ((1, 2), 2), (1, 4), (3, 4))$	$(T_{a_1}T_{b_1}T_{a_1})(T_{a_2}T_{b_2}T_{a_2})^{-1}$	4.4.4
3	$(3, 0; ((1, 3), 2), ((2, 3), 2)$	$(T_{a_1}T_{b_1})^2(T_{a_2}T_{b_2})^{-2}$	4.4.4
2	$(2, 0; ((1, 2), 6))$	$(T_{a_1}T_{b_1}T_{a_1})^2(T_{a_2}T_{b_2}T_{a_2})^{-2}$	4.3.3
2	$(2, 1; (1, 2), (1, 2))$	$(T_{a_2}T_{b_2}T_{x_1}T_{a_1}T_{b_1})^3$	4.3.3

Table 4.2: Words (in Dehn twists) representing the conjugacy classes of periodic elements in $\mathrm{Mod}(S_2)$.

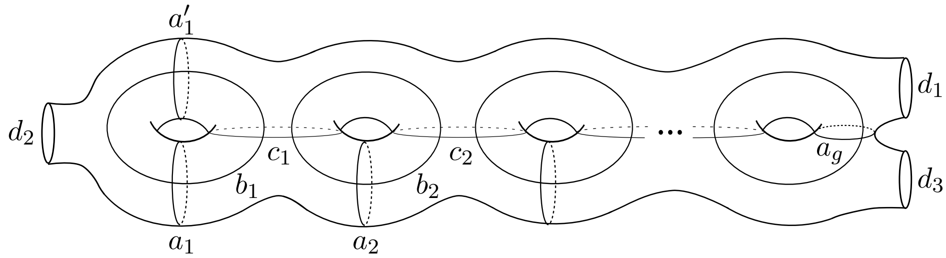


Figure 4.10: The curves S_g^3 involved in the generalized star relation.

method.

Theorem 4.5.1 (Generalized star relation)**.** *For $g \geq 2$ and $k = 2, 3$, the following relations hold in* $\mathrm{Mod}(S_g^k)$.

(i) When $k = 2$, we have:

$$(T_{a_1} T_{a_1'} \prod_{i=1}^{g-1} (T_{b_i} T_{c_i}) T_{b_g})^{4g} = T_{d_2}^{(2g-1)^+} T_{d_1},$$

where $2g - 1 \in \mathbb{Z}_{4g}^{\times}$.

(ii) When $k = 3$, we have:

$$(T_{a_1} T_{a_1'} \prod_{i=1}^{g-1} (T_{b_i} T_{c_i}) T_{b_g} T_{a_g})^{2g+1} = T_{d_2}^{(2g-1)^+} T_{d_1} T_{d_3},$$

where $2g - 1 \in \mathbb{Z}_{2g+1}^{\times}$.

Proof. To prove the result, we will use the well known Alexander method (see [9, Proposition 2.8]). For simplicity, we will only consider the case when $g = 2$, as our arguments easily generalize for $g > 2$. We provide our proofs for $k = 2$ and $k = 3$ through a series of pictures shown in Figures 4.12-4.13 below. The filling we consider (for the application of the Alexander method) is indicated in Figure 4.11.

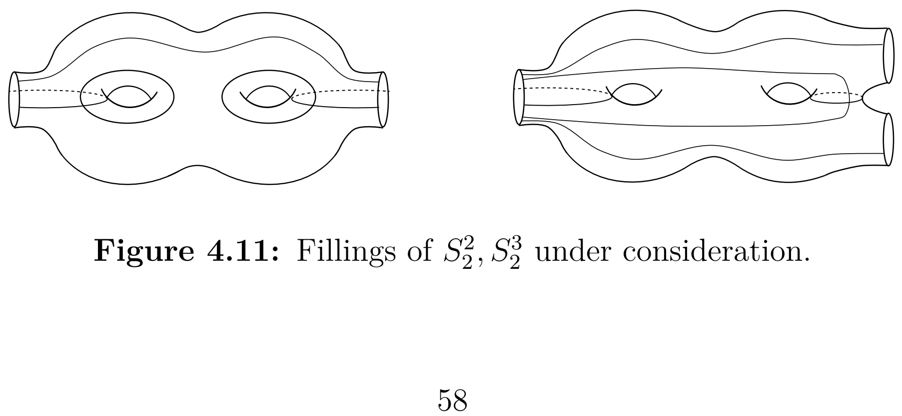


Figure 4.11: Fillings of S_2^2, S_2^3 under consideration.

□

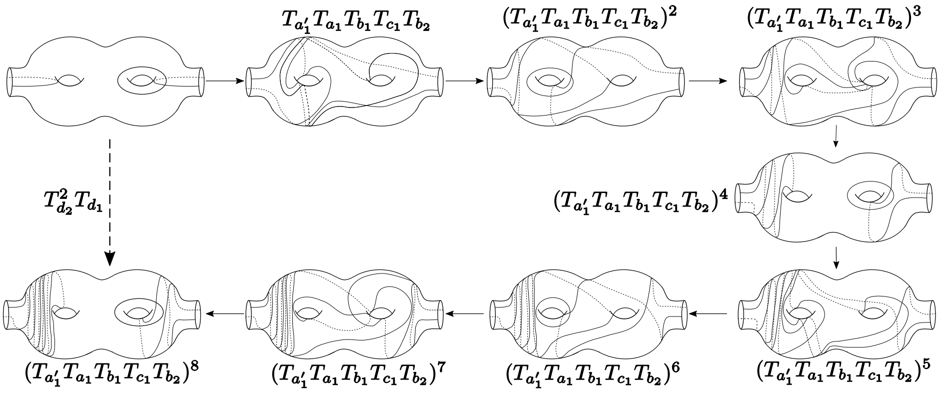


Figure 4.12: Proof of the generalized star relation in $\mathrm{Mod}(S_2^2)$.

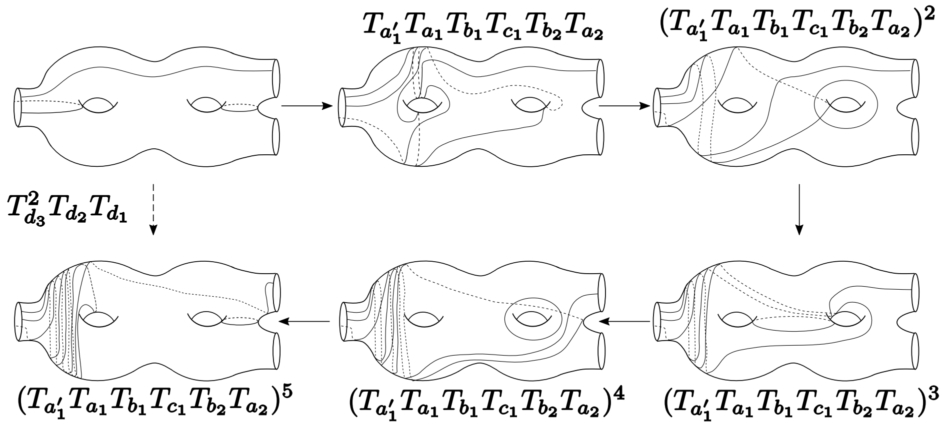


Figure 4.13: Proof of the generalized star relation in $\mathrm{Mod}(S_2^3)$.

Clearly, Theorem 4.5.1 is a generalization of Proposition 4.1.4. Moreover, by capping the boundary curve d_2, we can also recover Proposition 4.1.2. Following the notation from Section 4.4, we will now introduce a family of periodic mapping classes for which we will develop a method (of deriving $\mathcal{W}(F)$) using Theorem 4.5.1.

Definition 4.5.2. Let $F \in \mathrm{Mod}(S_g)$ be realizable as a linear s-tuple $(F_1, \ldots, F_s)$ of degree n and genus g as in Definition 2.2.7. Then F is said to be *star-realizable* if F admits a realization as a linear s-tuple $(F_1, \ldots, F_s)$ of genus g such that the following conditions hold.

(i) For $1 \leq i \leq s$, there exists an irreducible Type 1 mapping class $\tilde{F}_i \in \mathrm{Mod}(S_{g_i})$, a filling chain $\mathcal{C}(\tilde{F}_i)$ in S_{g_i}, and an $m_i \geq 1$ such that F_i is conjugate to $(W_{\mathcal{C}(\tilde{F}_i)})^{m_i}$. Then:

   (a) For each i, $D_{\tilde{F}_i}$ has one of the following forms on S_{g_i}

      1. $(2g_i + 2, 0; (g_i, g_i + 1), (1, 2g_i + 2), (1, 2g_i + 2))$,

2. $(4g_i, 0; (1,2), (1, 4g_i), (2g_i - 1, 4g_i))$,

3. $(4g_i + 2, 0; (1,2), (g_i, 2g_i + 1), (1, 4g_i + 2))$

4. $(2g_i + 1, 0; (2g_i - 1, 2g_i + 1), (1, 2g_i + 1), (1, 2g_i + 1))$

(b) For $1 \leq i \leq s-1$, $k_i = 1$, and for each pair (F_i, F_{i+1}), the 1-compatibility is across a pair of fixed points represented by pairs of the form (c_i, n) (in D_{F_i}) and $(n - c_i, n)$ (in $D_{F_{i+1}}$), where $(c_i, n) \in \{(1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i|/2 - 1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i| - 2, |\tilde{F}_i|)_{m_i, \tilde{F}_i}\}$ and $(n - c_i, n) \in \{(1, |\tilde{F}_{i+1}|)_{m_{i+1}, \tilde{F}_{i+1}}, (|\tilde{F}_{i+1}|/2 - 1, |\tilde{F}_{i+1}|)_{m_{i+1}, \tilde{F}_{i+1}}, (|\tilde{F}_{i+1}| - 2, |\tilde{F}_{i+1}|)_{m_{i+1}, \tilde{F}_{i+1}}\}$.

We will now fix the following notation:

$$W_j := \begin{cases} T_{a_1} \prod_{i=1}^{g-1} (T_{b_i} T_{c_i}) T_{b_g}, & \text{if } j = 4g+2, \\ T_{a_1} T_{a_1'} \prod_{i=1}^{g-1} (T_{b_i} T_{c_i}) T_{b_g}, & \text{if } j = 4g, \\ T_{a_1} \prod_{i=1}^{g-1} (T_{b_i} T_{c_i}) T_{b_g} T_{a_g}, & \text{if } j = 2g+2, \text{ and} \\ T_{a_1} T_{a_1'} \prod_{i=1}^{g-1} (T_{b_i} T_{c_i}) T_{b_g} T_{a_g}, & \text{if } j = 2g+1, \end{cases}$$

where

$$D_{W_j} = \begin{cases} (4g+2, 0; (1,2), (g, 2g+1), (1, 4g+2)), & \text{if } j = 4g+2, \\ (4g, 0; (1,2), (1,4g), (2g-1, 4g)), & \text{if } j = 4g, \\ (2g+2, 0; (g, g+1), (1, 2g+2), (1, 2g+2)), & \text{if } j = 2g+2, \text{ and} \\ (2g+1, 0; (1, 2g+1), (1, 2g+1), (2g-1, 2g+1)), & \text{if } j = 2g+1. \end{cases}$$

Let d_i denote the boundary curve of Σ_i involved in the 1-compatibility of F_i with F_{i+1}, and let γ_i represent the isotopy class of d_i in S_g after the compatibility. Let c^+ denote the unique integer in $[0, n]$ representing the multiplicative inverse of $c \in \mathbb{Z}_n^\times$. We further fix the following notation.

$$\begin{aligned}
\mu_{1,i} &= \frac{m_i}{\gcd(m_i, |\tilde{F}_i|)}, && \text{if } (c_i, n) = (1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, \\
\mu_{2,i} &= \frac{m_i}{\gcd(m_i, |\tilde{F}_i|)}(|\tilde{F}_i|/2 - 1)^+, && \text{if } (c_i, n) = (|\tilde{F}_i|/2 - 1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, \text{ and} \\
\mu_{3,i} &= \frac{m_i}{\gcd(m_i, |\tilde{F}_i|)}(|\tilde{F}_i| - 2)^+, && \text{if } (c_i, n) = (|\tilde{F}_i| - 2, |\tilde{F}_i|)_{m_i, \tilde{F}_i}.
\end{aligned}$$

By our notation, for each i, there exists a unique $z_i \in \{1,2,3\}$ such that we have $c_i = \mu_{z_i,i}^{-1} \pmod n$, and since $c_i + c_{i+1} \equiv 0 \pmod n$, we have $\mu_{z_i,i} + \mu_{z_{i+1},i+1} \equiv 0 \pmod n$. With this notation in place, we have the following lemma, which provides a word $\mathcal{W}(F)$ in Dehn twists that represents the conjugacy class of a star-realizable linear s-tuple F.

Lemma 4.5.3. *Let $F \in \mathrm{Mod}(S_g)$ be a star-realizable linear s-tuple of degree n as in Definition 4.5.2. For all i, let $\mathcal{W}(F_i) = (W_{|\tilde{F}_i|})^{m_i}$. Then:*

$$\mathcal{W}(F) = \left(\prod_{i=1}^{s} \mathcal{W}(F_i)\right) \prod_{i=1}^{s-1} (T_{\gamma_i})^{-\eta_i},$$

where, $\eta_i = \dfrac{\mu_{z_i,i} + \mu_{z_{i+1},i+1}}{n}$, is conjugate to F.

Proof. Since $\mathcal{W}(F_i)$ commutes with $\mathcal{W}(F_j)$ for $1 \le i, j \le s$, we have

$$(\prod_{i=1}^{s} \mathcal{W}(F_i))^n = \prod_{i=1}^{s} (\mathcal{W}(F_i))^n.$$

Since $\mathcal{W}(F_i) = (W_{|\tilde{F}_i|})^{m_i}$, the fact that

$$(c_i, n) \in \{(1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i|/2 - 1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i| - 2, |\tilde{F}_i|)_{m_i, \tilde{F}_i}\}$$

implies that

$$\prod_{i=1}^{s} (\mathcal{W}(F_i))^n = \prod_{i=1}^{s} ((W_{|\tilde{F}_i|})^{|\tilde{F}_i|})^{\frac{m_i}{\gcd(m_i, |\tilde{F}_i|)}}.$$

By Theorem 4.5.1, depending on $|\tilde{F}_i|$ and $D_{\tilde{F}_i}$, $(W_{|\tilde{F}_i|})^{|\tilde{F}_i|}$ one of:

$$T_{d_1},\ T_{d_1} T_{d_2}^{(|\tilde{F}_i|/2-1)^+},\ \text{or } T_{d_1} T_{d_2}^{(|\tilde{F}_i|-2)^+} T_{d_3}.$$

By the definition of $\mu_{z_i,i}$, we have

$$(\prod_{i=1}^{s} \mathcal{W}(F_i))^n = \prod_{i=1}^{s} (\mathcal{W}(F_i))^n = \prod_{i=1}^{s-1} \left(T_{\gamma_i}^{\mu_{z_i,i}+\mu_{z_{i+1},i+1}} \right) \tag{*}$$

As each T_{γ_i} commutes with every other Dehn twist appearing in $(*)$ and $\mu_{z_i,i}+\mu_{z_{i+1},i+1} \equiv 0 \pmod{n}$, we get

$$\mathcal{W}(F)^n = \left(\left(\prod_{i=1}^{s} \mathcal{W}(F_i) \right) \prod_{i=1}^{s-1} (T_{\gamma_i})^{-\eta_i} \right)^n = \left(\prod_{i=1}^{s} \mathcal{W}(F_i) \right)^n \prod_{i=1}^{s-1} (T_{\gamma_i})^{-\eta_i n} = 1,$$

from which the assertion follows. □

We will describe an algorithm to write a star-realizable linear s-tuple $F \in \text{Mod}(S_g)$ as a word in Dehn twists (up to conjugacy).

Algorithm 4.5.4. *Let $F \in \text{Mod}(S_g)$ be a star-realizable linear s-tuple of degree n and genus g.*

Step 1. Write $F = (F_1, \ldots, F_s)$ as in Definition 4.5.2.

Step 2. For each i, we set $\mathcal{W}(F_i) = W_{|\tilde{F}_i|}^{m_i}$, after appropriately relabeling the curves in Σ_i in order to ensure consistency with the (assumed) labeling in Theorem 4.5.1.

Step 3. Set

$$\mathcal{W}(F) = \left(\prod_{i=1}^{s} \mathcal{W}(F_i) \right) \prod_{i=1}^{s-1} (T_{\gamma_i})^{-\eta_i}.$$

Step 4. By Lemma 4.5.3, $\mathcal{W}(F)$ is the desired representation of F as a word in Dehn twists, up to conjugacy.

The method described in Algorithm 4.5.4 can be generalized to certain types of $(F, \mathcal{T})$-tuples.

Definition 4.5.5. A compatible $(F, \mathcal{T})$-tuple as in Definition 2.2.11 is said to be *star-realizable* if the following conditions hold.

(i) $v = w = 0$.

(ii) F is star-realizable.

(iii) For $1 \leq q \leq u$, $k_q = 1$

(iv) For $1 \leq q \leq u$, suppose the self 1-compatibility in $\mathcal{F}_{i_q,j_q}$ is along fix points represented by (c_{i_q}, n) (in $D_{F_{i_q}}$) and $(n - c_{i_q}, n)$ (in $D_{F_{j_q}}$), then

$$(c_{i_q}, n) \in \{(1, |\tilde{F}_{i_q}|)_{m_{i_q},\tilde{F}_{i_q}}, (|\tilde{F}_{i_q}|/2 - 1, |\tilde{F}_{i_q}|)_{m_{i_q},\tilde{F}_{i_q}}, (|\tilde{F}_{i_q}| - 2, |\tilde{F}_{i_q}|)_{m_{i_q},\tilde{F}_{i_q}}\}$$

and

$$(n - c_{i_q}, n) \in \{(1, |\tilde{F}_{j_q}|)_{m_{j_q},\tilde{F}_{j_q}}, (|\tilde{F}_{j_q}|/2 - 1, |\tilde{F}_{j_q}|)_{m_{j_q},\tilde{F}_{j_q}}, (|\tilde{F}_{j_q}| - 2, |\tilde{F}_{j_q}|)_{m_{j_q},\tilde{F}_{j_q}}\}.$$

Definition 4.5.6. A periodic mapping class $G \in \text{Mod}(S_g)$ is said to be *star-realizable* if there exists a star-realizable compatible $(F, \mathcal{T})$-tuple $F_{\mathcal{T}} \in \text{Mod}(S_g)$ and a nonzero integer m such that $G = F_{\mathcal{T}}^m$.

We will now extend Algorithm 4.5.4 to this broader class of periodic mapping classes. While doing so, we will retain the notation for γ_i and η_i, for $1 \leq i \leq s$ (for F) from Algorithm 4.5.4. To further simplify notation, we will denote the additional curves involved in the additional 1-self compatibilities (of $\mathcal{F}_{\mathcal{T}}$) by $\{\gamma_j\}_{j=s}^{u+s-1}$ and also extend the earlier definition of η_j to $s \leq j \leq u + s - 1$.

Algorithm 4.5.7. *Let $G \in \text{Mod}(S_g)$ be a star-realizable periodic mapping class.*

Step 1. Write $G = F_{\mathcal{T}}^m$, where $F_{\mathcal{T}}$ is a compatible $(F, \mathcal{T})$-tuple as in Definition 2.2.11.

Step 2. By Algorithm 4.5.4, we obtain

$$\mathcal{W}(F) = \left(\prod_{i=1}^{s} \mathcal{W}(F_i)\right) \prod_{i=1}^{s-1} (T_{\gamma_i})^{-\eta_i}.$$

Step 3. We set

$$\mathcal{W}(F_{\mathcal{T}}) = \mathcal{W}(F) \prod_{i=s}^{u+s-1} (T_{\gamma_i})^{-\eta_i}.$$

Step 4. By the same arguments from Lemma 4.5.3, $(\mathcal{W}(F_{\mathcal{T}}))^m$ is the desired representation of G as a word in Dehn twists (after an appropriate relabeling of curves to ensure consistency with Theorem 4.5.1).

We will now give three examples to demonstrate the application of Algorithms 4.5.4 and 4.5.7.

Example 4.5.8. Consider an $F \in \text{Mod}(S_7)$ with

$$D_F = (6, 0; ((1, 2), 2), (1, 3), (2, 3), (1, 6), (5, 6)).$$

Then F is a star-realizable linear 3-tuple (F_1, F_2, F_3), where

$$\begin{aligned} D_{F_1} &= (6, 0; ((1, 2), 2), (1, 6), (5, 6)) && \text{with } g_1(D_{F_1}) = 3 \\ D_{F_2} &= (6, 0; (1, 3), (5, 6), (5, 6)) && \text{with } g_2(D_{F_2}) = 2, \text{ and} \\ D_{F_3} &= (6, 0; (2, 3), (1, 6), (1, 6)) && \text{with } g_3(D_{F_3}) = 2. \end{aligned}$$

Note that the 1-compatibility of F_1 with F_2 is along fixed points represented by the pairs $(1, 6)$ (in D_{F_1}) and $(5, 6)$ (in D_{F_2}), while the compatibility of F_2 with F_3 is along the pairs $(5, 6)$ (in D_{F_2}) and $(1, 6)$ (in D_{F_3}). By Algorithm 4.5.4, we have $\mathcal{W}(F_1) = W_{4\cdot 3}^2, \mathcal{W}(F_2) = W_{2\cdot 2+2}^5, \mathcal{W}(F_3) = W_{2\cdot 2+2}$, and $\eta_1 = 1 = \eta_2$. Therefore, we have

$$\mathcal{W}(F) = (T_{a_1}^2 T_{b_1} T_{c_1} T_{b_2} T_{c_2} T_{b_3})^2 (T_{a_4} T_{b_4} T_{c_4} T_{b_5} T_{a_5'})^5 (T_{a_6} T_{b_6} T_{c_6} T_{b_7} T_{a_7})(T_{\gamma_1} T_{\gamma_2})^{-1}.$$

Example 4.5.9. Consider a periodic mapping class $G \in \text{Mod}(S_g)$ with $D_G = (2g - 2, 1; (1, 2), (1, 2))$. Then G is a star-realizable mapping class $F_{\mathcal{T}}$, where $\mathcal{T} = (1, 1, 0, 0)$ and $F = (F_1)$ with $D_{F_1} = (2g - 2, 0; (1, 2), (1, 2), (1, 2g - 2), (2g - 3, 2g - 2))$ and $g_1 = g(D_{F_1}) = g - 1$. Note that the self 1-compatibility of F is along a pair of fixed points of the $\langle \mathcal{F} \rangle$-action represented by the pairs $(1, 2g - 2)$ and $(2g - 3, 2g - 2)$ (in D_{F_1}). By Algorithm 4.5.7, we have $\mathcal{W}(F_1) = W_{4g_1}^2$, and so

$$\mathcal{W}(G) = (T_{a_2} \prod_{i=1}^{g-1} (T_{c_i} T_{b_{i+1}}))^2 T_{a_1}^{-1},$$

where we have relabeled γ_1 as a_1, and a_1' as c_1, so as to ensure consistency with Theorem 4.5.1.

Example 4.5.10. Consider a periodic mapping class $F \in \text{Mod}(S_{10})$ with $D_F = (7, 1; (1, 7), (3, 7), (3, 7))$. Then F is star-realizable linear 2-tuple (F_1, F_2), where $D_{F_1} = (7, 1; (3, 7), (4, 7))$ with $g_1 = g(D_{F_1}) = 7$ and $D_{F_2} = (7, 0; (1, 7), (3, 7), (3, 7))$ with $g_2 = g(D_{F_2}) = 3$. Note that F_1 is a rotational mapping class, and the 1-compatibility of F_1 with F_2 is along fixed points represented by the pairs $(4,7)$ (in D_{F_1}) and $(3,7)$ (in D_{F_2}). Following Algorithm 4.5.4, we have $\mathcal{W}(F_1) = W_{4\cdot 7}^{8}$, $\mathcal{W}(F_2) = W_{2\cdot 3+1}^{5}$, and $\eta_1 = 1$. Consequently,

$$\mathcal{W}(F) = (T_{a_1}^2 \prod_{i=1}^{6} (T_{b_i} T_{c_i}) T_{b_7})^8 (T_{a_8} T_{b_8} T_{c_8} T_{b_9} T_{c_9} T_{b_{10}} T_{a_{10}}^2)^5 T_{\gamma_1}^{-1},$$

where γ_1 is the separating curve involved in the 1-compatibility F_1 with F_2.

In general, the addition of a g'-permutation component to a periodic mapping class $F_2 \in \text{Mod}(S_g)$ of order n (as in Construction 2.2.8) can also be viewed as 1-compatibility of F_2 with the rotational mapping class $F_1 \in \text{Mod}(S_{ng'})$ with $D_{F_1} = (n, g'; (1,n), (n-1,n))$. (Note that this compatibility is along fixed points represented by $(n-1,n)$ (in D_{F_1}) and $(1,n)$ (in D_{F_2}). Moreover, it is not hard to see that $\mathcal{W}(F_1) = W_{4ng'}^{4g'}$, when n is odd. Thus, the ideas in Example 4.5.10 easily generalize to yield the following.

Proposition 4.5.11. *Let $F_2 \in \text{Mod}(S_g)$ be a periodic star-realizable mapping class of odd order with $D_{F_2} = (n, g_0; (c_1,n), (c_2,n_2), \ldots, (c_r,n_r))$. Let F be obtained through the addition of a 1-permutation component to F_2. Then viewing F as a 1-compatible pair (F_1, F_2) along a separating curve γ, where $D_{F_1} = (n, 1; (c_1,n), (n-c_1,n))$, we have*

$$\mathcal{W}(F) = \mathcal{W}(F_2) W_{4n}^{4c_1^+} T_{\gamma}^{-\eta},$$

where η is defined along the same lines as in Lemma 4.5.3.

4.6 Symplectic method

Let $F \in \text{Mod}(S_g)$ be of order n. In this section, we give a method by which one can use $\Psi(F)$ for finding a representation of F as a word $\mathcal{W}(F)$, up to conjugacy. (Here, we compute $\Psi(F)$ using Theorem 2.2.15 and Remark 2.2.16.) In other words, we have to find a suitable candidate for $\mathcal{W}(F)$ in

$$\mathcal{M}_F := \{G \in \text{Mod}(S_g) : \Psi(G) \text{ is conjugate to } \Psi(F)\}.$$

Let Ψ_m denote the composition of Ψ with the canonical projection $\text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z}_m)$. It is well known that $\ker \Psi_m$ (also known as the level-m subgroup $\text{Mod}(S_g)[m]$) is torsion-free for $m \geq 3$ (see [9, Theorem 6.9]). Considering that the conjugacy class of

$\Psi(F)$ can be infinite in $\mathrm{Sp}(2g,\mathbb{Z})$, for computational purposes, we consider the set

$$\widetilde{\mathcal{M}}_F := \{G \in \mathrm{Mod}(S_g) : \Psi_3(G) \text{ is conjugate to } \Psi_3(F)\}$$

in place of $\mathcal{M}(F)$. The key idea behind our method is to provide a systematic procedure for carefully and efficiently sifting through the elements in set $\widetilde{\mathcal{M}}_F$ to find a suitable candidate for $\mathcal{W}(F)$.

4.6.1 Structured searching for $\mathcal{W}(F)$

To standardize our procedure, we consider the Lickorish [20] generating set $\mathcal{L}_g$ for $\mathrm{Mod}(S_g)$ and assume that each element in $\mathcal{M}_F$ is a word in $\mathcal{L}_g$. To fix notation, let

$$\mathcal{L}_g = \{T_{a_1}, T_{b_1}, T_{c_1}, T_{b_2}, T_{a_2}, T_{c_2}, \ldots, T_{c_{g-1}}, T_{b_g}, T_{a_g}\}$$

with the a_i, the b_i and the c_i are indicated in Figure 4.14 below.

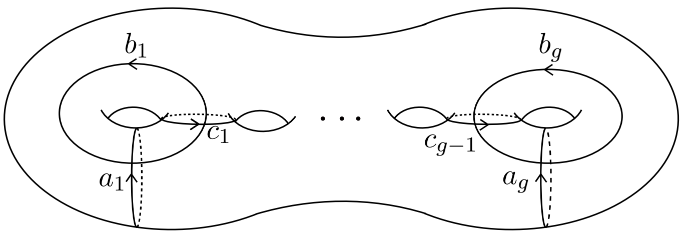


Figure 4.14: The curves in S_g involved in the Lickorish twists.

In order to make our search for $\mathcal{W}(F)$ in $\widetilde{\mathcal{M}}_F$ more efficient, we first have to ensure the implementation of a well-structured search process. For achieving this, we introduce the notion of the *depth* of a word. Let $\mathcal{W}$ be a reduced word in $\mathcal{L}_g$, and let n_i be the number of times the i^{th} generator in $\mathcal{L}_g$ appears in $\mathcal{W}$. Then the *depth* $d(\mathcal{W})$ of the word $\mathcal{W}$ is defined by $d(\mathcal{W}) = \max\{n_i : 1 \le i \le 3g-1\}$. For example, for the word $\mathcal{W} = T_{a_1}^5 T_{a_2}^4 T_{b_2} T_{a_2}^2 T_{b_1}$, $d(\mathcal{W}) = 2$. Further, we denote the largest power (in absolute value) of a Dehn twist in $\mathcal{L}_g$ appearing in a word W by $p(\mathcal{W})$, and fix the notation $\widetilde{\mathcal{M}}_F^{i,j} := \{W \in \widetilde{\mathcal{M}}_F : d(\mathcal{W}) = i \text{ and } p(\mathcal{W}) = j\}$. Thus, we will begin our search for $\mathcal{W}(F)$ in $\widetilde{\mathcal{M}}_F^{1,1}$, and then gradually broaden our search in an incremental manner to $\widetilde{\mathcal{M}}_F^{i,j}$ for $i, j > 1$.

4.6.2 Discarding redundant words

To begin with, we apply the basic property that Dehn twists about isotopically disjoint curves in S_g commute, we would like to discard redundant variants of words that are equivalent up to commutativity of the twists in $\mathcal{L}_g$. For this reason, we assign numbers 1 through $3g-1$ for the Dehn twists appearing (in sequence) in $\mathcal{L}_g$. A permutation σ of $\{1,\ldots 3g-1\}$ is said to be *good* if for $1 \leq i \leq 3g-1$, either $\sigma(i+1)-\sigma(i) \leq 1$ or $(\sigma(i),\sigma(i+1)) = (3k-2, 3k)$ for some k. Thus, in our process, we will filter out many (redundant) words in $\widetilde{\mathcal{M}}_F$ by considering only words as arise as good permutations of (powers of the) Dehn twists appearing in $\mathcal{L}_g$. We will further discard several non-periodic words in $\widetilde{\mathcal{M}}_F$ by applying Penner's construction [31] of pseudo-Anosov mapping classes.

Theorem 4.6.1. *Let $\mathcal{C} = \{\alpha_1,\ldots,\alpha_n\}$ and $\mathcal{D} = \{\alpha_{n+1},\ldots,\alpha_{n+m}\}$ be multicurves in S_g that together fill S_g. Then any product of positive powers of the T_{α_i}, for $i = 1,\ldots,n$ and negative powers of the $T_{\alpha_{n+j}}$, for $j = 1,\ldots,m$, where each α_i and each α_{n+j} appears at least once, is pseudo-Anosov.*

4.6.3 Searching for periodics

Let $i(\alpha,\beta)$ be the geometric intersection number of essential simple closed curves α,β in S_g. In order to identify the periodics in $\widetilde{\mathcal{M}}_F$, we will (in general) use the well-known Bestvina-Handel algorithm [1], which provides an effective way of identifying them. This brings us to the following remark

Remark 4.6.2. When F is irreducible, the elements $\widetilde{\mathcal{M}}_F$ are either irreducible periodics or pseudo-Anosovs. In this context, we have observed that it is easier to identify the periodics by simply determining whether the orbits (under F) of certain appropriately chosen curves are finite. In this regard, a software named Teruaki for Mathematica 7 (or TKM7) by Sakasai-Suzuki [36] designed for the visualization of actions of Dehn twists on curves (in S_g) really comes in handy.

This finally brings us to our method.

Algorithm 4.6.3 (Symplectic method)**.** *Let $F \in \mathrm{Mod}(S_g)$ be of order n.*

Step 1. Compute $\Psi(F)$ (up to conjugacy) using Theorem 2.2.15 and Remark 2.2.16.

Step 2. Set $i = 1$, $j = 1$, and $flag = 0$

Step 3. Repeat Steps 4-5 until $flag = 1$.

Step 4. Repeat Steps 4a - 4f, while $j \leq n$.

Step 4a. Compute the elements in $\widetilde{\mathcal{M}}_F^{i,j}$ *up to good permutations.*

Step 4b. Discard the words in $\widetilde{\mathcal{M}}_F^{i,j}$ *that are pseudo-Anosovs using Theorem 4.6.1 (or Remark 4.6.2 when* F *is irreducible).*

Step 4c. If $|\widetilde{\mathcal{M}}_F^{i,j}| > 0$*, repeat Steps 4d - 4e, for each* $W \in \widetilde{\mathcal{M}}_F^{i,j}$*. Else, proceed to Step 4f.*

Step 4d. Apply the Bestvina-Handel algorithm to determine the mapping class type of W*.*

Step 4e. If W *is periodic, then set* $\mathcal{W}(F) = W$, $flag = 1$*, and then proceed to Step 5. Else, set* $\widetilde{\mathcal{M}}_F^{i,j} = \widetilde{\mathcal{M}}_F^{i,j} \setminus \{W\}$ *and proceed to Step 4c.*

Step 4f. Set $j = j + 1$ *and proceed to Step 4.*

Step 5. If $flag = 0$*, set* $i = i + 1$ *and proceed to Step 3. Else, proceed to Step 6.*

Step 6. $\mathcal{W}(F)$ *is the desired representation of* F *as a word in Dehn twists, up to conjugacy.*

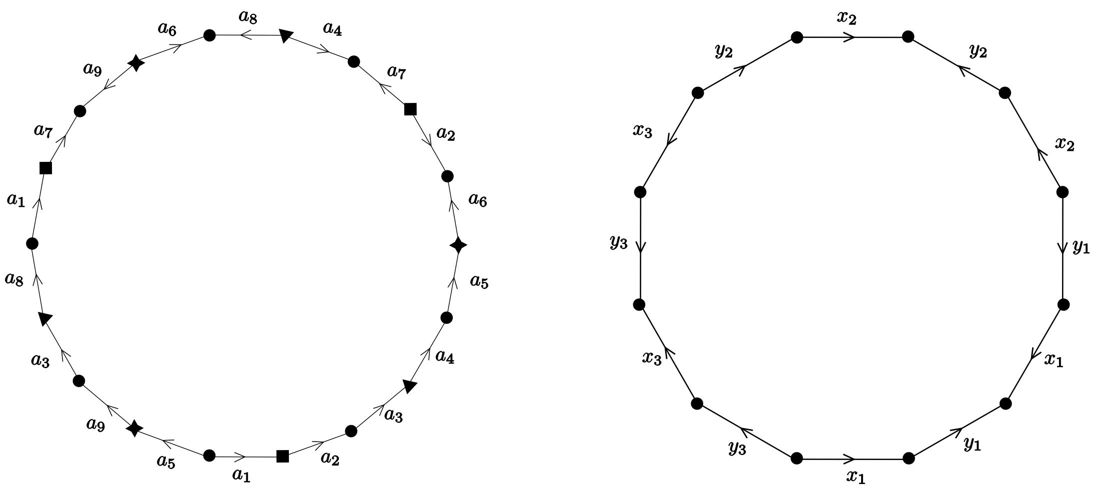
Example 4.6.4. Consider the irreducible Type 1 mapping class $F \in \mathrm{Mod}(S_3)$ with $D_F = (9, 0; (1, 3), (1, 9), (5, 9))$. Clearly, F is neither rotational nor star realizable, so we will use Algorithm 4.6.3 to find $\mathcal{W}(F)$. We will follow the notation from Subsections 2.2.1-2.2.2. By Theorem 2.2.4, F is realized as a rotation of the polygon $\mathcal{P}_F$, as shown in Figure 4.15 below, with

$$L(\mathcal{P}_F) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\} \text{ and}$$
$$W(\mathcal{P}_F) = a_1a_2a_3a_4a_5a_6a_2^{-1}a_7a_4^{-1}a_8a_6^{-1}a_9a_7^{-1}a_1^{-1}a_8^{-1}a_3^{-1}a_9^{-1}a_5^{-1}.$$

Let $\mathcal{P}' = \mathcal{P}_3$ be the standard 12-gon (realizing the surface S_3) with

$$L(\mathcal{P}_3) = \{x_1, y_1, x_2, y_2, x_3, y_3\} \text{ and } W(\mathcal{P}_3) = [x_1, y_1][x_2, y_2][x_3, y_3].$$

Denoting $\phi = \phi_{\mathcal{P}_F}$ and $f = f_{\mathcal{P}_F}$, we obtain $\varphi = f^{-1}\phi f$ (as in Theorem 2.2.15) in the following manner.



(a) $\mathcal{P}_F$ **(b)** $\mathcal{P}_3$

Figure 4.15: The polygons $\mathcal{P}_F$ and $\mathcal{P}_3$.

$$
\begin{array}{llllll}
[x_1] & \xrightarrow{f} & [b_1]+[b_2] & \xrightarrow{\phi} & -[b_1]-[b_3]-[b_6] & \xrightarrow{f^{-1}} -[y_1]+[x_2]-[y_2] \\
[y_1] & \xrightarrow{f} & [b_1]+[b_3] & \xrightarrow{\phi} & [b_2]-[b_3]+[b_4] & \xrightarrow{f^{-1}} [x_1]-[y_1]+[x_3]-[y_3] \\
[x_2] & \xrightarrow{f} & [b_2]+[b_4]+[b_5] & \xrightarrow{\phi} & -[b_4]-[b_6] & \xrightarrow{f^{-1}} [x_2]-[y_2]-[x_3]+[y_3] \\
[y_2] & \xrightarrow{f} & [b_4]+[b_5]+[b_6]+[b_2] & \xrightarrow{\phi} & [b_3]-[b_4]+[b_5]-[b_6] & \xrightarrow{f^{-1}} -[x_1]+[y_1]+2[x_2] \\
 & & & & & \quad -[y_2]-2[x_3]+2[y_3] \\
[x_3] & \xrightarrow{f} & -[b_5] & \xrightarrow{\phi} & [b_4] & \xrightarrow{f^{-1}} [x_3]-[y_3] \\
[y_3] & \xrightarrow{f} & -[b_4]-[b_5] & \xrightarrow{\phi} & [b_4]-[b_1] & \xrightarrow{f^{-1}} -[x_1]+[x_2]+[x_3]
\end{array}
$$

Here, $[b_1]=[a_8^{-1}a_3^{-1}], [b_2]=[a_9^{-1}a_5^{-1}], [b_3]=[a_2^{-1}a_1^{-1}], [b_4]=[a_6^{-1}a_9], [b_5]=[a_7^{-1}a_2], [b_1]=[a_4^{-1}a_8]$. Thus, the matrix M_φ representing the conjugacy class of $\Psi(F)$ in $\mathrm{Sp}(2g,\mathbb{Z})$ is given by

$$
M_\varphi = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 1 \\ 0 & -1 & 1 & 2 & -1 & 0 \end{bmatrix},
$$

and so we have

$$\Psi_3(F) = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

Following Algorithm 4.6.3, we begin our search for $\mathcal{W}(F)$ in $\widetilde{\mathcal{M}}_F^{1,1}$. As it turns out, even after considering only the words among the good permutations that not Penner-type pseudo-Anosovs, we were still left with numerous (at least 150) possible candidates for $\mathcal{W}(F)$ in $\widetilde{\mathcal{M}}_F^{1,1}$. For brevity, we will demonstrate the algorithm on a small subcollection of words

$$\widetilde{\mathcal{M}}_F^{1,1}(\sigma) = \{T_{a_1}^{-1}T_{c_1}^{-1}T_{a_2}^{-1}T_{b_3}^{-1}T_{c_2}^{-1}T_{b_2}^{-1}T_{b_1}^{-1}, T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1},$$
$$T_{c_1}T_{a_2}T_{b_3}T_{a_3}T_{c_2}T_{b_2}T_{b_1}, T_{c_1}^{-1}T_{a_2}^{-1}T_{b_3}^{-1}T_{a_3}^{-1}T_{c_2}^{-1}T_{b_2}^{-1}T_{b_1}^{-1}\}$$

corresponding to the permutation of $\sigma = (2\,3\,5\,8)(4\,7)$ on $\mathcal{L}_3$. Using Remark 4.6.2 (and the Teruaki software [36]), we can easily deduce that all words in $\widetilde{\mathcal{M}}_F^{1,1}(\sigma)$ are finite order. Thus, we may choose $\mathcal{W}(F) = T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1}$.

Note that the symplectic method can be applied to any periodic mapping class. However, as the method is computationally intense, we recommend its application only for non-rotational periodics that are neither star-realizable nor chain-realizable. It is apparent that while the earlier methods were more restrictive in terms of their applicability, they work quite efficiently for the specific families of periodic mapping classes they were designed for.

4.7 Applications

4.7.1 Periodic mapping classes in $\mathrm{Mod}(S_3)$ as words in Dehn twists

Using Algorithms 4.3.3, 4.4.4, 4.5.7, and 4.6.3, in Table 4.3 below, we provide a word $\mathcal{W}(F)$ (in Dehn twists) representing the conjugacy class of each periodic mapping class $F \in \mathrm{Mod}(S_3)$.

4.7.2 Roots of Dehn twists

For $g \geq 2$, let c be a nonseparating curve in S_g. A *root of* T_c *of degree* n is an $F \in \mathrm{Mod}(S_g)$ such that $F^n = T_c$. Margalit-Schleimer [23] gave the first example of such a root of degree $2g-1$ in $\mathrm{Mod}(S_g)$. A complete classification of such roots was obtained [27], where it was also shown that the Margalit-Scleimer root (of degree $2g-1$) had the largest possible degree in $\mathrm{Mod}(S_g)$. A periodic mapping class $\bar{F} \in \mathrm{Mod}(S_{g-1})$ is said to be *root-realizing* if the $\langle \bar{\mathcal{F}} \rangle$-action on S_{g-1} has two distinguished fixed points where the induced local rotation angles add up to $2\pi/n \pmod{2\pi}$. Given a root-realizing $\bar{F} \in \mathrm{Mod}(S_{g-1})$ of order n with distinguished fixed points P_1 and P_2, one can remove $\langle \bar{\mathcal{F}} \rangle$-invariant neighborhoods around the P_i and then attach an annulus A with an $(1/n)^{th}$-twist connecting the resulting boundary components to realize a root $F \in \mathrm{Mod}(S_g)$ of a Dehn twist T_c about the nonseparating curve c in A. Conversely, for $g \geq 2$, given a root $F \in \mathrm{Mod}(S_g)$ of T_c of degree n, one can reverse this process to recover a root-realizing periodic mapping class $\bar{F} \in \mathrm{Mod}(S_{g-1})$. Thus, the conjugacy class of a typical root realizing $\bar{F} \in \mathrm{Mod}(S_{g-1})$ that corresponds to the conjugacy class of a root $F \in \mathrm{Mod}(S_g)$ of degree n has the form

$$D_{\bar{F}} = (n, g_0; (a, n), (b, n), (c_1, n_1), \ldots, (c_\ell, n_\ell)),$$

where $a + b \equiv ab \pmod{n}$. Here the pairs (a, n) and (b, n) represent the fixed points (of the $\langle \bar{\mathcal{F}} \rangle$-action) involved in the construction of the root F. We will now describe a family of roots that can be represented as words in Dehn twists by using a minor modification of the method described in Algorithm 4.5.7.

Definition 4.7.1. A root $F \in \mathrm{Mod}(S_g)$ of T_c of degree n is said to be *star-realizable* if the following conditions hold.

$\lvert F\rvert$	D_F	$\mathcal{W}(F)$	Algorithm
14	$(14,0;(1,2),(3,7),(1,14))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})$	4.4.4
14	$(14,0;(1,2),(2,7),(3,14))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^5$	4.4.4
14	$(14,0;(1,2),(1,7),(5,14))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^3$	4.4.4
14	$(14,0;(1,2),(6,7),(9,14))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^{11}$	4.4.4
14	$(14,0;(1,2),(5,7),(11,14))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^9$	4.4.4
14	$(14,0;(1,2),(4,7),(13,14))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^{13}$	4.4.4
12	$(12,0;(1,2),(5,12),(1,12))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})$	4.4.4
12	$(12,0;(1,2),(7,12),(11,12))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^7$	4.4.4
12	$(12,0;(2,3),(1,4),(1,12))$	$(T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})$	4.6.3
12	$(12,0;(1,3),(1,4),(5,12))$	$(T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})^5$	4.6.3
12	$(12,0;(2,3),(3,4),(7,12))$	$(T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})^7$	4.6.3
12	$(12,0;(1,3),(3,4),(11,12))$	$(T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})^{11}$	4.6.3
9	$(9,0;(1,3),(5,9),(1,9))$	$(T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})$	4.6.3
9	$(9,0;(2,3),(1,9),(2,9))$	$(T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})^5$	4.6.3
9	$(9,0;(1,3),(2,9),(4,9))$	$(T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})^7$	4.6.3
9	$(9,0;(2,3),(7,9),(5,9))$	$(T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})^2$	4.6.3
9	$(9,0;(1,3),(8,9),(7,9))$	$(T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})^4$	4.6.3
9	$(9,0;(2,3),(4,9),(8,9))$	$(T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1})^8$	4.6.3
8	$(8,0;(3,4),(1,8),(1,8))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})$	4.4.4
8	$(8,0;(1,4),(3,8),(3,8))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^3$	4.4.4
8	$(8,0;(3,4),(5,8),(5,8))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^5$	4.4.4
8	$(8,0;(1,4),(7,8),(7,8))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^7$	4.4.4
8	$(8,0;(1,4),(5,8),(1,8))$	$(T_{a_1}T_{c_1}T_{b_3}T_{a_3}T_{c_2}T_{b_2}T_{b_1})$	4.6.3
8	$(8,0;(3,4),(7,8),(3,8))$	$(T_{a_1}T_{c_1}T_{b_3}T_{a_3}T_{c_2}T_{b_2}T_{b_1})^3$	4.6.3
7	$(7,0;(5,7),(1,7),(1,7))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})$	4.4.4
7	$(7,0;(3,7),(2,7),(2,7))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^4$	4.4.4
7	$(7,0;(1,7),(3,7),(3,7))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^5$	4.4.4
7	$(7,0;(6,7),(4,7),(4,7))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^2$	4.4.4
7	$(7,0;(4,7),(5,7),(5,7))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^3$	4.4.4
7	$(7,0;(2,7),(6,7),(6,7))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^6$	4.4.4
7	$(7,0;(4,7),(2,7),(1,7))$	$(T_{b_1}T_{b_2}T_{c_2}T_{a_3}T_{b_3}T_{c_1}T_{a_1}^2)^{-1}$	4.6.3
7	$(7,0;(3,7),(5,7),(6,7))$	$(T_{b_1}T_{b_2}T_{c_2}T_{a_3}T_{b_3}T_{c_1}T_{a_1}^2)^{-6}$	4.6.3
6	$(6,0;(1,2),(1,2),(1,6),(5,6))$	$(T_{a_1}^2T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^2$	4.4.4
6	$(6,0;(1,2),(2,3),(2,3),(1,6))$	$(T_{a_1}T_{b_1})^{-1}(T_{a_2}T_{b_2}T_{c_2}T_{b_3}T_{a_3})$	4.4.4
6	$(6,0;(1,2),(1,3),(1,3),(5,6))$	$(T_{a_1}T_{b_1})(T_{a_2}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^{-1}$	4.4.4
4	$(4,1;(1,2),(1,2))$	$(T_{a_2}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^2T_{a_1}^{-1}$	4.5.7
4	$(4,0;(1,2),(1,2),(1,2),(1,4),(1,4))$	$(T_{a_1}^2T_{b_1})(T_{a_2}T_{a_2'}T_{b_2})^3(T_{a_3}^2T_{b_3})(T_{s_1}T_{s_2})^{-1}$	4.5.4
4	$(4,0;(1,2),(1,2),(1,2),(3,4),(3,4))$	$(T_{a_1}^2T_{b_1})^3(T_{a_2}T_{a_2'}T_{b_2})(T_{a_3}^2T_{b_3})^3(T_{s_1}T_{s_2})^{-1}$	4.5.4
4	$(4,0;(1,4),(1,4),(3,4),(3,4))$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3}T_{a_3})^2$	4.4.4
4	$(4,0;(1,4),(1,4),(1,4),(1,4))$	$(T_{a_1}T_{c_1}T_{b_3}T_{a_3}T_{c_2}T_{b_2}T_{b_1})^2$	4.6.3
4	$(4,0;(3,4),(3,4),(3,4),(3,4))$	$(T_{a_1}T_{c_1}T_{b_3}T_{a_3}T_{c_2}T_{b_2}T_{b_1})^6$	4.6.3
3	$(3,0;(1,3),(1,3),(1,3),(1,3),(2,3))$	$(T_{a_1}^3T_{b_1})(T_{a_2}^2T_{a_2'}T_{b_2})^2(T_{a_3}^3T_{b_3})(T_{s_1}T_{s_2})^{-1}$	4.5.4
3	$(3,0;(2,3),(2,3),(2,3),(2,3),(1,3))$	$(T_{a_1}^3T_{b_1})^2(T_{a_2}^2T_{a_2'}T_{b_2})(T_{a_3}^3T_{b_3})^2(T_{s_1}T_{s_2})^{-1}$	4.5.4
3	$(3,1;(1,3),(2,3))$	$(T_{a_1}^2T_{c_1}T_{b_1})(T_{a_3}^2T_{c_2}T_{b_3})^2(T_{a_2}T_{a_2'})^{-1}$	4.5.7
2	$(2,2,1;)$	$(T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2})^3T_{a_3}^{-1}$	4.5.7
2	$(2,1;((1,2),4))$	$(T_{a_2}T_{b_2}T_{x_1}T_{a_1}T_{b_1})^3(T_{a_3}T_{b_3}T_{a_3})^{-2}$	4.3.3
2	$(2,0;((1,2),8))$	$(T_{a_1}T_{b_1}T_{a_1})^2(T_{a_2}T_{b_2}T_{a_2'})^{-2}(T_{a_3}T_{b_3}T_{a_3})^2$	4.3.3

Table 4.3: Words (in Dehn twists) representing the conjugacy classes of periodic elements in $\mathrm{Mod}(S_3)$.

(i) The root-realizing periodic mapping class $\bar{F} \in \mathrm{Mod}(S_{g-1})$ with

$$D_{\bar{F}} = (n, g_0; (a,n), (b,n), (c_1,n_1), \ldots, (c_\ell, n_\ell)),$$

$a + b \equiv ab \pmod{n}$, is star-realizable.

(ii) Suppose that $\bar{F} = F_{\mathcal{T}}^m$, for some star-realizable $F_{\mathcal{T}}$ as in Definition 4.5.5 so that the pairs (a,n) (resp. (b,n)) belong to D_{F_i} (resp. D_{F_j}). Then

$$(a,n) \in \{(1, |\tilde{F}_i|)_{m_i,\tilde{F}_i}, (|\tilde{F}_i|/2-1, |\tilde{F}_i|)_{m_i,\tilde{F}_i}, (|\tilde{F}_i|-2, |\tilde{F}_i|)_{m_i,\tilde{F}_i}\}$$

and

$$(b,n) \in \{(1, |\tilde{F}_j|)_{m_j,\tilde{F}_j}, (|\tilde{F}_j|/2-1, |\tilde{F}_j|)_{m_j,\tilde{F}_j}, (|\tilde{F}_j|-2, |\tilde{F}_j|)_{m_j,\tilde{F}_j}\}$$

Denoting $\eta = \frac{\mu_{z_i i} + \mu_{z_j j} - 1}{n}$, we will now we give an algorithm to represent a star-realizable root as a word in Dehn twists.

Algorithm 4.7.2. *Consider a star-realizable root $F \in \mathrm{Mod}(S_g)$ of T_c as in Definition 4.7.1.*

Step 1. Apply Algorithm 4.5.7 to obtain $\mathcal{W}(\bar{F})$.

Step 2. Set

$$\mathcal{W}(F) = \mathcal{W}(\bar{F})(T_c)^{-\eta}.$$

Step 3. By Lemma 4.5.3, $\mathcal{W}(F)$ is the desired representation of F as a word in Dehn twists, up to conjugacy.

Let $G \in \mathrm{Mod}(S_g)$ be the Margalit-Schleimer root (of T_c) of degree $2g-1$. In [23] an expression for $\mathcal{W}(G)$ was derived using the chain relation, and in [27] it was shown that $D_{\bar{G}} = (2g-1, 0; (2, 2g-1), (2, 2g-1), (-4, 2g-1))$. In the following example, we will apply Algorithm 4.7.2 to derive the $\mathcal{W}(F)$ for a root $F \in \mathrm{Mod}(S_g)$ of degree $2g-1$ for which $D_{\bar{F}}$ is different from $D_{\bar{G}}$.

Example 4.7.3. Consider a root $F \in \mathrm{Mod}(S_g)$ of degree $2g-1$, where $D_{\bar{F}} = (2g-1, 0; (g, 2g-1), (g, 2g-1), (2g-2, 2g-1))$. Since $\mathcal{W}(\bar{F}) = W_{2g-1}^2$, by applying Algorithm 4.7.2, we get

$$\mathcal{W}(F) = T_{a_1}^{-1}(T_{c_1}T_{a_2}\prod_{i=2}^{g-1}(T_{b_i}T_{c_i})T_{b_g}T_{a_g})^2.$$

For $g \geq 2$, a *fractional root of T_c of degree* (m,n) is an $F \in \text{Mod}(S_g)$ such that $F^n = T_c^m$. It is known [33] that such a root of T_c may either preserve or reverse the two sides of c, and for a side-preserving root of degree (m,n), $n \leq 4g-4$. Further, a side-preserving fractional root of degree $(2g-2, 4g-4)$ always exists in $\text{Mod}(S_g)$. As in the case of roots of Dehn twists, a side-preserving fractional root $F \in \text{Mod}(S_g)$ of degree (m,n) corresponds to an $\bar{F} \in \text{Mod}(S_{g-1})$ of order n such that the $\langle \bar{\mathcal{F}} \rangle$-action has two distinguished fixed points where the induced rotation angles add up to $2\pi m/n \pmod{2\pi}$. This brings us to the final result in this subsection, in which we assume the notation of Theorem 4.5.1.

Proposition 4.7.4. *Let $F \in \text{Mod}(S_g)$ be a side-preserving fractional root of T_{a_1} of degree $(2g-2, 4g-4)$. Then*

$$F = T_{a_2} \prod_{i=1}^{g-1} (T_{c_i} T_{b_{i+1}}).$$

Proof. Assume that F is realized from $\bar{F} \in \text{Mod}(S_{g-1})$ by attaching an annulus (with a $2\pi m/n$-twist) connecting the two boundary components (d_2 and d_1) of the subsurface S_{g-1}^2. By Theorem 4.5.1 (for $k = 2$), we have

$$(T_{a_2} \prod_{i=1}^{g-1} (T_{c_i} T_{b_{i+1}}))^{4g-4} = T_{a_1} T_{a_1}^{(2g-3)^+}.$$

But, as $(2g-3)^+ \equiv 2g-3 \pmod{4g-4}$, this further simplifies to

$$(T_{a_2} \prod_{i=1}^{g-1} (T_{c_i} T_{b_{i+1}}))^{4g-4} = T_{a_1}^{2g-2}$$

□

4.7.3 Construction of pseudo-Anosov mapping classes

In this subsection, we show that for $g \geq 2$, there exists conjugates of the periodic mapping classes W_{4g} and W_{4g+2} (from Subsection 4.4.1) whose product is pseudo-Anosov. In this connection, we will use the following symplectic (sufficient) criterion [24, Proposition 2] for a given mapping class to be pseudo-Anosov (originally due to Casson-Bleiler [6]).

Theorem 4.7.5. *Let $F \in \text{Mod}(S_g)$ and let $P_F(x)$ be the characteristic polynomial of $\Psi(F)$. Suppose that each of the following conditions hold.*

(i) $P_F(x)$ is symplectically irreducible over $\mathbb{Z}$.

(ii) $P_F(x)$ is not a cyclotomic polynomial.

(iii) $P_F(x)$ is not a polynomial in x^k for any $k > 1$.

Then F is pseudo-Anosov.

We now consider the conjugates

$$W'_{4g+2} = (T_{a_1} \prod_{i=1}^{g-1} (T_{b_i} T_{c_i}) T_{b_g}) \text{ and } W'_{4g} = (T_{b_1}^2 \prod_{i=1}^{g-1} (T_{c_i} T_{b_{i+1}}) T_{a_g})$$

of W_{4g+2} and W_{4g}, respectively. (It is not hard to check that these are indeed conjugates). Let $W = W'_{4g} W'_{4g+2}$. A direct computation shows that

$$P_W(x) = x^{2g} + 2x^{2g-1} + 3x^{2g-2} + \cdots + gx^{g+1} + (g+3)x^g + gx^{g-1} + \cdots + 2x + 1.$$

We will require the following technical lemma.

Lemma 4.7.6. *No complex root of unity can be a root of $P_W(x)$.*

Proof. It is apparent that $x = \pm 1$ is not a root of $P_W(x)$. Suppose that an n^{th} root of unity $e^{\iota\theta}$ for some $\theta \in \mathbb{R}$ and $n \geq 3$, is a root of $P_W(x)$. Then we have $P_W(e^{\iota\theta}) = 0$, which yields the following two equations:

$$\sum_{j=0}^{2g} \cos(j\theta) + \sum_{j=1}^{2g-1} \cos(j\theta) + \cdots + \sum_{j=g-1}^{g+1} \cos(j\theta) + 3\cos(g\theta) = 0$$
$$\sum_{j=0}^{2g} \sin(j\theta) + \sum_{j=1}^{2g-1} \sin(j\theta) + \cdots + \sum_{j=g-1}^{g+1} \sin(j\theta) + 3\sin(g\theta) = 0$$

Applying the formulas

$$\sum_{j=0}^{n-1} \cos(\alpha + j\beta) = \frac{\cos(\alpha + (n-1)\beta/2) \sin(n\beta/2)}{\sin(\beta/2)} \text{ and}$$
$$\sum_{j=0}^{n-1} \sin(\alpha + j\beta) = \frac{\sin(\alpha + (n-1)\beta/2) \sin(n\beta/2)}{\sin(\beta/2)}$$

in the pair of equations above, we obtain the pair of equations

$$\cos(g\theta)(\sin^2((g+1)\theta/2) + 2\sin^2(\theta/2)) = 0$$
$$\sin(g\theta)(\sin^2((g+1)\theta/2) + 2\sin^2(\theta/2)) = 0$$

These equations yield a contradiction, as there does not exist any θ such that $\cos(g\theta) = 0 = \sin(g\theta)$, from which our assertion follows. □

This leads us to the following proposition.

Proposition 4.7.7. *For $g \geq 2$, there exists conjugates W'_{4g} and W'_{4g+2} of W_{4g} and W_{4g+2}, respectively, such that $W'_{4g}W'_{4g+2}$ is pseudo-Anosov in $\mathrm{Mod}(S_g)$.*

Proof. It is apparent that $P_W(x)$ satisfies condition (iii) of Theorem 4.7.5. Further, condition (ii) of Theorem 4.7.5 holds true in view of Lemma 4.7.6. Finally, to show condition (i) it suffices to show that W does not preserve any subsurface of S_g with genus greater than 0. To show this, we consider the chain of simple closed curves $C = \{b_1, c_1, b_2, c_2, \cdots, c_{g-1}, b_g, a_g\}$ in S_g (as indicated in Figure 4.14). For simplicity, we relabel the curves (appearing in sequence) in C by $\{\alpha_1, \cdots, \alpha_{2g}\}$. By the properties of chain maps, we have

$$W(\alpha_i) = \alpha_{i+2}, \text{ for } 1 \leq i < 2g-1$$

and further it is easily seen that there exists a k such that each component of $\overline{S_g \setminus \cup_{i=0}^{k} W^i(\alpha_j)}$ has genus 0. Consequently, it follows that W cannot preserve a subsurface of S_g of genus greater than 0, from which our assertion follows. □

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