INTRODUCTION TO GEOMETRIC GROUP THEORY

A REPORT

submitted in partial fulfillment of the requirements

for the award of the dual degree of

Bachelor of Science-Master of Science

in

MATHEMATICS

by

MAYANK JAIN

(14092)

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL
BHOPAL - 462066

April 2019

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान भोपाल
Indian Institute of Science Education and Research Bhopal
(Estb. By MHRD, Govt. of India)

CERTIFICATE

This is to certify that **Mayank Jain**, BS-MS (Dual Degree) student in the Department of Mathematics, has completed bona fide work on the dissertation entitled **Introduction to Geometric Group Theory** under my supervision and guidance.

April 2019 **Dr. Kashyap Rajeevsarathy**
IISER Bhopal

Committee Member	Signature	Date

ACADEMIC INTEGRITY AND COPYRIGHT DISCLAIMER

I hereby declare that this MS-Thesis is my own work and, to the best of my knowledge, that it contains no material previously published or written by another person, and no substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Bhopal or any other educational institution, except where due acknowledgement is made in the document.

I certify that all copyrighted material incorporated into this document is in compliance with the Indian Copyright (Amendment) Act (2012) and that I have received written permission from the copyright owners for my use of their work, which is beyond the scope of that law. I agree to indemnify and saveguard IISER Bhopal from any claims that may arise from any copyright violation.

April 2019 **Mayank Jain**
IISER Bhopal

ACKNOWLEDGMENT

First of all, I would like to thank my project advisor Dr. Kashyap Rajeevsarathy for his expert advice and encouragement throughout the project. It would have been almost impossible to carry out the project without his help and valuable comments. I would also like to thank my project committee members, Dr. Siddhartha Sarkar, and Dr. Sanjay Kumar Singh for their positive comments and for showing a keen interest in my project through careful evaluation of my presentations. I would like to thank my batch mates and seniors at the department for constantly motivating me and carrying out wonderful discussions on various topics of interest during the project. Thanks to Shivani, my best friend at IISER Bhopal, for always supporting me and making life at IISER joyous and wonderful. I would also like to express my gratitude to my numerous friends, especially Mohit and Piyush for always being there by my side throughout this journey. Finally, I would like to thank my family for their love, support and faith in me.

Mayank Jain

ABSTRACT

In this project, we will try to perceive groups as geometric objects to study their properties relatively easily. We begin with introducing the notions of *Cayley graphs*, and the action of groups on trees endowed with a path metric. By studying the action of SL(2,$\mathbb{Z}$) on the *Farey tree*, we show that for $m \geq 3$, a level m congruence subgroup of SL(2,$\mathbb{Z}$) is free. Further, we show that if a group acts freely and transitively on the edges of a tree, then it is isomorphic to the free product of the stabilizers of the vertices under the action. Finally, applying this result to the action of PSL(2,$\mathbb{Z}$) on the *Farey tree*, we prove that $\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$.

We will go on to define the geometric realizations of Cayley graphs, and look at how Cayley graphs for the same group with two different generating sets are equivalent to each other through the notion of *quasi-isometry.* Further, we discuss the word problem for a group and how *Dehn functions* can be used to measure the complexity of its solvability. We will then try to validate the importance of Dehn functions, as a measure for the complexity of the word problem.

Finally, we will discuss hyperbolic groups and the word problem for these groups.

LIST OF SYMBOLS OR ABBREVIATIONS

$\tau(G, S)$	the Cayley graph of a group G w.r.t. a generating set S.
F_n	the free group of n letters.
$\mathrm{SL}(2, \mathbb{Z})$	the special linear group of 2×2 matrices.
$\mathrm{SL}(2, \mathbb{Z}[m])$	the level m congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$.
$\mathrm{PSL}(2, \mathbb{Z})$	the quotient group of $SL(2, \mathbb{Z})$ w.r.t subgroup generated by $-I$.
$M * N$	the free product of two groups M and N.
$\langle S \mid R \rangle$	the presentation of a group with generating set S and defining relations R.
$\pi_1(G)$	the fundamental group of G.
$e(\tau)$	the number of ends of a graph τ.
$e(G)$	the number of ends of a group G.
$\dim X$	the dimension of a metric space X.
$\mathrm{asdim}(X)$	the asymptotic dimension of a metric space X.
$\mathrm{asdim}(G)$	the asymptotic dimension of a group G.
$\mathrm{diam}(X)$	the max distance between two points of a metric space X.

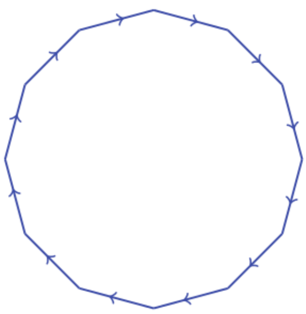
Contents

Chapter 1

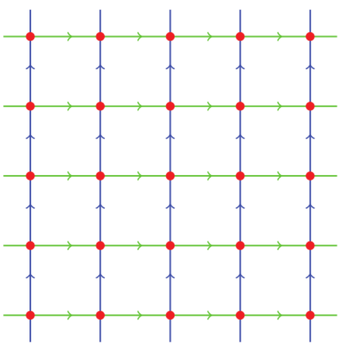
Introduction

The emergence of geometric group theory as a distinct area of mathematics is usually traced to the late 1980s and early 1990s. The 1987 monograph of Mikhail Gromov titled "Hyperbolic groups" [1] introduced the notion of a hyperbolic group, which captures the idea of a finitely generated group having large-scale negative curvature, and by his subsequent monograph "Asymptotic Invariants of Infinite Groups" [2], that outlined Gromov's program of understanding discrete groups up to quasi-isometry. The work of Gromov had a transformative effect on the study of discrete groups and the phrase "geometric group theory" started appearing soon afterward [3].

An important aspect of mathematics consists of the study of symmetries of an object, whether the object is a simple 3-dimensional object seen in daily life such as a cube or a complicated abstract object such as a group. Geometric group theory tries to study every group as a group of symmetries of some object so that we can infer some of its properties which are relatively harder to study with respect to the abstract group structure [4]. This idea is illustrated in Figure 1.1.



(a) The group $\mathbb{Z}/n\mathbb{Z}$ as rotations of a regular n-gon.



(b) The group $\mathbb{Z}^2$ as translations in $\mathbb{R}^2$.

Figure 1.1: Representation of two groups as groups of symmetries of mathematical objects.

The object that captures the symmetries of a group is called a Cayley graph associated with the group.

Using the notion of group actions on spaces, geometric group theory tries to understand the properties of that group by studying the geometric properties of its associated Cayley graphs. That is why it is important for us to study how a group acts on graphs and trees. We will study this notion in the next Chapter. Later, we will study the notions of quasi-isometries, word problems, ends of groups and asymptotic dimensions to work with such groups and graphs. These chapters in the thesis are mostly based on [4] “Office hours with a geometric group theorist.”

Chapter 2

Group actions on trees

2.1 Group action on sets

We start by introducing the action of a group on a set.

Definition 2.1. An **action of a group G on a set X** is a function $G \times X \to X$ where the image of (g, x) is written $g \cdot x$ and where

(i) $1 \cdot x =$ x for all $x \in X$ and

(ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all g, $h \in G$ and $x \in X$.

If the group S_X is the group of symmetries of X, thought of as a set. An action of G on X is the same thing as a homomorphism $G \to S_X$. So an action of G on X is the formal way to realize G as a group of symmetries of the set X.
We need the notions of graphs which can be naturally perceived as metric spaces that helps us understand the properties of groups via their actions on them.

Definition 2.2. A **graph** G is a pair (V, E) where:

(i) $V \neq \phi$ is a set of vertices and

(ii) every $e \in E$ joins a pair of (not necessarily distinct) $v_1, v_2 \in V$.

A type of graph which encodes the information about a group is called a Cayley graph. Before diving into the notion of Cayley graphs, we first understand the action of a group on a graph.

2.2 Cayley graphs and group actions on graphs

Definition 2.3. Let G be an arbitrary group and let S be a generating set for G. The **Cayley graph for** G **with respect to** S is a directed, labeled graph $\tau(G, S) := (V, E)$ where, $V = G$, and $E = \{(g, gs) : g \in G \text{ and } s \in S\}$.

Theorem 2.4. *Let S be a generating set for G. Then the map $f : G \to Aut(\tau(G, S))$ such that $f(g) = f_g$ maps $s \in G$ to $gs \in G$ is an isomorphism.*

Proof. The map f is a homomorphism. It is injective as $\forall\ g \in G$ and $1 \in V$ of τ, $f_g(1) = g$. Therefore $g_1(1) \neq g_2(1)\ \forall\ g_1 \neq g_2$. Let $\phi \in Aut(\tau)$, that is, ϕ is an isomorphism $\tau \to \tau$. Suppose that $\phi(1) = g$. We will show that $\phi = \phi_g$.

We use induction on word length in G with respect to S. The base case is word length 0, which is just the statement $\phi(1) = g = \phi_g(1)$. Assume ϕ agrees with ϕ_g on all elements of G of word length n with respect to S. Suppose $v \in G$ has word length $n + 1$. This means that $v = ws$, where the word length of w is n and $s \in S \cup S^{-1}$. For simplicity, we assume that $s \in S$, as the other case is similar. By assumption, $\phi(w) = \phi_g(w)$. There is a unique edge labeled s from w, namely, the edge from w to $ws = v$. Similarly, there is a unique edge labeled s from $\phi(w)$, with ending point $\phi(w)s$. Since ϕ and ϕ_g both respect edge labels, it must be that $\phi(v)$ $= \phi_g(v) = \phi(w)s$. In particular, $\phi(v) = \phi_g(v)$. □

We define a group action more specifically on graphs in the following manner.

Definition 2.5. An **action of a group G on a graph** (V, E) is a homomorphism $G \to Aut((V, E))$ with the following properties.

(i) Any $g \in G$ acting on $v \in V$ takes it to some $g \cdot v \in V$;

(ii) Any $g \in G$ acting on $e \in E$ takes it to some $g \cdot e \in E$;

(iii) For any $x \in V$ or $x \in E$, we have $1 \cdot x = x$;

(iv) For $g, h \in G$ and $x \in V$ or $x \in E$, $g.(h.x) = (g.h).x$;

(v) If $e \in E$ connects $v, w \in V$ then $g \cdot e$ connects $g \cdot v$ and $g \cdot w$.

Example 2.5.1. $\mathbb{Z}/3\mathbb{Z}$ acts on a regular 3–valent tree T_3 such that image of $1 \in \mathbb{Z}/3\mathbb{Z}$ is the identity map, the image of $2, 3 \in \mathbb{Z}/3\mathbb{Z}$ maps each vertex to its two descendants.

2.3 Free groups

Definition 2.6. A **free group** F_n of 'n' letters is $\langle a_1, \ldots, a_n\rangle$ with no defining relations.
The set $\{a_1, \ldots, a_n\}$ is said to be **the generating set for the free group**.

Definition 2.7. An action of G on a set X is said to be **free** if $\forall g \in G$ and $\forall x \in X$, if $g \cdot x = x$ then $g = 1$.

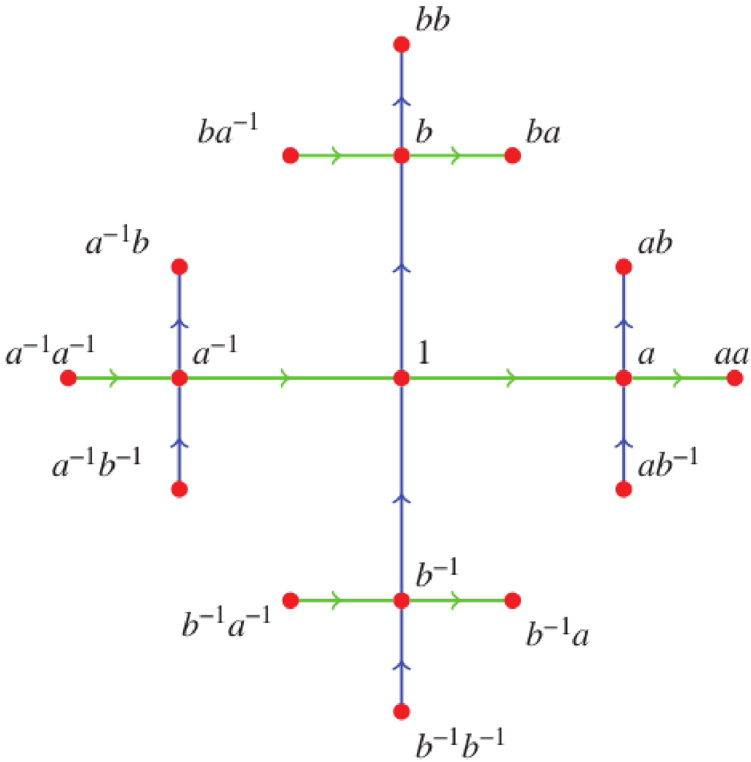


Figure 2.1: Cayley graph for a free group of two letters.

Theorem 2.8. *If a group G acts freely on a tree, then G is a free group.*

Before trying to prove this theorem, we need to define the tiling of a tree.

Definition 2.9. A **tile** is a subtree T_0 of the barycentric subdivision T' of T (the **barycentric subdivision** of a graph is the graph obtained by subdividing each edge; that is, we place a new vertex at the center of each edge of the original graph). A **tiling** of T is a collection of tiles with the following properties:

(i) No two tiles share an edge, so two tiles can only intersect at one vertex.

(ii) The union of the tiles is the entire tree T'.

Proof. The key to the proof is to obtain a tiling of our tree T . For each $g \in G$, let T_g be the subtree of the barycentric subdivision whose vertex set is the set of vertices w of T' so that $d(w, gv) \leq d(w, g'v)$ for all $g' \in G$ and whose edge set is the set of edges e of T' so that both vertices of e lie in T_g.

We claim that the following set is a generating set for our group G

$$S = \{g \in G | (gT_0) \cap T_0 = \phi\}.$$

We need to show that our set S is a symmetric generating set for G. Let $s \in S$. This means that $(sT_0) \cap T_0 = w$ for some vertex w of T' . Applying s^{-1} we can

conclude that $T_0 \cap s^{-1}T_0 = \{s^{-1}(w)\}$. This means that $s^{-1} \in S$, as desired.
Now, we need to show that S generates G. Let $g \in G$. We want to write g as a product of elements of S. We look at the vertex gv. We can draw the unique path from gv back to v and keep track of the tiles encountered along this path:
$T_{gn}, T_{gn-1}, \ldots, T_{g1}, T_{g0}$, where $g_n = g$ and $g_0 = e$. If a path travels through tiles T_{gi+1} and T_{gi} without traveling through any tiles in between, then $T_{gi+1} \cap T_{gi}$ must be nonempty. Applying g_i^{-1} , we see that $(g_i^{-1}T_{gi+1}) \cap (g_i^{-1}T_gi) = T_{gi-1gi+1} \cap T_0 \neq \phi$. But this means exactly that every $g_i^{-1}g_{i+1}$ is in S, and g can be written as a product of elements in S. Thus, S is a symmetric generating set of G.

Since there is a unique (non-backtracking) path from gv to v in T $\forall g = s_1 s_2 \ldots s_k$ we can argue that that the unique path from gv to v completely determines the word g, $\forall g \in G$. $\square$

We will apply this theorem for the action of a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ on a Farey tree to see how a free action on a tree implies that the group is free.

Definition 2.10. We say that $(m, n) \in \mathbb{Z}^2$ is **primitive** if $\gcd(|m|, |n|) = 1$. We define an equivalence relation $\sim$ on the primitive elements of $\mathbb{Z}^2$ by declaring (m, n) to be equivalent to $-(m, n) := (-m, -n)$. We represent their equivalence class by $\pm(m, n)$.

Definition 2.11. The **Farey graph** is the graph whose vertex set is the set $\{\pm(m, n) \in \mathbb{Z}^2 \mid \pm(m, n) \text{ is primitive}\}$. We denote a vertex $\pm(m, n)$ by (m, n) in Figure 2.2. Two vertices $\pm(p, q)$ and $\pm(r, s)$ are connected by an edge if $ps - qr = \pm 1$. The Farey graph thus obtained can be seen in Figure 2.2.

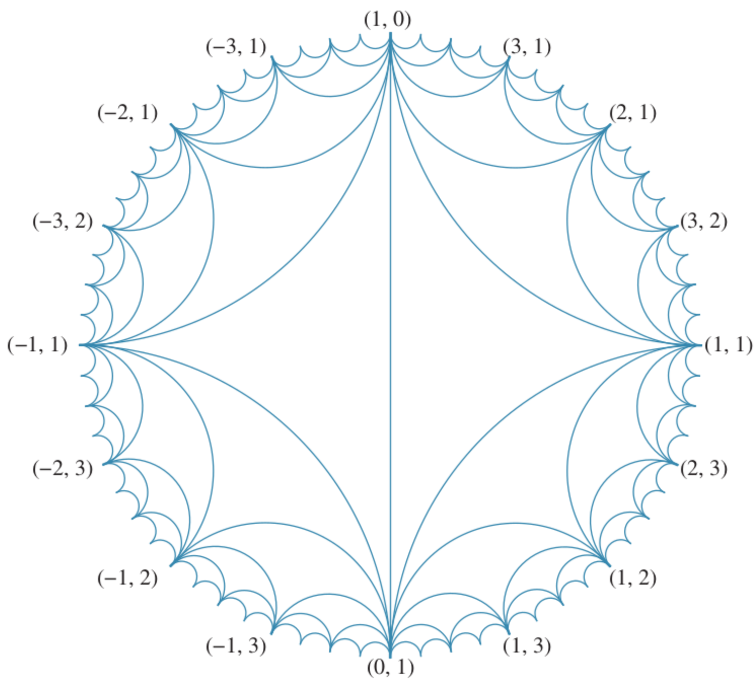


Figure 2.2: The Farey graph.

Now, we can obtain the **Farey tree** by the following steps.

(i) Mark the centre of each edge of the Farey graph as a vertex for the Farey tree.

(ii) Mark the centroid of each triangle formed by the edges of the Farey graph as a vertex for the Farey tree.

(iii) An endpoint function for the Farey tree connects the triangle vertices to the corresponding edge vertices if and only if that edge is a side of that triangle.

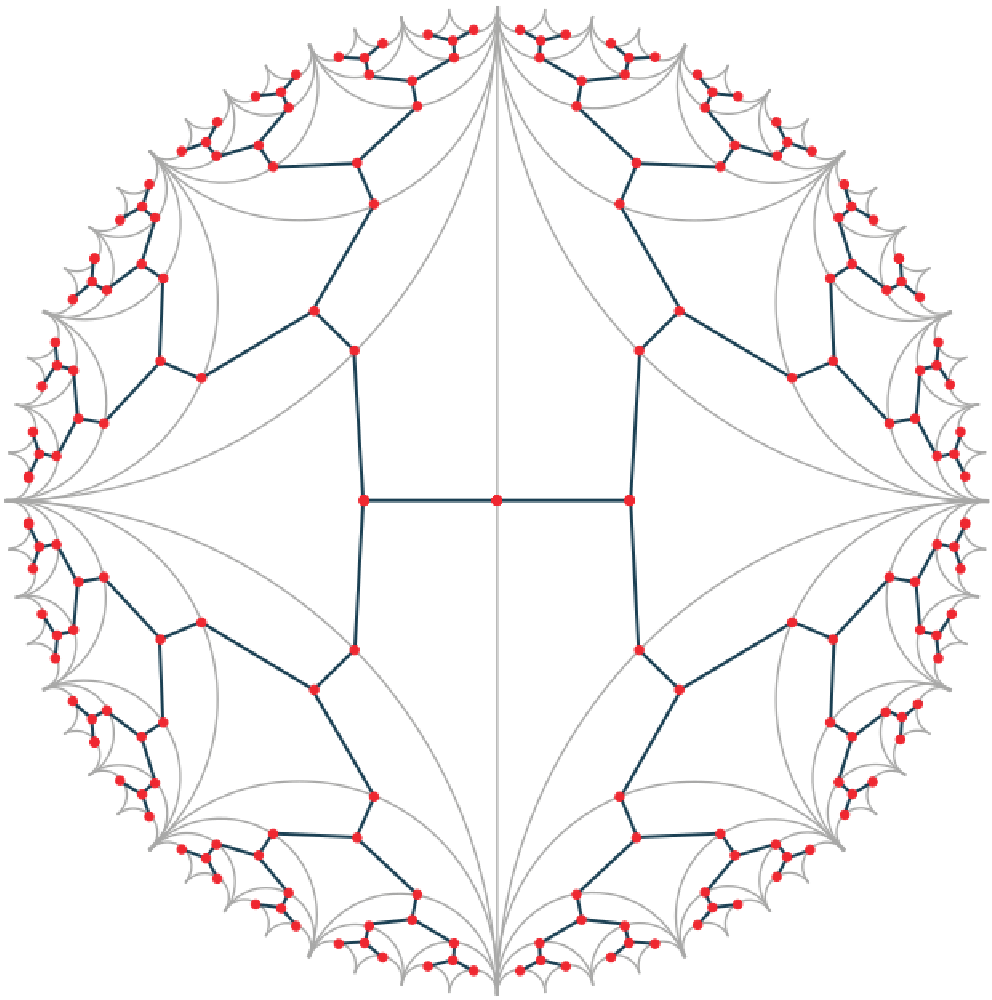


Figure 2.3: The Farey tree.

Definition 2.12. The level m congruence subgroup of SL(2,$\mathbb{Z}$) denoted by $\mathrm{SL}(2,\mathbb{Z})[m]$ is the kernel of the homomorphism $\phi : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/m\mathbb{Z})$ which reduces the entries of all the matrices in SL(2,$\mathbb{Z}$) modulo m.

Corollary 2.13. *For $m \geq 3$, the group $SL(2,\mathbb{Z})[m]$ is isomorphic to a free group.*

Proof. Since the action of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey complex cannot interchange an edge and a triangle, it means the action cannot interchange the two endpoints of the same edge. In other words, $\mathrm{SL}(2,\mathbb{Z})$ acts on the Farey tree without inversions. It remains to understand the stabilizer in $\mathrm{SL}(2,\mathbb{Z})$ of each vertex of the Farey tree. Let us first consider stabilizers of vertices corresponding to edges of connecting the vertices $\pm(1,0)$ and $\pm(0,1)$. For an element of $\mathrm{SL}(2,\mathbb{Z})$ to stabilize v, it simply must preserve the set $S = \{\pm(1,0), \pm(0,1)\}$. As the columns of a matrix are just the images of the standard basis vectors under the action of that matrix, the columns of our stabilizers must lie in S. That gives exactly 6 matrices to think about. But we cannot choose a vector and its negative, for then the determinant will be 0. Therefore, we have the following 4 matrices as candidates.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These are the elements of the cyclic group generated by the second matrix on the list. So the stabilizer of v is a cyclic group of order 4. The only matrices that lie in $\mathrm{SL}(2,\mathbb{Z})[m]$ for $m \geq 2$ are identity and its negative. However, only the identity lies in $\mathrm{SL}(2,\mathbb{Z})[m]$ for $m \geq 3$. But, we have proven that the required condition holds true only for one vertex. But, as the action of $\mathrm{SL}(2,\mathbb{Z})[m]$ is transitive on Farey tree, the stabilizer of any vertex can be calculated as follows,

$$\mathrm{Stab}(Mv) = M\,\mathrm{Stab}(v)M^{-1},$$

Therefore the stabilizer for all the vertices will be identity, and hence the action is free. Using Theorem 2.8, $\mathrm{SL}(2,\mathbb{Z})[m]$ is a free group. □

Theorem 2.14. *Suppose that a group G acts without inversions on a tree T in such a way that G acts freely and transitively on edges. Choose one edge e of T and say that the stabilizers of its vertices are H_1 and H_2. Then $G \cong H_1 * H_2$.*

Proof. Since G acts without inversions, we avoid the barycentric subdivision, so the definition of the tiling is the same as before except that the tiles are subgraphs of T itself. A path in T from e to ge will give us a unique alternating word in the elements of H_1 and K_1 , and an alternating word in the elements of H_1 and K_1 gives a unique path in T from e to gv. Since there is only one path from e to ge, it follows that the product is a free product. □

Corollary 2.15. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \cong PSL(2,\mathbb{Z})$

Proof. The action of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey tree can be restricted to the action of $PSL(2,\mathbb{Z})$ since (m,n) and $-(m,n)$ represents the same vertex. This means that $\mathrm{PSL}(2,\mathbb{Z})$ also acts without any inversions and transitively on the Farey tree.

We, name the vertex corresponding to $\{\pm(1,0), \pm(0,1)\}$ as v_0 and the vertex corresponding to $\{\pm(1,0), \pm(0,1), \pm(1,1)\}$ as w_0. Now, applying Theorem 2.14, we know that $PSL(2,\mathbb{Z})$ is the free product of the stabilizers of v_0 and w_0. $\mathrm{SL}(2,\mathbb{Z})$, the stabilizers were isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$. The negative of the identity corresponds to 2 and 3 in these two groups. Thus the images of these stabilizers in $PSL(2,\mathbb{Z})$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$. Therefore,

$$\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

□

Corollary 2.16. *Every finitely generated subgroup of a free group is free.*

Proof. As a consequence of Theorem 2.8, if we have a subgroup H of a free group G, then H acts freely on a tree as well. We can take any free action of G on a tree and restrict the action to H. Again using Theorem 2.8, H is also a free group. □

Chapter 3

Quasi-isometries

One of the problems we encounter when using Cayley graphs as a geometric representation of a group is that there can be different Cayley graphs for the same group with respect to different generating sets. These graphs are not isometric to each other. So, we need to define another form of equivalence called quasi-isometries or "coarse isometries".

3.1 The need for defining Quasi-isometries

Definition 3.1. The **geometric realization of a Cayley graph** is defined as follows

(i) Each edge in the graph is associated to an inteval $[0, 1]$

(ii) All points on the edges are now a part of the metric space.

(iii) Distance between two points on the same edge is the corresponding on the real line w.r.t Euclidean metric.

(iv) Distance between two points not on the same edge can be calculated by summing up the smallest distance between the endpoints of the two concerned edges and the distance of the points from those two endpoints.

Definition 3.2. If (X, d_X) and (Y, d_Y) are metric spaces, a function $f : X \to Y$ is called an **isometric embedding** if f preserves distances, that is, for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

An isometric embedding f is called an **isometry** if it is also surjective.

Example 3.2.1. A map from $f : \mathbb{Z} \to \mathbb{R}^2$ such that $f(x) = (x, 0)$ is an isometric embedding, but not an isometry.

Definition 3.3. If (X, d_X) and (Y, d_Y) are metric spaces, a function $f : X \to Y$ is called a **bi-Lipschitz embedding** if f preserves distances by allowing distances to be stretched and compressed by bounded amounts, that is, $\exists\ K > 0$ such that $\forall$ $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

A bi-Lipschitz embedding f is a **bi-Lipschitz equivalence** if it is also surjective.

Example 3.3.1. A map from $f : \mathbb{Z} \to \mathbb{R}^2$ defined by $f(x) = (3x, 0)$ is a bi-Lipschitz embedding, but not a bi-Lipschitz equivalence.

Theorem 3.4. *Let G be a finitely generated group and let S and S' be two finite generating sets for G. Then the identity map $f : G \to G$ is a bi-Lipschitz equivalence from the metric space (G, d_S) to the metric space $(G, d_{S'})$.*

Proof. Since S is finite, we can define a constant $M \geq 1$ by

$$M = \max\{d_{S'}(1, s) | s \in S \cup S^{-1}\}.$$

Now consider any $g \in G$. Suppose g has word length n with respect to S, so we can write

$$g = s_1 s_2 \dots s_n,$$

where each s_i is in $S \cup S^{-1}$. Using the triangle inequality, we get:

$$\begin{aligned} d'_S(1, g) &= d_{S'}(1, s_1 s_2 \dots s_n) \\ &\leq d_{S'}(1, s_1) + d_{S'}(s_1, s_1 s_2 \dots s_n) \\ &\leq d_{S'}(1, s_1) + d_{S'}(s_1, s_1 s_2) + d_{S'}(s_1 s_2, s_1 s_2 \dots s_n) \\ &\leq d_{S'}(1, s_1) + d_{S'}(s_1, s_1 s_2) + d_{S'}(s_1 s_2, s_1 s_2 s_3) + \dots d_{S'}(s_1 s_2 \dots s_{n-1}, s_1 s_2 \dots s_n). \end{aligned}$$

Since the action of a group element preserves the distances between the elements of the same group with respect to word metric we can rewrite this as

$$\begin{aligned} d_{S'}(1, g) &\leq d_{S'}(1, s_1) + d_{S'}(1, s_2) + \dots + d_{S'}(1, s_n) \\ &\leq M + M + \dots + M \\ &= Mn. \end{aligned}$$

But n is the word length of g with respect to S, that is, $d_S(1, g) = n$. Therefore, we have shown that $\forall g \in G$, $d_{S'}(1, g) \leq M d_S(1, g)$. We can just reverse the roles of S and S' to find another bound N and the larger of two values can be assigned to K. $\square$

We have shown that the Cayley graphs of a group with respect to two different generating sets will be equivalent to each other under the bi-Lipschitz equivalence.

Example 3.4.1. Consider the Cayley graphs of $\mathbb{R}^2$ and $\mathbb{Z}^2$ with respect to the generating set $\{(1,0),(0,1)\}$ with the natural Taxicab metric for their geometric realizations. We can take a function $f : \mathbb{Z}^2 \to \mathbb{R}^2$ such that each point is marked onto itself, that is, $f(x,y) = (x,y)$. The distances are preserved by Taxicab metric and we get an isometric embedding.But this map is not surjective. Hence this is not an isometry. Intuitively there should be some equivalence relation between the two metric spaces which make $\mathbb{Z}^2 \in \mathbb{R}^2$ different from $\mathbb{Z} \in \mathbb{R}^2$

We have a similar problem with regards to the Cayley graphs and their geometric realizations not being equivalent to each other in any sense.

3.2 Quasi-isometric embeddings and euivalences

3.2.1 Notion of quasi-isometries

Definition 3.5. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called a **quasi-isometric embedding** if there are constants $K \geq 1$ and $C \geq 0$ so that

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C.$$

A quasi-isometric embedding $f : X \to Y$ is called a **quasi-isometric equivalence**, or just a **quasi-isometry**, if there is a constant $D > 0$ so that for every point $y \in Y$, there is an $x \in X$ so that

$$d_Y(f(x), y) \leq D.$$

Example 3.5.1. A map from $f : \mathbb{R}^2 \to \mathbb{Z}^2$ such that $f(x,y) = (\lfloor x \rfloor, \lfloor y \rfloor)$ is a quasi-isometry, but it is neither an isometry or bi-Lipschitz equivalence.

3.2.2 Geometric realizations of Cayley graphs of a group

Theorem 3.6. *Let G be a finitely generated group and let S and S' be two finite generating sets for G. Then the geometric realization of the Cayley graph $\tau(G,S)$ is quasi-isometric to the geometric realization of the Cayley graph $\tau(G,S')$*

Proof. First, there is a quasi-isometry from the geometric realization of any graph to its set of vertices (with the path metric) obtained by sending every point on an edge to the nearest vertex. In Theorem 3.4, we showed that there is bi-Lipschitz equivalence between two Cayley graphs of the same group. Therefore, we can obtain a quasi-isometry $\tau(G,S) \to \tau(G,S')$ as a composition of three quasi-isometries. $\square$

Since we have proven that all Cayley graphs and their geometric realizations for a given group are equivalent to each other, we can now talk about the geometric representation of a group being equivalent to some other group without being concerned about the choice of generating set influencing our geometric image of that group. In the next chapter, we will try to solve the other problem we encounter while perceiving groups as geometric objects.

Chapter 4

Word problem and its solvability

When we have a finitely presented group, we face the problem of being able to tell if two words w_1, and w_2 represent the same group element. This can be equivalently stated as the problem of being able to tell whether or not $w_1^{-1}w_2$ is the identity element. This complexity is captured by the notion of a word problem of a group.

Definition 4.1. A group $G = \langle X \mid R \rangle$ is a **finitely presented** if it is finitely generated, i.e., $|X| < \infty$ and has a finite number of relations defined on them, i.e. $|R| < \infty$.

Definition 4.2. The **word problem** for a finitely generated and presented group is the problem of determining whether or not a given word in the group represents identity.
A **word problem is called solvable** in finite number of steps if we are able to devise an algorithm to find whether or not a word of certain length represents identity.

The complexity of word problem for a group presentation is governed by the Dehn function. The faster the Dehn function grows, the greater the number of times relations must be used to reduce the problem.

4.1 Dehn Function

A word w on $S \cup S^{-1}$ represents the identity in G when it can be converted to the empty word via

- a finite sequence of free reductions ($aa^{-1} \to 1$)
- free expansions ($1 \to aa^{-1}$ for some $a \in S \cup S^{-1}$)
- applications of defining relators (elements of R) or their cyclic permutations

Such a sequence is called a **null-sequence** for w. The number of applications of defining relators or its permutations is the **length** of that null sequence.
Counting the number of such moves it can take for the word of a given length to be reduced to identity gives us a measure of the difficulty in working with the particular presentation for a group. The Dehn function estimates this measure thereby measuring the complexity of the solution to the word problem.

Definition 4.3. The **Dehn function** $f : \mathbb{N} \to \mathbb{N}$ maps n to the minimal number N such that, if w_i is a word of length $\leq n$ that represents the identity, l_i is the length of minimal null-sequence for w_i, then $N = \max\{l_i\}$.

Since there are only finitely many words of length at most n (S is finite) the Dehn function is well–defined.

Example 4.3.1. Consider the group $\mathbb{Z}/m\mathbb{Z} = \langle a \,|\, a^m \rangle$. The Dehn function $f(n)$ is given by $\lfloor \frac{m}{n} \rfloor$

Example 4.3.2. The Dehn function $f(n)$ of $\mathbb{Z} \times \mathbb{Z} = \langle a, b \,|\, a^{-1}b^{-1}ab \rangle$ satisfies

$$f(n) \leq \frac{n^2}{16}.$$

This bound is realized by the words $a^{-k}b^{-k}a^kb^k$.

Proof. To see this, first note that $a^{-k}b^{-k}a^kb^k$ is the hardest word to reduce of a given length n since all pairs of aa^{-1} and bb^{-1} are as far apart as possible. The permutations of defining relators tell us that the multiplication on a, b, a^{-1}, b^{-1} is commutative. If we apply $b^{-1}a = ab^{-1}$ in the middle of the word, we will get $a^{-k}b^{-k+1}ab^{-1}a^{k-1}b^k$.

We can continue this operation on the first a to reduce the first pair of $a^{-1}a$ to identity. This will take k applications of the operation $b^{-1}a = ab^{-1}$. Since the number of such a is k, the total number of moves required will be k^2. As $k = \frac{n}{4}$, where n is the word length, we have

$$f(n) \leq \frac{n^2}{16}.$$

□

Lemma 4.4. Any finitely generated abelian group has a quadratically growing Dehn function.

Proof. If there are h elements in the generating set, then $a^{-k}b^{-k} \cdot h^{-k}a^kb^k \ldots h^k$ will be the hardest word to reduce. It will take us $(h-1)k$ operations for each h to reach

h^{-1} and so, total operations only for h will be $(h-1)k^2$. Repeating the process for each element of the generating set, we have

$$f(n) \leq (h-1)k^2 + (h-2)k^2 + \cdots + k^2.$$

Since, $k = \frac{n}{2h}$,

$$f(n) \leq (h-1)(\frac{n}{2h})^2 + (h-2)(\frac{n}{2h})^2 + \cdots + (\frac{n}{2h})^2.$$

$\square$

Example 4.4.1. Consider $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$ which has a quadratically growing Dehn function. To tell whether or not a word on $\{a^{\pm 1}, b^{\pm 1}\}$ represents the identity, it is enough just to add up the exponents of the $a^{\pm 1}$ and $b^{\pm 1}$ present and check whether both are zero. Therefore, we can solve the word problem in linear time, which shows that Dehn function is not a good measure of the difficulty of the word problem. It's more of a worst case scenario.

4.1.1 The importance of the Dehn Function

Since the Dehn function measures the complexity of the solution to the word problem for a group, it gives us an upper bound of the time required to solve the word problem.

Definition 4.5. A **recursive** function $f : M \to N$ is a function defined on a discrete, well ordered domain with a least element that calls itself, that is, $f(n)$ is defined in terms of $f(m)$, where $m < n$ and $m, n \in M$.

Theorem 4.6. *For a finitely presented group $\langle S \mid R \rangle$ of a group with Dehn function $f : \mathbb{N} \to \mathbb{N}$, the following are equivalent.*

(i) There is an algorithm that, given the input of a word on $S^{\pm 1}$, will compute whether or not that word represents the identity (i.e., the presentation has solvable word problem).

(ii) There is a recursive function $g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq g(n)$, $\forall n$.

(iii) f itself is a recursive function.

Proof. ($ii \implies i$) Given an upper bound $g(n)$ on the Dehn function, it is always possible to reduce a word of length w that represents the identity to the empty word using a null-sequence with at most $g(n)$ applications of defining relators. So, if $g(n)$ is recursive and $f(n) \leq g(n)$, we can reduce the problem for words of length n to words of length $< n$ using the recursion on $g(n)$. Thus, we have $g(n) = k(g(n-1))$ and $f(n-1) \leq g(n-1)$. We can repeat this process until the word length reduces to a very small number and the word problem becomes trivial. Thus, we have devised an algorithm to solve the word problem given a recursive upper bound for Dehn function.

($i \implies iii$) If, on the other hand, we have an algorithm that solves the word problem, then we can calculate $f(n)$ by the following procedure.

- First list all words on $S^{\pm 1}$ of length at most n
- Discard from the list all that fail to represent the identity
- For each word w that remains, calculate the minimal number of application of defining relator moves necessary to reduce w to identity.

Since we have devised using an algorithm to solve the problem for words of length n, we can simply use some free reductions to write the Dehn function $f(n)$ as a function of some $f(n-k)$. Thus the Dehn function is recursive if we have an algorithm and the word problem is solvable.

($iii \implies i$) If $f(n)$ is a recursive function, we can simply define

$$g(n) = f(n) + 1,$$

and we will get a recursive upper bound for $f(n)$. Thus, we have shown that

$$ii \implies i \implies iii \implies ii.$$

Hence, our proof for the theorem is complete. □

4.2 A semigroup with an unsolvable word problem

It is not common to encounter a group with an unsolvable word problem. Alan Turing [5] studied the importance of semigroups with unsolvable word problems in cryptography and cryptanalysis. We conclude this chapter with an example of one such semigroup.

Example 4.6.1. One of the simplest example of a semigroup with an unsolvable word problem is $G = \langle a, b, c, d, e | ac = ca, ad = da, bc = cb, bd = db, ce = eca, de = edb, cdca = cdcae, caaa = aaa, daaa = aaa \rangle$ given by Cétjin [5]. Collins [5] came up with another presentation for this semigroup.

Recently, Wang [6] reduced the number of generators to two. But, there were 27 defining relators for that presentation the smallest of which had a word length of about 100.

Chapter 5

Hyperbolic groups

Curvature is a fundamental way of understanding the intrinsic geometry of manifolds. There are three curvatures on 2-dimensional manifolds, namely zero, positive, and negative. The most trivial examples for these three are the plane, the sphere, and the saddle.

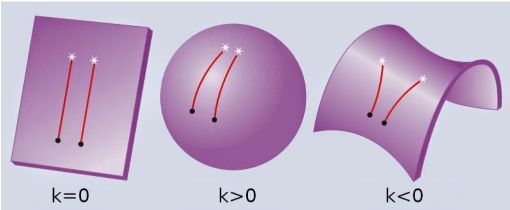


Figure 5.1: 2-dimensional manifolds with zero, positive and negative curvatures.

A nice space that exhibits saddle point behavior at each of its points is the hyperbolic plane $\mathbb{H}^2$. This space can be interpreted by multiple models isometric to each other. To understand one of them, we consider the Möbius transformation

$$z \to \frac{i-z}{i+z}$$

This transformation takes the upper half-plane $\mathbb{U}$ to the open unit disk $\mathbb{D} \in \mathbb{C}$, and it takes the real line to the unit circle. Since we have already identified $\mathbb{U}$ with the hyperbolic plane, we now have an identification of the hyperbolic plane with $\mathbb{D}$. We refer to $\mathbb{D}$ as the Poincaré disk. If we denote the hyperbolic metric on $\mathbb{D}$ by du^2 , it turns out that $du^2 = 4\frac{dE^2}{(1-r^2)^2}$, where r denotes the distance from the origin and dE^2 is the Euclidean metric. It is known that surfaces with constant negative Gaussian curvature admit a metric locally modeled on $\mathbb{H}^2$. But to study such groups, we want to capture this behavior in a discrete model, which earns them the name "hyperbolic groups."

5.1 δ–hyperbolicity

In the Euclidean metric, we define the **incircle** of a triangle to be the largest inscribed circle. The points of tangency are called **inpoints**, and they cut the three sides of the triangle into six pieces that come in length-matched pairs. Now in a more general space, we cannot necessarily find an inscribed circle in any nice way, but we can generalize the other property. Let the inpoints of a geodesic triangle be the uniquely determined three points that divide the sides into pairs of equal lengths, as shown in Figure 5.2.

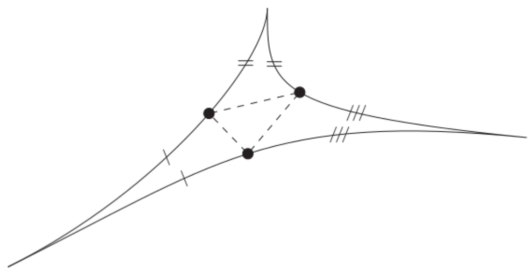


Figure 5.2: Inpoints of a geodesic triangle.

They are uniquely determined because we are just solving the system $a = r + s$, $b = s + t$, $c = r + t$, and the triangle inequality guarantees a solution. If we consider our space to be a tree, any 'triangle' has actually the same inpoints. So the 'insize' of our 'triangle' is zero.

Definition 5.1. We will call a metric space $\delta-$**hyperbolic** if all geodesic triangles have insize $\leq \delta$, where $\delta \in \mathbb{R},$ and $\delta \geq 0$.

Example 5.1.1. A tree is a $0-$hyperbolic space because if we try to take any geodesic triangle in a tree, the insize would be zero as the tree has no closed loops and thus no triangles.

This definition works fine on geodesic spaces. For a general space, we say a space is $\delta-$hyperbolic if all four-tuples for any four points x, y, z, w in that space satisfy

$$(x\Delta y)_w \geq \min\{(y\Delta z)_w, (x\Delta z)_w\} - \delta,$$

where $(x.y_w)$ is the shortest distance between the line joining x, y and w.

A crucial property is that $\delta-$hyperbolicity is stable under quasi-isometry but, it does not preserve the constant δ. So a quasi isometry on one $\delta-$hyperbolic space can take it to some other $\delta'-$hyperbolic space.

5.2 Hyperbolic groups

Definition 5.2. A finitely generated group is called **hyperbolic** if any of its Cayley graphs (for a finite generating set) is $\delta-$hyperbolic.

Example 5.2.1. Since trees are always $0-$hyperbolic, natural examples of hyperbolic groups are free groups F_n of any rank, as their Cayley graphs with respect to any finite generating set S with $|S| = n$ are just $(2n)-$regular trees.

Example 5.2.2. As proven in Chapter 2, $\mathrm{PSL}(2, \mathbb{Z}) = \langle v, w \mid v^2 w^3 \rangle$ With this presentation, the Cayley graph looks just like a tree of triangles as in Figure 5.3, a graph that clearly has the δ–hyperbolic property with $\delta = 1$.

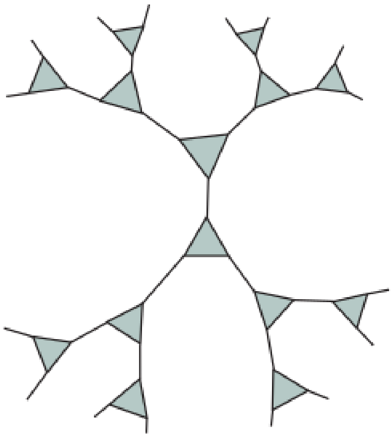


Figure 5.3: Cayley graph of $\mathrm{PSL}(2, \mathbb{Z})$.

Since $\mathrm{PSL}(2, \mathbb{Z})$ is a quotient group derived from $\mathrm{SL}(2, \mathbb{Z})$ and this induces a quasi-isometry between the two, this implies that $\mathrm{SL}(2, \mathbb{Z})$ should also be a hyperbolic group.

5.3 Surface groups

Surface groups are fundamental groups of closed hyperbolic surfaces. Each manifold S has an associated group $\pi_1(S)$, called the fundamental group, which can be said to be group-theoretically encoding information about the topology of S. The idea started as an attempt by Poincaré [8] to classify manifolds by associating a group to them which could be distinguished from other groups relatively easily.

Definition 5.3. The **fundamental group** of a topological space X with some chosen base point x_0 , denoted by $\pi_1(X, x_0)$, is the group of homotopy classes of loops, which are closed paths on X starting and ending at x_0 .

Example 5.3.1. A tree has no loops, so its fundamental group is trivial.

Example 5.3.2. All loops on a circle from a point to itself are just complete rotations around the circle, so the fundamental group will be isomorphic to $\mathbb{Z}$.

Example 5.3.3. Let Σ_g denote the closed orientable surface of genus g. $\pi_1(\Sigma_2)$ can be generated by the four loops in Figure 5.4.

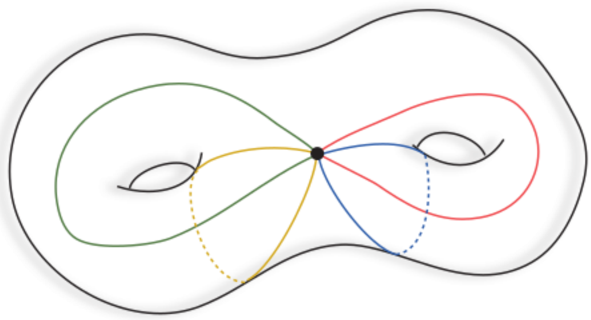


Figure 5.4: Generating set for the fundamental group of Σ_2

Its fundamental group can be written as

$$\pi_1(\Sigma_2) \cong \langle a, b, c, d \,|\, [a, b][c, d] \rangle.$$

In general, for a surface of genus $g \geq 1$,

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, a_2, b_2 \dots a_g, b_g \,|\, [a_1, b_1][a_2, b_2] \dots [a_g, b_g] \rangle.$$

Since these surfaces are closed hyperbolic surfaces for $g \geq 2$, their groups are surface groups. Surface groups are important examples of hyperbolic groups themselves and their construction can lead us to an important result about word problem for hyperbolic groups.

5.4 Word problem for hyperbolic groups

Definition 5.4. A presentation $G = \langle a_1, ..., a_n \,|\, r_1, ..., r_m \rangle$ for a group G is called a **Dehn presentation of G** if the following conditions hold true:

(i) There is a set of strings $u_1, v_1, ..., u_m, v_m$ and each relator r_i is of the form $r_i = u_i v_i^{-1}$. (Relator r_i encodes the equivalence in the group of u_i and v_i.)

(ii) For each i, the word length of v_i is shorter than the word length of u_i .

(iii) For any nonempty string w in $S = \{a_i\}$ that represents the identity element, if w has been reduced by canceling all occurrences $a_i a_i^{-1}$, then at least one of u_i or u_i^{-1} must appear as a substring.

Theorem 5.5. *Hyperbolic groups admit Dehn presentations.*

Proof. Fix any $K > 8\delta$ and consider a Cayley graph for G with respect to a (finite) generating set $S = \{a_i \,|\, i \in \mathbb{N}\}$ for G. Now consider the list of all reduced words t_i with word length at most K. Now, we can check which of the t_i represent the same word by just following them in the graph. Let u_i be the non-geodesic words from that list, and for each u_i , let v_i be some geodesic word from that list reaching the same point in the graph, so it is guaranteed to be strictly shorter. Then put $R = \{r_i = u_i v_i^{-1} \,|\, i \in \mathbb{N}\}$. We claim $G = \langle S \,|\, R\rangle$ is a Dehn presentation. Conditions $(i) - (ii)$ of Definition 5.4 are satisfied by construction. For Condition (iii), we can rule out $8\delta-$local geodesic loops of length at least 8δ. So any long loop has a non-geodesic subsegment of length at most K, which is one of u_i. This completes the proof. □

Corollary 5.6. *Hyperbolic groups have solvable word problems.*

Proof. By construction of Dehn presentations, they have a solvable word problem, and by Theorem 5.5 all hyperbolic groups have a solvable word problem. □

Appendix A

Ends of groups

When we study the properties a real-valued function on $\mathbb{R}$, we often ask ourselves, what is happening to the function at infinity. It helps us predict properties such as if the function is going to be negligibly small when our argument if large enough even if it may never actually be zero. Here, we will study what happens to a finitely generated group G at infinity, using its Cayley graph $\tau(G, S)$ with respect to a generating set S as its geometric representation.

A.1 Number of ends of a group

Definition A.1. Let τ be a connected, locally finite (having a finite number of edges for each vertex), infinite graph. Let C be a subgraph of τ. We define

$$||\tau \backslash C||$$

to be the number of disjoint, connected and unbounded components we get if we remove C from τ.

Example A.1.1. If we remove any vertex from a free group of n letters, we will get 2^n components.

Definition A.2. For a locally finite graph τ, the **number of ends of the graph** is

$$e(\tau) = sup\{||\tau \backslash C|| \,|\, \text{C is a finite subgraph of } \tau \backslash C\}$$

Definition A.3. The **number of ends of a finitely generated group** G is equal to the number of ends of a Cayley graph $\tau(G, S)$ where S is a finite generating set for G

For this definition to be consistent, we need that the number of ends of G should not depend on the choice of S.

Theorem A.4. *The number of ends of a finitely generated group is independent of the choice of the finite generating set.*

Before trying to prove this theorem, we look at how the number of ends of groups can be related for groups that are quasi-isometric to each other.

A.2 Ends of groups and quasi-isometries

Theorem A.5. *If two finitely generated groups are quasi-isometric, then they have the same number of ends.*

It can be clearly seen that this statement is stronger than the previous theorem. To prove these statements; we will require the following result from Freudenthal and Hopf [4].

Theorem A.6. *If G is a finitely generated group, then G has zero, one, two, or infinitely many ends.*

Proof. To prove this statement, all we need two show that if $\exists C \subset \tau$ such that $||\tau \backslash C|| \geq 3$, then $\exists D \subset \tau$ such that $||\tau \backslash D|| = \infty$. (See Figure A.1)

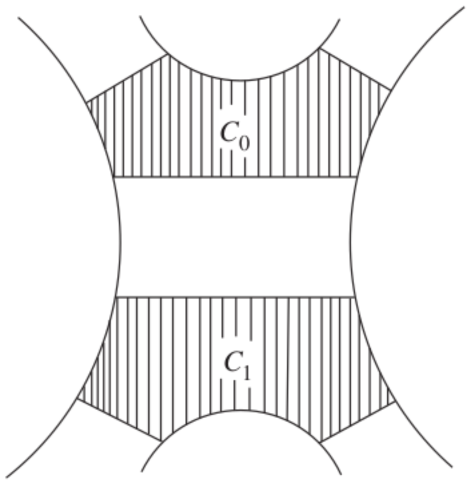


Figure A.1: Construction of a group with more than 2 ends.

Let τ be a Cayley graph for a group G and let C_0 be a finite subset of τ such that $||\tau \backslash C_0|| = 3$. Then there is then an element g in the group so that $C_1 = g \cdot C_0$ is disjoint from C_0 . Then if $D_1 = C_0 \cup C_1$, it follows that $||\tau \backslash D_1|| > 3$. But the same argument can be applied inductively, and we can have a finite $d_\infty \subset \tau$ such that $||\tau \backslash D_\infty|| = \infty$. $\square$

Now, we can provide the proof of Theorem A.4 [7].

Proof of Theorem A.4. Let S_1 and S_2 be two finite generating sets for a group G, and let $\tau(G, S_1)$ and $\tau(G, S_2)$ be the corresponding Cayley graphs. Then the two Cayley graphs must have a bi-Lipschitz eqivalence as proved earlier. Now, if we have a finite subgraph $C_1 \subset S_1$, then the bi-Lipschitz function will map it to some finite subgraph $C_2 \subset S_2$, such that $||\tau(G, S_1)\backslash C_1|| = ||\tau(G, S_2)\backslash C_2||$. Hence, we are done. □

This proves that the number of ends is the property of a group and not of its graphs. The same logic can be applied while proving that two quasi-isometric groups have the same number of ends, as all their Cayley graphs will have a quasi-isometric equivalence relation between them. Now, we move onto some examples.

Example A.6.1. The number of ends of a $\mathbb{Z} \times \mathbb{Z}$ is 1. We can have two connected components using a finite subgraph of its standard Cayley graph, but only one of them will be connected.

Example A.6.2. Any free group of n letters($n \geq 2$) has infinitely many ends, as we can remove just the identity vertex from the Cayley graph and get more than 4 ends. The result then follows from Theorem A.6.

Corollary A.7. *Suppose that G is a finitely generated group with a finite index subgroup N. Then $e(G) = e(N)$.*

Proof. The proof follows from Theorem A.5 and the fact that if G is quasi-isometric to some group P, then it must have a finite index subgroup isometric to P and vice versa. □

This concludes our study for ends of groups. In the final Chapter, we look at the topological notion of dimensions for finitely generated groups we have been working with.

Appendix B

Asymptotic dimension

One of the most basic notions we study in topology is dimension. We think of the building blocks of topology, such as points, lines and planes as being 0–dimensional 1–dimensional and 2–dimensional respectively. Then our notion expands to the amount of information required to represent a space. For instance, $\mathbb{R}^2$ is two dimensional as it requires two parameters to be presented, whether it is the standard coordinate system or the polar coordinate system.

But, while defining the notion of dimension of groups, we cannot take the exact same approach, as the groups we work with are often discrete objects having multiple defining relations between their elements. So, once again we use the Cayley graphs as a metric space representation of our groups and use the topological notion of dimensions to construct the theory.

We can begin to think about the dimensions of groups with one of the simplest examples, the group of integers $\mathbb{Z}$. We can guess, that it should be 1–dimensional as it sits in $\mathbb{R}$. Similarly we can define the dimension for a free abelian group of two letters $\mathbb{Z}^2$ to be 2. (We also saw how $\mathbb{Z} \in \mathbb{R}^2$ is different from $\mathbb{Z}^2 \in \mathbb{R}^2$ during our study of quasi-isometries.) Similarly, the dimension of $\mathbb{Z}^n$ can be defined as n. Now, if we talk about the nonabelian free group of two letters F_2, we can see that it does not follow a metric similar to the euclidean metric in $\mathbb{R}^2$. Thus, it appears that it might have a dimension different than 2, which seems non–intuitive. Also, if we look at nonstandard Cayley graphs for $\mathbb{Z}$, it appears that it no longer sits inside $\mathbb{R}$.

So, if we are going to use Cayley graphs as a representation of our groups, we must make sure that the defined notion of dimensions for our groups is invariant under quasi-isometries.

B.1 Topology and dimension

Definition B.1. We say that the **dimension of a metric space** X does not exceed n and write $\dim X \leq n$ if for every open cover U of X there is a refinement V with order at most $n+1$.

We will first characterize what it means to be zero dimensional and then use zero dimensional sets to compute higher dimensions.

Lemma B.2. Suppose X is a separable metric space that is nonempty. Then $\dim X = 0$ if and only if for all $p \in X$ and every open set U containing p, there is an open set $V \subset U$ so that $X - V$ is also open.

Example B.2.1. By definition, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}^2$ are 0 dimensional.

This seems counter-intuitive until we recall that all these spaces are actually countable unions of discrete points, each having 0 dimension. This poses the challenge to appreciate the fact that upon a change of perspective, if we look from far enough, $\mathbb{Z}$ seems like $\mathbb{R}$ and not like $\mathbb{R}^2$. This notion rather covers the dimensions on a smaller scale. We need a notion for large scale geometry which can look over the small nuances of these spaces and differentiate between the dimensions of $\mathbb{Z}$ and $\mathbb{Z}^2$.

B.2 Large-scale dimension

Definition B.3. Let $r > 0$. A metric space X is said to have dimension 0 at scale r if it can be expressed as a union $X = \cup X_i$ where:

1. $\sup\{\mathrm{diam}(X_i)\} < \infty$
2. $\inf\{d(x, x') | x \in X_i, x' \in X_j\} > r$ whenever $i \neq j$.

B.2.1 Changing the scale

Considering $\mathbb{Z}$ as a metric space with the Euclidean metric, we see that $\mathbb{Z}$ has dimension 0 at every scale $r < 1$. However, if $r \geq 1$, then $\mathbb{Z}$ does not have dimension 0 at scale r. This agrees with our intuition: on small scales, $\mathbb{Z}$ looks like a discrete collection of points and hence is $0-$dimensional, but on large scales, $\mathbb{Z}$ no longer looks $0-$dimensional.

B.2.2 Asymptotic dimension of metric spaces and groups

Definition B.4. We say that the **asymptotic dimension of a metric space** X **does not exceed** n, and write $\mathrm{asdim}(X) \leq n$ if for each $r > 0$, there exist subsets

$X_0, X_1, \ldots, X_n$ with $X = X_0 \cup X_1 \cup \cdots \cup X_n$, and for each i, X_i has dimension 0 at scale r.

Definition B.5. We say that **asymptotic dimension of a metric space** X **is** n written $\operatorname{asdim}(X) = n$ if and only if $\operatorname{asdim}(X) \leq n$ and there is no integer $q < n$ such that as $asdim(X) \leq q$, i.e., it exceeds every integer less than n.

Since this definition is constructed in such a manner that the asymptotic dimension is preserved under quasi-isometries, we can define:

Definition B.6. The **asymptotic dimension of a group** G is the asymptotic dimension of one of its Cayley graphs constructed with respect to a generating set of G.

Example B.6.1. The asymptotic dimension of $\mathbb{Z}^n$ which does not exceed n, but exceeds $n-1$ and hence $asdim(\mathbb{Z}) = n$.

Example B.6.2. Consider the free group of n letters F_n. The Cayley graph is a tree $T = (V, E)$ in which each vertex is incident to four edges. We will show that any infinite tree has asymptotic dimension 1. Since T contains an infinite geodesic $\operatorname{asdim}(T) \geq 1$. Thus, it remains to show that $\operatorname{asdim}(T) \leq 1$. To prove this, for each r we must find two families of uniformly bounded, r–disjoint sets whose union covers T . Let $r > 0$ be given. Fix some vertex of the tree and call it x_0 . We will use the notation $|x|$ to denote the distance $d(x, x_0)$ from x to the fixed vertex x_0 . As a first step in the construction of the cover, for each positive integer n, let

$$A_n = \{v \in V \mid 2r(n-1) \leq |v| \leq 2rn\}.$$

Although the collections $A_0 = \bigcup_{n\in\mathbb{N}} A_{2n+1}$ and $A_e = \bigcup_{n\in\mathbb{N}} A_{2n}$ each consist of r–disjoint subsets, they are not uniformly bounded. So, we subdivide them further. Clearly, the equivalence classes of A_o and A_e are uniformly bounded and r–disjoint. Hence $\operatorname{asdim}(F_n) = 1$.

We conclude this thesis with an illustration of the asymptotic dimension of F_2 as shown in Figure B.1.

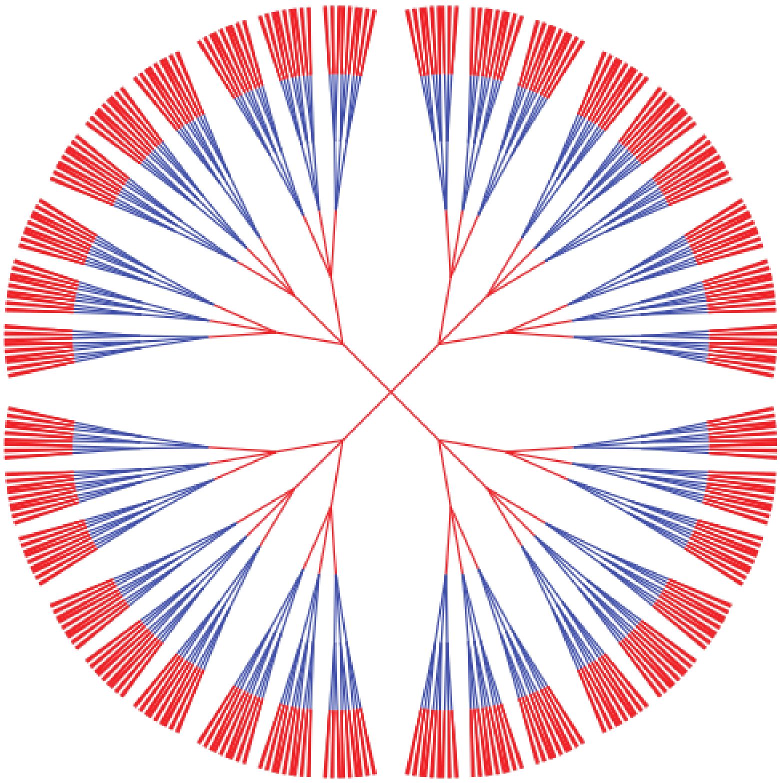


Figure B.1: $asdim(F_2) = 1$.

Bibliography

[1] Mikhail Gromov *“Hyperbolic Groups”* in “Essays in Group Theory” (Steve M. Gersten, ed.), MSRI Publ. 8, 1987

[2] Mikhail Gromov *“Asymptotic invariants of infinite groups”* Vol. 2 (Sussex, 1991), London Mathematical Society Lecture Note Series, 182, Cambridge University Press

[3] Graham A. Niblo and Martin A. Roller *”Geometric group theory. Vol. 1.”* Mathematical Society Lecture Note Series, 181. Cambridge University Press, Cambridge, 1993

[4] Matt Clay and Dan Margalit *“Office Hours with a Geometric Group Theorist”* Princeton University Press, 2017

[5] Donald J. Collins *“A Simple Presentation of a Group with Unsolvable Word Problem”* Illinois Journal of Mathematics, 1986

[6] Xiaofeng Wang, Chen Xu, Guo Li, and Hanling Lin *“Groups With Two Generators Having Unsolvable Word Problem And Presentations of Mihailova Subgroups”* School of Mathematics, Shenzhen University, 2016

[7] John Meier *“Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups”* Cambridge University Press, 2008

[8] Henri Poincaré *“Papers on Topology Analysis Situs and Its Five Supplements (Translations by John Stillwell)”* Edinburgh University Press, 2009