QUADRATIC UNITARY CAYLEY GRAPHS

A REPORT

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CERTIFICATE

This is to certify that **Hricha Acharya**, BS-MS (Dual Degree) student in Department of Mathematics, has completed bonafide work on the thesis entitled **'Quadratic unitary Cayley graphs'** under my supervision and guidance.

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Hricha Acharya

ABSTRACT

The primary objective of this project is to study the results in [13] pertaining to a family of Cayley graphs on finite commutative rings, called quadratic unitary Cayley graphs (QUCG). Let R be such a ring and $R^\times$ be its set of units. Let $Q_R = \{u^2 : u \in R^\times\}$ and $T_R = Q_R \cup (-Q_R)$. We define the QUCG of R, denoted by $\mathcal{G}_R$, to be the Cayley graph on the additive group of R with respect to T_R. It is well known that any finite commutative ring R can be decomposed as $R = R_1 \times R_2 \times \cdots \times R_s$, where each R_i is a local ring with maximal ideal M_i [3]. Let R_0 be a local ring with maximal ideal M_0 such that $|R_0| / |M_0| \equiv 3 \pmod 4$. We study the spectra of $\mathcal{G}_R$ and $\mathcal{G}_{R_0 \times R}$ under the condition that $|R_i| / |M_i| \equiv 1 \pmod 4)$ for $1 \leq i \leq s$. We also understand the conditions under which these graphs are Ramanujan.

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1. INTRODUCTION

Algebraic graph theory [7] is a field of mathematics that focuses on studying various properties of graphs using algebraic techniques. One such approach known as spectral graph theory [5] employs widely known concepts of linear algebra on graphs, which includes the study of the Laplacian and the spectrum of a graph (i.e. the eigenvalues of its adjacency matrix). Thus, spectral graph theory lends a unique perspective to the study of the fundamental properties of graphs such as graph-connectivity, which has broad applications in the fields of communication systems and computer science.

It is well known the spectra of graphs are linked with many of their extremal properties. One such connection arises during the study of graph-connectivity. A graph is well-connected if it has large *spectral gap* (i.e. the difference between largest two eigenvalues of the adjacency matrix is large). The Alon-Boppana theorem [10, Chapter 3] asserts that for a large, k-regular graph G, the strongest upper bound for the second highest eigenvalue of its adjacency matrix is $2\sqrt{k-1}$. Consequently, k-regular graphs which have this property (also known as Ramanujan graphs) are of great importance. In the past few decades, the expander families of Ramanujan graphs, particularly those arising from families of *Cayley graphs* have been extensively studied (see [11, 14, 15, 16] and the references therein).

In Chapter 1, we begin by building a basic understanding of algebraic graph theory. We start by understanding the basic notion of a graph isomorphism. We then understand the interplay between linear algebra and graph theory by studying the properties of graph invariant matrices such as the adjacency matrix and the Laplacian. Further more we examine the correlation between properties of a k-regular graph and the eigenvalues of its adjacency matrix. We then introduce two types of graph products, namely, dot product and tensor product, and also study the relation between spectra of graphs and their products. Since QUCGs are generalizations of Paley graphs, which are a strongly regular family of Cayley graphs, we conclude this chapter with a study of the basic theory of these special families of graphs.

In Chapter 2, we see a widely used graph invariant that quantifies the connectivity of a graph G known as the isoperimetric constant [10] or the Cheeger constant, denoted by $h(G)$. Interestingly, for a k-regular graph the isoperimetric constant is closely related to its spectral gap. In this context, we will also study the Rayleigh-Ritz theorem. Moreover, for a k-regular graph G with second-largest eigenvalue $\lambda_1(G)$, the following inequality holds:

$$\frac{k-\lambda_1(G)}{2} \leq h(G) \leq \sqrt{2k(k-\lambda_1(G))}.$$

Thus, k-regular graphs with large spectral gaps $k-\lambda_1(G)$ have higher Cheeger constant and hence higher connectivity. Additionally, we study a combinatorial proof of the Alon-Boppana theorem which further motivates the study of Ramanujan graphs [14, 16].

In Chapter 3, we begin by introducing QUCG as in [13] and look at some properties of QUCG for finite fields. We then determine the spectra of QUCGs associated with local rings which we then use to determine the spectra of QUCGs for certain finite commutative rings. Finally, we explicitly determine the conditions under which the QUCG of a finite commutative ring is also a Ramanujan graph.

2. INTRODUCTION TO ALGEBRAIC GRAPH THEORY

In this chapter, we will start by introducing basic notions from algebraic graph theory that are relevant to this thesis. In Sections 1.2 and 1.3 we introduce matrices associated with graphs called adjacency matrix and the Laplacian. We also study the set of eigenvalues for their matrices for k-regular graphs. In Section 1.4, we look at two different kinds of products on graphs and also see their relations to their eigenvalues. In the last two sections, we introduce two special families of graphs namely strongly regular graphs and Cayley graphs and study their spectral properties. The results in this chapter are based on [2, Chapters 3,4], [4, Chapter 9], [7, Chapter 10], and [10, Chapter 1,3].

2.1 Preliminaries

Definition 2.1. An *undirected graph* X is a pair $(V(X), E(X))$, where $V(X)$ is a set called the *vertex set* of X and $E(X)$ is a multiset of pairs unordered pairs $\{x, y\}$ for $x, y \in V(X)$, called an *edge set* of X.

We say that $x, y \in V(X)$ are *adjacent* if $\{x, y\} \in E(X)$.

Definition 2.2. A graph X is said to be *finite* if $|V(X)| < \infty$.

The *order* of a finite graph, denoted by $|X|$, is defined by $|X| := |V(X)|$.

Definition 2.3. A graph X is said to be *simple* if each unordered pair in $E(X)$ has multiplicity one.

In this thesis, we will be only concerned with finite simple graphs.

Definition 2.4. Let X, Y be a graphs, then we say graph Y is *subgraph* of X if $V(X) \subseteq V(Y)$ and $E(X) \subseteq E(Y)$.
A subgraph Y of X is called *spanning subgraph* if $V(X) = V(Y)$.

Definition 2.5. For $v \in V(X)$ we define order of the vertex, denoted as $|v|$, to be the number of vertices adjacent to v. We say a graph X is k-regular if for all $v \in V(X)$ we have $|v| = k$.

Example 2.6. Fig 2.1(a) shows a 2-regular graph of order four. Every 2-regular graph of order $n \geq 3$ is called a *cycle graph* denoted by C_n.

(a) The 2-regular graph C_4. **(b)** The complete graph K_4.

Fig. 2.1: Example of a k-regular and a complete graph.

Definition 2.7. A graph X is called a *complete graph* if for every pair of distinct vertices $x, y \in V(X)$ there exists an edge $e = \{x, y\} \in E(X)$.
A complete graph with n vertices is denoted as K_n.

Note that a complete graph K_n is always $(n-1)$-regular.

Example 2.8. Fig 2.1(b) shows a complete graph K_4 which is also a 3-regular graph of order four.

Definition 2.9. A graph X is called *bipartite graph* if there exist disjoint subsets V_1, V_2 of $V(X)$ such that $V(X) = V_1 \cup V_2$ and for every $x, y \in E(X)$ we have $x \in V_1$ and $y \in V_2$.
A complete bipartite graph (i.e. bipartite graph where every vertex in one subset is adjacent to every vertex in other set) with $|V_1| = n$ and $|V_2| = m$ is denoted as $K_{n,m}$.

Example 2.10. The graph $K_{3,4}$ is shown in Figure 2.2 below.

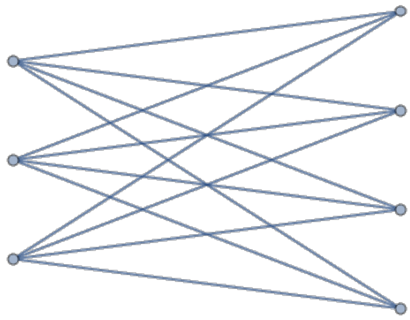


Fig. 2.2: A bipartite graph, $K_{3,4}$.

Definition 2.11. Let X be a graph.

1. A *walk* in X is a sequence $w = (v_0, e_1, v_1, \ldots, e_k, v_k)$ of vertices $v_i \in V(X)$ and edges $e_i \in E(X)$ such that for $1 \leq i \leq k$, the edge $e_i = \{v_{i-1}, v_i\}$. The number of edges in the sequence is called the length of the walk.

2. The walk $w = (v_0, e_1, v_1, \ldots, e_k, v_k)$ is called a *closed walk* if $v_0 = v_k$.

3. A *trail* is a walk with no repeated edges. A closed trail is called a *circuit*.

4. A *path* is a walk with no repeated vertex. A closed path is called a *cycle*.

5. A walk $w = (v_0, e_1, v_1, \ldots, e_k, v_k)$ is called *non backtracking* if $e_i \neq e_{i+1}$ for all $1 \leq i \leq k-1$ otherwise, the walk is called *backtracking*.

6. A closed walk $u = (v_0, e_1, v_1, \ldots, e_k, v_0)$ is called *unfactorable* if $v_i \neq v_0$ for all $1 \leq i \leq k-1$ otherwise, it is called a *factorable* walk.

Note that every path is a trail but the opposite need not be true.

Definition 2.12. A graph is called *connected* if for any two vertices of the graph there is a walk in the graph joining them. A non-connected graph is said to be *disconnected*.

Definition 2.13. A graph X is called a *tree* if it is connected and has no circuits.

Example 2.14. A tree of order 7 is shown in Figure 2.3 below.

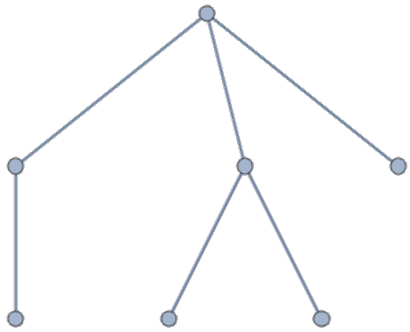


Fig. 2.3: A tree graph of order 7.

Definition 2.15. The *distance* between any two vertices u, v, denoted by $dist(u, v)$, is the length of a shortest path joining u and v in the graph, where $dist(u, v) = 0$ if and only if $u = v$.
If no such path exist then we say that $dist(u, v)$ is infinite.

Definition 2.16. The diameter of a graph X, denoted by $diam(X)$, is defined as

$$diam(X) = \max\{dist(u, v) : u, v \in V(X)\}.$$

Definition 2.17. Let X, H be graphs. A map $\phi : V(X) \to V(H)$ is said to be a *graph homomorphism* if $\{\phi(v), \phi(u)\} \in E(H)$, whenever $\{v, u\} \in E(X)$.

Example 2.18. We exhibit a graph homomorphism between $K_{2,3}$ and C_2. Let $V_1, V_2 \subset V(K_{2,3}) = \{y_1, y_2, y_3, y_4, y_5\}$ such that $|V_1| = 2$ and $|V_2| = 3$. If $V(C_2) = \{x_1, x_2\}$, then $\phi : V(K_{2,3}) \to V(C_2)$ such that

$$\phi(y_i) = \begin{cases} x_1, & \text{if } y_i \in V_1, \text{ and} \\ x_2, & \text{if } y_i \in V_2. \end{cases}$$

is a graph homomorphism as $\{\phi(v), \phi(u)\} \in E(C_2)$, whenever $\{v, u\} \in E(K_{2,3})$.

Definition 2.19. Two graphs X and H are *isomorphic* if there exists a bijective map $\phi : V(X) \to V(H)$ such that $\{\phi(v), \phi(u)\} \in E(H)$ if and only if $\{v, u\} \in E(X)$.

Example 2.20. Consider the following graphs X and Y as seen in Figure 2.4 below. Then $\phi : V(X) \to V(Y)$ such that $\phi(x_1) = y_1, \phi(x_2) = y_4, \phi(x_3) = y_2, \phi(x_4) = y_5$, and $\phi(x_5) = y_3$ is a graph isomorphism as $\{\phi(x), \phi(y)\} \in E(X)$ if and only if $\{x, y\} \in E(Y)$.

(a) X **(b)** Y

Fig. 2.4: Two isomorphic graphs.

2.2 The Adjacency Operator

In the section, we will study the spectra of k-regular graphs.

Definition 2.21. Let X be a graph with $V(X) = \{v_1, \ldots, v_n\}$. The *adjacency matrix* of X, denoted by $A(X)$, is defined by $A(X) = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E, \text{ and} \\ 0, & \text{otherwise.} \end{cases}.$$

Remark 2.22. Note that $A(X)$ of an undirected graph X is a real and symmetric matrix. Hence, by the Spectral theorem [10, Theorem A.53] for symmetric matrices, all of its eigenvalues are real.

Example 2.23. For the cycle graph C_4 shown in Figure 2.1a(a), we have

$$A(C_4) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Definition 2.24. The *spectrum* of a graph X is said to be the multiset of eigenvalues of $A(X)$.
We write the spectrum of X as

$$Spec(X) = \begin{pmatrix} \lambda_0(X) & \lambda_1(X) & \cdots & \lambda_k(X) \\ n_0 & n_1 & \cdots & n_k \end{pmatrix},$$

where for each $0 \leq i \leq k, \lambda_i(X)$ are distinct eigenvalues of A, and $\sum_{i=0}^{k} n_i = n$ i.e. the order of graph X.

From here on, we will fix an ordering on $V(X)$ so that $\lambda_0(X) \geq \lambda_1(X) \geq \ldots \geq \lambda_{n-1}(X)$. (Note that the spectrum of the graph is independent to ordering of its vertices.)

Example 2.25. A complete pseudo graph of order n, denoted by $\mathring{K}_n$, is a graph obtained by attaching a loop at each vertex of K_n. Then $\mathring{K}_n$ has adjacency matrix $A(\mathring{K}_n) = J_n$ where J_n is a matrix with each entry as 1. Since, J_n is real and symmetric, by the Spectral theorem, we know that it is diagonalisable and has n real eigenvalues. But, note that $rank(J_n) = 1$, so it has only one non-zero eigenvalue. Consider the vector $v = (1, 1, \ldots, 1) \in R$. Clearly, $J_n v = \lambda v$, which implies the λ is an eigenvalue of J_n. So,

$$Spec(\mathring{K}_n) = \begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}.$$

For a finite set S we consider the complex vector space $L^2(S) := \{f : S \to \mathbb{C}\}$. Then $L^2(S)$ is an inner product space and for $f, g \in L^2(S)$ and $\alpha \in \mathbb{C}$:

1. The vector sum in $L^2(S)$ is given by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$.

2. Scalar multiplication is given by $f(\alpha x) = \alpha f(x)$ for all $x \in S$.

3. The standard inner product are given is by $\langle f, g \rangle_2 = \sum_{x \in S} f(x)\overline{g(x)}$.

Let X be a graph with $V(X) = \{v_1, \ldots, v_n\}$. Given $f \in L^2(V)$, we may think of f as a vector in $\mathbb{C}^n$ and we $A(X)$ as a linear transformation from $L^2(V) \to L^2(V)$ given by the formula:

$$(Af)(v) = \sum_{w \in V} A_{v,w} f(w).$$

The linear operator A defined above is called the *adjacency operator* of X.

Proposition 2.26. *If X is a k-regular of order n. Then*

1. $\lambda_0(X) = k$,

2. $|\lambda_i(X)| \leq k$ *for* $i = 0, 1, \ldots, n-1$,

3. $\lambda_1(X) < \lambda_0(X,)$ *if and only if X is a connected graph, and*

4. *If X is bipartite, then $Spec(X)$ is symmetric about 0 i.e. if $\lambda \in Spec(X)$ with multiplicity m then $-\lambda \in Spec(X)$ with multiplicity m.*

Proof. Throughout this proof, let $V = V(X)$ and A denote the adjacency operator of X.

1. We show that there exists a eigenfuction associated with k. Let $f_0 \in L^2(V)$ be defined as $f_0(x) = 1$ for all $x \in V$. Then

$$(Af_0)(x) = \sum_{y \in V} A_{x,y} f_0(y) = \sum_{y \in V} A_{x,y} = k = k \cdot f_0(x)$$

Thus, k is an eigenvalue of A.

2. Let λ be an eigenvalue of A and f be a real-valued eigenfunction of A associated with λ (By Spectral theorem [10, Theorem A.53] we know such an f exists). Pick an $x \in V$ such that $|f(x)| = \max_{y \in V} |f(y)|$. Note that

$f(x) \neq 0$ since f is an eigenfunction of A. By the definition of x, we see that

$$\begin{aligned}|\lambda||f(x)| = |(Af)(x)| = \left|\sum_{y\in V} A_{x,y} f(y)\right| &\leq \sum_{y\in V} |A_{x,y}|\,|f(y)| \\ &\leq |f(x)| \sum_{y\in V} |A_{x,y}| = k|f(x)|.\end{aligned}$$

Thus, $|\lambda| \leq k$.

3. Suppose that X is connected. We want to show that $|\lambda_1| < k$. Let $v = (v_1, v_2, \cdots, v_n)^t \in \mathbb{R}^n$ be an eigenvector corresponding to k. Again, let $1 \leq i \leq n$ be such that

$$|v_i| = \max\left\{|v_j| \mid 1 \leq j \leq n\right\}.$$

Replacing v by $-v$ if necessary, assume that v_i is positive. Then we have

$$kv_i = \sum_{j=1}^{n} a_{i,j} v_j \Longrightarrow v_i = \sum_{j=1}^{n} \frac{a_{i,j}}{k} v_j.$$

This implies v_i is a convex linear combination of $v_j, 1 \leq j \leq n$. Since for each j, $|v_j| \leq v_i$, we get $v_j = v_i$ for all j such that $a_{i,j} = 1$. Since X is connected, any two distinct vertices are connected by a walk, which eventually gives $v_j = v_i$ for all j. Hence, v is a scalar multiple of $(1, 1, \cdots, 1)^t$, which implies eigenvalue k has multiplicity one.

To prove the converse, suppose that X is disconnected. Let $v \in V(X)$, and let V_1 be the set of all vertices $w \in V(X)$ such that there exists a walk in X connecting v and w. We note that, if $w \in V$ is adjacent to a vertex in V_1, then $w \in V_1$. Since, $V \backslash V_1 \neq \phi$, X splits into two k-regular graphs with vertex sets V_1 and $V_2 = V \backslash V_1$, respectively. Hence, k is an eigenvalue of both these graphs, which implies that $\lambda_1(X) = k$.

4. Suppose that X is a bipartite graph and $V = V_1 \cup V_2$ is a bipartition of V. Let λ be an eigenvalue of A with multiplicity m. By the Spectral theorem, there exist linearly independent real-valued eigenfunctions $f_1, \ldots, f_m$ of A

associated with λ. Consider the functions

$$g_i(x) = \begin{cases} f_i(x) & x, \in V_1 \text{ and} \\ -f_i(x) & x, \in V_2, \end{cases}$$

for $i = 1, \ldots, m$.

We will now show that each g_i is an eigenfunction of A associated with $-\lambda$. Suppose that $x \in V_1$. Then, since every y adjacent to x is in V_2, we see that

$$\begin{aligned}(Ag_i)(x) &= \sum_{y \in V_2} A_{x,y} g_i(y) = -\sum_{y \in V} A_{x,y} f_i(y) = -(Af_i)(x) \\ &= -\lambda f_i(x) = -\lambda g_i(x).\end{aligned}$$

Similarly, if $x \in V_2$, then $(Ag_i)(x) = -\lambda g_i(x)$. One can check that $g_1, \ldots, g_m$ form a linearly independent set. Hence, $-\lambda$ is an eigenvalue of A with multiplicity $l \geq m$. The same argument, on reversing the roles of λ and $-\lambda$, shows that $m \geq l$. □

2.3 The Laplacian

In this section, we will discuss another linear operator associated to a graph, called the Laplacian. The Laplacian on a graph is the discrete analogue of the Laplacian $\Delta = \text{div}(\text{grad}(f))$ from multivariable calculus.

Let X be a graph. Give edges in $E(X)$ an arbitrary orientation. In particular, for each edge $e \in E(X)$, labelling one endpoint e^+ and the other e^-, we call e^+ the *origin* of e, and e^- the *extremity* of e (as illustrated in Figure 2.5 below).

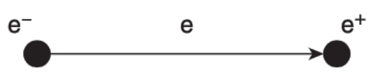


Fig. 2.5: An oriented edge $e \in E(X)$.

Definition 2.27. Let X be a graph with $V = V(X)$ and $E = E(X)$. We define:

1. $d : L^2(V) \to L^2(E)$ for each $f \in L^2(V)$ as

$$(df)(e) = f\left(e^+\right) - f\left(e^-\right), \text{ and} \tag{2.1}$$

2. $d^* : L^2(E) \to L^2(V)$ for each $f \in L^2(E)$ as

$$(d^*f)(v) = \sum_{\substack{e\in E\\ v=e^+}} f(e) - \sum_{\substack{e\in E\\ v=e^-}} f(e). \tag{2.2}$$

If we think of the function f as a flow on the edges of the graph G, $(df)(e)$ measures the change of f along the edge e of the graph and $(d^*f)(v)$ measures the total inward flow at the vertex v. Hence, we have the following definition of the Laplacian for a graph.

Definition 2.28. Let $X = (V, E)$ be a graph with a orientation on its edges. We define *laplacian operator* $\Delta : L^2(V) \to L^2(V)$ to be $\Delta = d^*d$.

We will see in the following lemma that the Laplacian operator is independent of the orientation of the edges.

Lemma 2.29. *If $X = (V, E)$ is a k-regular graph with adjacency operator A, then $\Delta = kI - A$.*

Proof. Let $f \in L^2(V)$ and $x \in V$. Then:

$$\begin{aligned}
(\Delta f)(x) &= (d^*(df))(x) \\
&= \sum_{\substack{e\in E\\ x=e^+}} (df)(e) - \sum_{\substack{e\in E\\ x=e^-}} (df)(e) \\
&= \left(\sum_{\substack{e\in E\\ x=e^+}} f(x) - \sum_{\substack{e\in E\\ x=e^+ \text{ and } y=e^-}} f(y) \right) - \left(\sum_{\substack{e\in E\\ x=e^- \text{and } y=e^+}} f(y) - \sum_{\substack{e\in E\\ x=e^-}} f(x) \right) \\
&= kf(x) - \sum_{y\in V} A_{x,y} f(y) \\
&= ((kI - A)f)(x).
\end{aligned}$$

□

Theorem 2.30. *Suppose $X = (V, E)$ is a k-regular graph of order n. For a fixed ordering of vertices and orientation for edges the following holds:*

1. *The eigenvalue of Δ are given by*

$$0 = k - \lambda_0(X) \leq k - \lambda_1(X) \leq \ldots \leq k - \lambda_n - 1(X).$$

In particular, the eigenvalues of Δ lie in the interval $[0, 2k]$.

2. *Let $f \in L^2(V)$ and $g \in L^2(E)$. Then $\langle df, g\rangle_2 = \langle f, d^*g\rangle_2$ and*

$$\langle \Delta f, f\rangle_2 = \sum_{e\in E} \left|f(e^+) - f(e^-)\right|^2 \tag{2.3}$$

Proof. 1. Let f be an eigenfuction of A corresponding to the eigenvalue λ. Then we have

$$\Delta f = (kI - A)f = kf - Af = (k - \lambda)f.$$

Thus, f is an eigenfuction of Δ corresponding to eigenvlaue $(k - \lambda)$. Then (1) follows from Proposition 2.30 and Proposition 2.26.

2. Note that

$$\begin{aligned}
\langle df, g\rangle_2 &= \sum_{e\in E} (df)(e)\overline{g(e)} = \sum_{e\in E} \left[f\left(e^+\right) - f\left(e^-\right)\right] \overline{g(e)} \\
&= \sum_{e\in E} f\left(e^+\right) \overline{g(e)} - \sum_{e\in E} f\left(e^-\right) \overline{g(e)} \\
&= \sum_{v\in V} f(v) \sum_{\substack{e\in E \\ v=e^+}} \overline{g(e)} - \sum_{v\in V} f(v) \sum_{\substack{e\in E \\ v=e^-}} \overline{g(e)} \qquad . \\
&= \sum_{v\in V} f(v)\overline{\left(d^*g\right)(v)} \\
&= \langle f, d^*g\rangle_2
\end{aligned}$$

Thus,

$$\langle \Delta f, f\rangle_2 = \langle d^* df, f\rangle_2 = \overline{\langle f, d^* df\rangle_2} = \overline{\langle df, df\rangle_2} = \langle df, df\rangle_2 = \|df\|_2^2,$$

so it follows that

$$\langle df, df\rangle_2 = \sum_{e\in E} \left(f\left(e^+\right) - f\left(e^-\right)\right)\overline{\left(f\left(e^+\right) - f\left(e^-\right)\right)} = \sum_{e\in E} \left|f\left(e^+\right) - f\left(e^-\right)\right|^2.$$

□

2.4 Product and Powers of Graphs

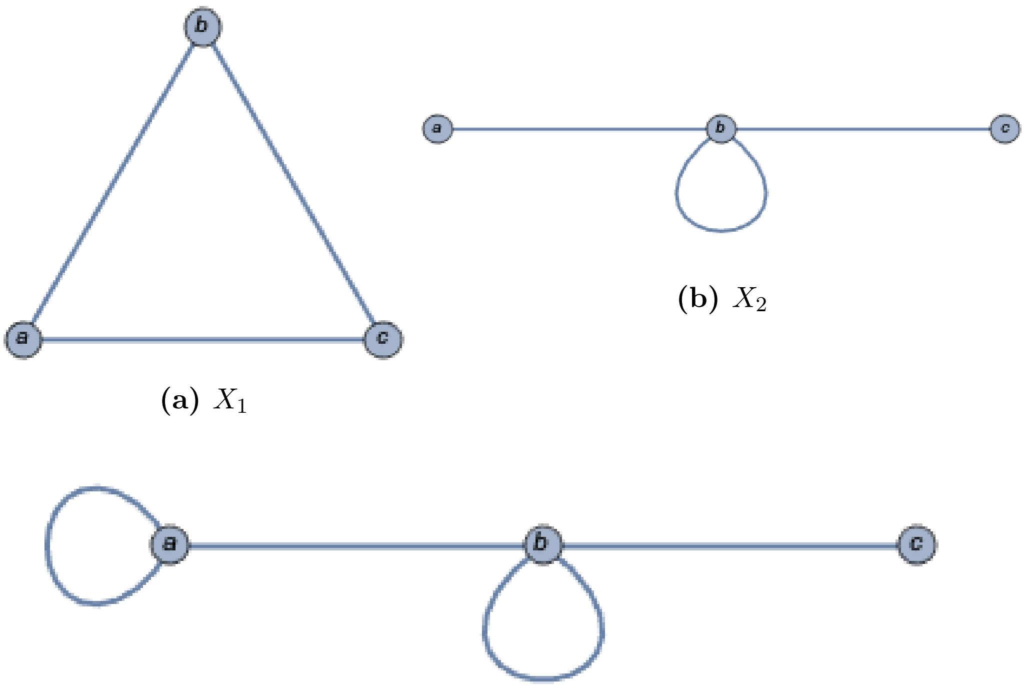
Definition 2.31. Let $X_1 = (V, E_1)$ and $X_2 = (V, E_2)$ be two finite graphs. We define the *dot product* of graphs $X_1 \cdot X_2$ to be a graph (V, E), where the multiplicity of the edge in E from v_1 to v_2 equals the number of pairs (e_1, e_2) such that $e_1 \in E(X_1)$ is an edge with end point v_1 and e_2 is an edge from the terminal point of e_1 to v_2.

The n^{th} *power of a graph* $X(V, E)$ is defined to be $X^n = X \cdot X \ldots \cdot X$. In this case, edge of X^n from v_1 to v_2 is a walk of length n from v_1 to v_2 in X and mutliplicity of edge is the number of walks of length n.

Example 2.32. Let $X_1 = (V, E_1)$ and $X_2 = (V, E_2)$ be two finite graphs as seen in the Figure 2.6, where $V = \{a, b, c\}$, $E_1 = \{\{a, b\}, \{a, c\}, \{b, c\}\}$, and $E_2 = \{\{a, b\}, \{b, c\}, \{b, b\}\}$. Then the dot product $X_1 \cdot X_2$ is as shown in the third subfigure.

Theorem 2.33. *Let $X_1(V, E_1)$ and $X_2(V, E_2)$ be two finite graphs.Choose an ordering of V. Let A_1 and A_2 be adjacency matrices of graphs X_1 and X_2 with respect to this ordering. Then the adjacency matrix of $X_1 \cdot X_2$ with respect to this ordering is $A_1 \cdot A_2$.*

Proof. Choose an ordering of V, let it be $\{v_1, v_2, \ldots, v_n\}$. Let A be adjacency matrix of $X_1 \cdot X_2$, then by definition of adjacency matrix we get $a_{ij_{n\times n}} = k_{ij}$,



(a) X_1

(b) X_2

(c) $X_1 \cdot X_2$

Fig. 2.6: A dot product of graphs.

where k_{ij} is multiplicity of edge between v_i and v_j. Then,

$$\begin{aligned}
k_{ij} &= |\{(e_1, e_2) \mid e_1 \in E_1 \text{ and } e_2 \in E_2 \text{ such that } e_1 = v_i, v_k \text{ and } e_2 = v_k, v_j\}| \\
&= \sum_{\substack{e_1 \in E_1 \\ e_1 = v_i, v_k}} \sum_{\substack{e_2 \in E_2 \\ e_2 = v_k, v_j}} 1 \\
&= \sum_{k=1}^{n} a_{ik} b_{kj} \\
&= A_1 A_2.
\end{aligned}$$

□

Corollary 2.34. *For a X be a finite graph. Then $A(X^k) = A^k$. Consequently, the eigenvalues of X^k are given by $\lambda(X)^j$, for $1 \le j \le n-1$.*

Theorem 2.35. *Let X be a finite graph with eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$, where*

$\lambda_i \leq \lambda_{i-1}$ for $1 \leq i \leq n-1$. Then the eigenvalues for X^j are $\lambda_0^j, \ldots, \lambda_{n-1}^j$.

Proof. Let A be an adjacency matrix of X. Let O such that OAO^{-1} is the diagonal matrix with diagonal entries $\lambda_0, \ldots, \lambda_{n-1}$, as in [10, Theorem A.56]. Then OA^jO^{-1} is the diagonal matrix with diagonal entries $\lambda_0^j, \ldots, \lambda_{n-1}^j$. The result follows from [10, Theorem A.61]. □

Definition 2.36. The *tensor product* of two graphs X_1 and X_2, denoted by $X_1 \otimes X_2$, is the graph with $V(X_1 \otimes X_2) = V(X_1) \times V(X_2)$ and (u, v) is adjacent to (x, y) in $E(X_1 \otimes X_2)$ if and only if u is adjacent to x in X_1 and v is adjacent to y in X_2.

Since the tensor product on graphs is associative [8], a finite tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_s$ of graphs $X_1, X_2, \ldots, X_s$ (where $s \geq 1$) is well-defined.

Theorem 2.37. *Let X_1 and X_2 be graphs with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_m$, respectively. Then the eigenvalues of $X_1 \otimes X_2$ are $\lambda_i\mu_j, 1 \leq i \leq n, 1 \leq j \leq m$.*

Proof. Let A be adjacency matrix of X_1 and B be adjacency matrix of X_2. It is easy to see that adjacency matrix of $X_1 \otimes X_1$ is tensor product of matrices, $A \otimes B$. Let λ be an eigenvalue of A corresponding to eigenvector $x = (x_1, x_1, \ldots, x_m)$. Similarly, μ be an eigenvalue of B corresponding to eigenvector $y = (y_1, y_1, \ldots, y_n)$. Then we have,

$$(A \otimes B)(x \otimes y) = Ax \otimes By = \lambda x \otimes \mu y = \lambda\mu(x \otimes y).$$

□

2.5 Strongly regular graph

In the case of a k-regular graph, the regularity only tells us about the order of each vertex in the graph. In this section, we introduce *strongly regular graphs*, which also give information about a number of adjacent and non-adjacent vertices.

We will also show that the spectrum of connected strongly regular graphs comprises exactly three distinct eigenvalues and study how the eigenvalues, as well as their multiplicities, are only dependent on certain parameters associated with such graphs. Finally, we will end with an example of a Paley graph. This section is based on [4, Chapter 9] and [7, Chapter 10]. We will start by introducing the parameters related to such graphs and study their properties.

Definition 2.38. Let X be a regular graph that is neither complete nor empty. We say that X is *strongly regular graph* with parameters (n, k, λ, τ) if it is a k-regular graph of order n, in which every pair of adjacent vertices has λ common neighbors and every pair of non adjacent vertices have μ common neighbors.

Example 2.39. The simplest example of strongly regular graph is the graph C_4 as seen in Fig 2.1(a). Clearly, the graph is 2-regular or order 4. Every pair of adjacent vertices has 1 neighbor in common and every pair of non-adjacent vertices has 0 neighbor in common. Thus, C_4 is a $(4, 2, 1, 0)$ strongly regular graph.

Remark 2.40. If X is a (n, k, λ, μ) strongly regular graph and $\bar{X}$ be its graph complement with $V(\bar{X}) = V(X)$ and $E(\bar{X}) = E(K_n)\backslash E(X)$. Then $\bar{X}$ is also a strongly regular graph with parameters $(n, \bar{k}, \bar{\lambda}, \bar{\mu})$ given by:

$$\begin{aligned}
\bar{k} &= n - k, \\
\bar{\lambda} &= n - 2k + \mu - 2, \text{ and} \\
\bar{\mu} &= n - 2k + \lambda.
\end{aligned}$$

A strongly regular graph X is called *primitive* if X and its compliment are complete. A non-primitive graph is called *imprimitive*. The lemma provides a characterization of imprimitive strongly regular graphs.

Lemma 2.41. *Let X be an (n, k, λ, μ) strongly regular graph. Then the following statments are equivalent:*

1. *X is not connected,*

2. $\mu = 0$,

3. $\lambda = k - 1$, *and*

4. *X is isomorphic to graph with m number of components as* K_{k+1} *for some* $m > 1$.

Thus, in the case of imprimitive strongly regular graph X, finding the spectrum for X is equivalent to finding spectrum of K_{k+1}. We will now show that for any primitive strongly regular graph there are exactly three eigenvalues in its spectrum.

Theorem 2.42. *Let X be a* (n, k, λ, μ) *primitive strongly regular graph. Then*

$$Spec(X) = \begin{pmatrix} k & \theta & \tau \\ 1 & m_\theta & m_\tau \end{pmatrix},$$

where

$$\theta, \tau = \frac{(\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2},$$

and

$$m_\theta, m_\tau = \frac{1}{2}\left(\frac{1}{n} \mp \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right).$$

Proof. Let $A = A(X)$ the $(uv)^{th}$-entry of the matrix A^2 is the number of walks of length two from the vertex u to the vertex v. In a strongly regular graph, this number is determined only by whether u and v are equal, adjacent, or distinct and nonadjacent. Therefore, we get the equation

$$A^2 = kI + \lambda A + \mu(J - I - A). \tag{2.4}$$

Here the $(uv)^{th}$-entry of kI counts the number of walks length 2 starting and ending at u. The $(uv)^{th}$-entry of λA counts the number of walks of length 2 starting from u to its adjacent vertex. Finally the $(uv)^{th}$-entry of $\mu(J - I - A)$ counts the number of walks of length 2 starting from u to its non-adjacent verter.

We rewrite (2.4) as

$$A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J$$

and can use it to determine the eigenvalues of A. Since X is regular with valency k, it follows that k is an eigenvalue of A with eigenvector 1. We know by [7, Lemma 8.4.1] that any other eigenvector of A is orthogonal to 1. Let z be an eigenvector for A with eigenvalue $\theta \neq k$. Then

$$A^2z - (\lambda - \mu)Az - (k - \mu)Iz = \mu Jz = 0,$$

so

$$\theta^2 - (\lambda - \mu)\theta - (k - \mu) = 0.$$

Therefore, the eigenvalues of A different from k must be zeros of the quadratic $x^2 - (\lambda - \mu)x - (k - \mu)$. If we set $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$ and denote the two zeros of this polynomial by θ and μ, we get

$$\theta = \frac{(\lambda - \mu) + \sqrt{\Delta}}{2} \text{ and}$$
$$\tau = \frac{(\lambda - \mu) - \sqrt{\Delta}}{2}.$$

Now, $\theta\tau = (\mu - k)$, and so, provided that $\mu < k$, we get that θ and τ are nonzero with opposite signs. Assuming that $\theta > 0$, we see that the eigenvalues of a strongly regular graph are determined by its parameters.

To see the multiplicities of the eigenvalues, let m_θ and m_τ be the multiplicities of θ and τ, respectively. Since k has multiplicity equal to one and the sum of all the eigenvalues is the trace of A (which is 0), we have

$$m_\theta + m_\tau = n - 1, \quad m_\theta\theta + m_\tau\tau = k.$$

Hence,

$$m_\theta = -\frac{(n-1)\tau + k}{\theta - \tau} \quad \text{and} \quad m_\tau = \frac{(n-1)\theta + k}{\theta - \tau},$$

and so,

$$(\theta-\tau)^2 = (\theta+\tau)^2 - 4\theta\tau = (\lambda-\mu)^2 + 4(k-\mu) = \Delta.$$

Substituting the values for θ and τ into the expressions for the multiplicities, we get

$$m_\theta = \frac{1}{2}\left((n-1) - \frac{2k+(n-1)(\lambda-\mu)}{\sqrt{\Delta}}\right),$$

and

$$m_\tau = \frac{1}{2}\left((n-1) + \frac{2k+(n-1)(\lambda-\mu)}{\sqrt{\Delta}}\right).$$

□

We will now look at a well-known example of a primitive strongly regular graph.

Definition 2.43. Let p be a prime number and n be a positive integer such that $p^n \equiv 1 \pmod 4$. The graph $P = (V, E)$ with

$$V(P) = \mathbb{F}_{p^n} \text{ and } E(P) = \left\{\{x, y\} : x, y \in \mathbb{F}_{p^n}, x - y \in \left(\mathbb{F}_{p^n}^{\times}\right)^2\right\}$$

is called the *Paley graph* of order p^n.

Note that the set $E(P)$ in the Definition 2.43 of Paley graph is well-defined because $x-y \in \left(\mathbb{F}_{p^n}^{\times}\right)^2$ if and only if $y-x \in \left(\mathbb{F}_{p^n}^{\times}\right)^2$. Since $x-y = -1(y-x)$, we need only to show that $-1 \in \left(\mathbb{F}_{p^n}^{\times}\right)^2$.

Example 2.44. Let P be a Payley graph as defined above of order $q = p^n$, then P is strongly regular graph with the parameters

$$\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right).$$

Then by Theorem 2.42 we get

$$Spec(P) = \begin{pmatrix} q & \frac{-1+\sqrt{q}}{2} & \frac{-1-\sqrt{q}}{2} \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}.$$

2.6 Cayley Graph

In this section, we will introduce an important class of regular graphs called Cayley graphs, which are graphs that capture the abstract structures of groups. Given a group and a subset of that group, we can construct a Cayley graph with respect to the subset. The properties of the group often determine the properties of its Cayley graph and vise versa.

Definition 2.45. Let G be a group and $\Gamma \subseteq G$. We say that Γ is a *symmetric* subset of G if $\gamma \in \Gamma$, then $\gamma^{-1} \in \Gamma$.

Example 2.46. Consider the group $\mathbb{Z}_6$. Consider subsets of $\mathbb{Z}_6$, $\Gamma_1 = \{1, 2, 5\}$ and $\Gamma_2 = \{1, 3, 5\}$. Clearly, Γ_1 is not symmetric since $-2 = 3 \notin \Gamma_1$, but Γ_2 is symmetric.

Definition 2.47. Let G be a finite group and $\Gamma \subseteq G$ be symmetric. The *Cayley graph* on G with respect to Γ is defined by $Cay(G, \Gamma) := X(V, E)$, where

1. $V = G$, and
2. for $x, y \in G$, $\{x, y\} \in E$ if and only if $y^{-1}x \in \Gamma$.

Example 2.48. Consider the group $\mathbb{Z}_6$ and $\Gamma = \{1, 3, 5\}$. Then the $Cay(\mathbb{Z}_6, \Gamma)$ has vertex set $V = \mathbb{Z}_6$ and edge set $E = \{\{0, 1\}, \{0, 3\}, \{0, 5\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 5\}\}$.

Proposition 2.49. *Let G be a finite group and $\Gamma \subseteq G$ be symmetric. Then:*

1. *$Cay(G, \Gamma)$ is $|\Gamma|$-regular, and*
2. *$Cay(G, \Gamma)$ is connected if and only if Γ generates G.*

Proof. 1. For a vertex g of the graph $Cay(G, \Gamma)$, let E_g denote the set of edges incident to g. By definition, it is clear that

$$E_g = \{\{g, g\gamma\} \mid \gamma \in \Gamma\} \implies |E_g| = |\Gamma|.$$

2. Suppose that $Cay(G, \Gamma)$ is connected. Then for any two vertices $a, b \in G$, there is a path in $Cay(G, \Gamma)$ joining them, that is, there exists $\gamma_1, \gamma_2, \cdots, \gamma_r \in \Gamma$ such that

$$a = b\gamma_1\gamma_2 \cdots \gamma_r \Longrightarrow b^{-1}a = \gamma_1\gamma_2 \cdots \gamma_r.$$

Since the vertices chosen were arbitrary, it proves that Γ generates G. Conversely, suppose that Γ generates G. Then for any $g \in G$, there exists $\gamma_1, \gamma_2, \cdots, \gamma_r \in \Gamma$ such that $g = \gamma_1\gamma_2 \cdots \gamma_r$. This implies that all the vertices are connected to the identity element of G, which proves the assertion.

□

3. THE ALON-BOPPANA THEOREM

Study of graph spectra [4, 5] has wide applications in the areas of computer science [9], chemistry [12] as well as communication systems [10]. In this chapter, we define a discreet analog of the Cheeger constant for a graph X, also known as the isoperimetric constant ($h(X)$), measures the connectivity of X. In Section 3.2 we study the proof of the Rayleigh-Ritz Theorem, which provides a method to calculate the second-largest eigenvalue of a graph. Additionally, we see the relation between $h(X)$ and $Spec(X)$. In section 3.3, we look at a combinatorial proof of the Alon-Boppana theorem as stated by A. Lubotsky, P. Sarnak, and R. Philips [14] in their seminal paper on Ramanujan graphs.

3.1 Isoperimetric constant

One can see any communication network as a graph, where entities that want to communicate are vertices, and the connection between them is represented by edges. Then two entities can communicate if there is a path from one of them to the other, and the larger the length of the path, the longer it takes for the communication. Thus, a communication network is efficient if it is reliable and fast, i.e., if the graph is "well-connected".

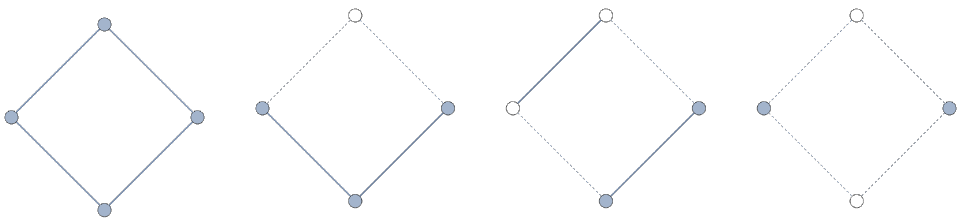
Definition 3.1. Let X be a graph and $F \subset V(X)$. The *boundary* of F, denoted by ∂F, is defined to be the set of edges with one end point in F and one endpoint in $V(X) \setminus F$.

Definition 3.2. The *isoperimetric constant* or the *Cheeger constant* of a graph X, denoted by $h(X)$, is defined as follows

$$h(X) = \min\left\{ \frac{|\partial F|}{|F|} \;\middle|\; F \subset V(X) \text{ and } |F| \le \frac{|V(X)|}{2} \right\}.$$

The boundary of a subset F of $V(X)$ tells us the number of connections between the set and its complement. Thus, the larger the boundary, the more the number of connections between the subset F and its complement. Thus, the higher the value of the isoperimetric constant, the more "well-connected" the graph is.

Example 3.3. In Figure (2.1) below, we see all possible $F \subset V(C_4)$ such that $|F| \le |V(C_4)|/2$.



C_4 $\qquad \frac{|\partial F_1|}{|F_1|} = 2$ $\qquad \frac{|\partial F_2|}{|F_2|} = 1$ $\qquad \frac{|\partial F_3|}{|F_3|} = 2$

Fig. 3.1: Computing $h(C_4)$.

Thus $h(C_4) = 1$.

Definition 3.4. Let (a_n) be a sequence of nonzero real numbers. We say that (a_n) *is bounded away from zero* if there exists $\epsilon > 0$ such that $a_n \ge \epsilon$ for all n.

Definition 3.5. Let k be a positive integer. Let (X_n) be a sequence of k-regular graphs such that $|X_n| \to \infty$ as $n \to \infty$. We say that (X_n) is an *expander family* if the sequence $(h(X_n))$ is bounded away from zero.

Remark 3.6. One can think of expander families as infinite families of "well-connected" graphs, particularly from the viewpoint of communication networks.

Example 3.7. Consider (C_n), a sequence of 2-regular connected graphs. If m is a fixed integer such that $1 \leq m \leq n/2$ then,

$$\min\left\{\frac{|\partial F|}{|F|} \;\middle|\; F \subset V \text{ and } |F| = m\right\} = \frac{2}{m}.$$

Note that there exists a $F \subset V(C_n)$ such that $|F|$ is $n/2$ or $(n-1)/2$, depending on whether n is even or odd. Therefore,

$$h(C_n) = \min_{\substack{F \subset V \\ 0<|F|<n/2}} \left\{\frac{|\partial F|}{|F|}\right\} = \begin{cases} 4/n, & \text{if } n \text{ is even and} \\ 4/n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore $h(C_n) \to 0$ as $n \to \infty$. Thus, (C_n) is not an expander family.

3.2 The Rayleigh-Ritz Theorem

Definition 3.8. Let X be a finite set and f_0 be the function that is equal to 1 for all of X. Define,

$$L^2(X,\mathbb{R}) = \{f : X \to \mathbb{R}\}, \tag{3.1}$$

$$\text{and} \qquad L^2_0(X,\mathbb{R}) = \{f \in (X,\mathbb{R}) : \langle f, f_0\rangle_2 = 0\}$$
$$= \{f \in (X,\mathbb{R}) : \sum_{x\in X} f(x) = 0\}. \tag{3.2}$$

Theorem 3.9. *(Rayleigh-Ritz) Let $X = (V,E)$ be a k-regular graph with $A = A(X)$. Then,*

$$\lambda_1(X) = \max_{f \in L^2_0(V,\mathbb{R})} \frac{\langle Af, f\rangle_2}{\|f\|_2^2} = \max_{\substack{f\in L^2_0(V,\mathbb{R}) \\ \|f\|_2^2=1}} \langle Af, f\rangle_2. \tag{3.3}$$

Consequently,

$$k - \lambda_1(X) = \min_{f \in L^2_0(V,\mathbb{R}} \frac{\langle \Delta f, f\rangle_2}{\|f\|_2^2} = \min_{\substack{f\in L^2_0(V,\mathbb{R}) \\ \|f\|_2^2=1}} \langle \Delta f, f\rangle_2. \tag{3.4}$$

Proof. By Spectral theorem, there exists an orthanormal basis $\{f_0, f_1, \ldots, f_n\}$ for $L^2(V, \mathbb{R})$, such that every f_i is a real valued eigenfuntion of A associated with the eigenvalue $\lambda_i = \lambda_i(X)$ where $i = 0, 1, \ldots, n-1$.
Consider $f \in L^2_0(V, \mathbb{R})$ with $\|f\|_2^2 = 1$. Then we have $f = c_0 f_1 + c_1 f_1 + \cdots + c_{n-1} f_{n-1}$, where $c_i \in \mathbb{R}$ for all $i = 0, 1, \ldots, n-1$. So,

$$0 = \langle f, f_0 \rangle_2 = \ c_0 \langle f_0, f_0 \rangle_2 + c_1 \langle f_1, f_0 \rangle_2 + \cdots + c_{n-1} \langle f_{n-1}, f_0 \rangle_2 = c_0.$$

Therefore,

$$f = c_1 f_1 + \cdots + c_{n-1} f_{n-1}.$$

Now,

$$\langle Af, f \rangle_2 = \left\langle A \sum_{i=1}^{n-1} c_i f_i, \sum_{i=1}^{n-1} c_i f_i \right\rangle_2 = \left\langle \sum_{i=1}^{n-1} c_i \lambda_i f_i, \sum_{i=1}^{n-1} c_i f_i \right\rangle_2$$

$$\begin{aligned} &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_i \lambda_i c_j \ \langle f_i, f_j \rangle_2 = \sum_{i=1}^{n-1} {c_i}^2 \ \lambda_i \\ &\leq \lambda_1 \sum_{i=1}^{n-1} {c_i}^2 = \lambda_1 \|f\|_2^2 = \lambda_1. \end{aligned}$$

Since this is holds for any $f \in L^2_0(V, \mathbb{R})$ with $\|f\|_2^2 = 1$, we have

$$\lambda_1 \geq \max_{\substack{f \in L^2_0(V, \mathbb{R}) \\ \|f\|_2^2 = 1}} \langle Af, f \rangle_2.$$

Taking $f = f_1$, we get $\langle Af, f \rangle_2 = \langle \lambda_1 f, f \rangle_2 = \lambda_1$, and so it follows that

$$\lambda_1(X) = \max_{f \in L^2_0(V, \mathbb{R})} \frac{\langle Af, f \rangle_2}{\|f\|_2^2} = \max_{\substack{f \in L^2_0(V, \mathbb{R}) \\ \|f\|_2^2 = 1}} \langle Af, f \rangle_2.$$

Consequently, by Theorem 2.30, we get that

$$k - \lambda_1(X) = \min_{f \in L_0^2(V,\mathbb{R}} \frac{\langle \Delta f, f \rangle_2}{\|f\|_2^2} = \min_{\substack{f \in L_0^2(V,\mathbb{R}) \\ \|f\|_2^2 = 1}} \langle \Delta f, f \rangle_2.$$

□

By applying Rayleig-Ritz, theorem we obtain a lower bound on $h(X)$ in terms of the *spectral gap* of X given by $k - \lambda_1(X)$.

Theorem 3.10. *Let X be a k-regular graph. Then,*

$$\frac{k - \lambda_1(X)}{2} \leq h(X) \leq \sqrt{2k(k - \lambda_1(X))}. \tag{3.5}$$

Thus, we get an alternate condition to check whether a sequence of graphs given by (X_n) forms an expander family.

Corollary 3.11. *Let (X_n) be a sequence of k-regular graphs with $|X_n| \to \infty$ as $n \to \infty$. Then (X_n) is a family of expanders if and only if the sequence $(k - \lambda_1(X_n))$ is bounded away from zero.*

3.3 Alon-Boppana Theorem

3.3.1 Catalan Numbers

Definition 3.12. Let $a = (a_1, a_2, \ldots, a_{2k})$ be a sequence where $a_i = \pm 1$ for $i = 1, 2, \ldots, 2k$. We say that the sequence a is *balanced* if $\sum_{i=1}^{2k} a_i = 0$ and $\sum_{i=1}^{n} a_i \geq 0$ for $n = 1, 2, \ldots, 2k$.

Example 3.13. The sequence $a = (1, 1, -1, 1, -1, -1)$ is balanced since the sum is zero and $\sum_{i=1}^{n} a_i \geq 0$ for $n = 1, 2, \ldots, 6$.

However, the sequence $a = (1, 1, -1, -1, -1, 1)$ is unbalanced, as even though the sum is zero at $n = 5$, we have $\sum_{i=1}^{n} a_i = -1 \leq 0$.

Remark 3.14. Let X be a connected graph. Then for a fixed vertex $v_0 \in V(X)$, every walk $w = (v_0, e_1, v_1, \ldots, e_k, v_k)$ can be seen as a sequence a as above, where $a_i = +1$ when e_i takes a step away from v_0 and $a_i = -1$ when e_i takes a step towards v_0. Thus, when the sequence a is balanced, w is a unfactorable walk.

Definition 3.15. Let n be a positive integer. The n^{th} *Catalan number* C_n is the number of balanced sequences of length $2n$ consisting of n positive ones and n negative ones.
By convention $C_0 = 1$.

Remark 3.16. It is easy to see that definition above is equivalent to the recurring relation as stated below.

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \quad \text{and} \quad C_0 = 1.$$

Lemma 3.17. *[10, Lemma 3.23] The n^{th} Catalan number is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

3.3.2 Universal Covering Graphs

Definition 3.18. Let x be a connected k-regular graph. Let $v_0 \in V(X)$ be a fixed vertex. The *universal covering graph*, denoted by U_{v_0}, of X using v_0 as a base point is constructed as follows.

1. Each vertex of U_{v_0} is a non backtracking walk of X that begins at v_0.
2. Two vertices are adjacent via an edge of multiplicity 1 if one walk extends the other by a single step.

Example 3.19. Consider the graph X as shown in Figure 3.2. Then universal covering graph of X corresponding to fixed vertex v_0 is U_{v_0} as in the Figure 3.3. Notice that U_{v_0} is an infinte tree.

Proposition 3.20. *Let X be a connected k-regular graph . Let U_{v_0} be the universal covering graph of X constructed using some fixed vertex $v_0 \in V(X)$*

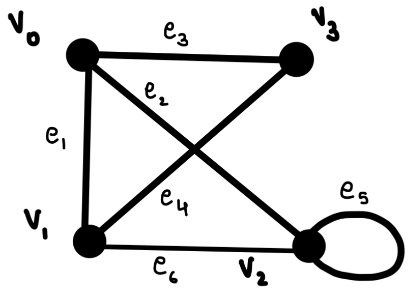


Fig. 3.2: Graph X.

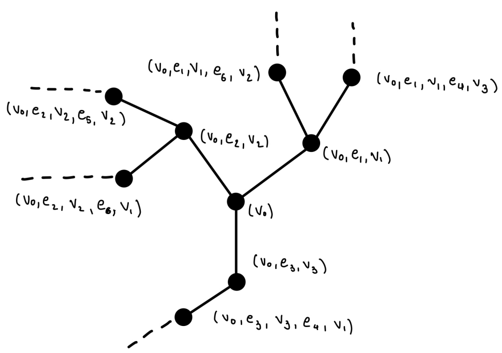


Fig. 3.3: Universal covering graph U_{V_0}.

as a base point. Let d be a fixed positive integer. Then the number of unfactorable walks of length $2d$ *in* U_{v_0} *that begin and end at* (v_0) *equals*

$$\frac{1}{d}\binom{2d-2}{d-1}k(k-1)^{d-1}.$$

Definition 3.21. Let X be a connected k-regular graph U_{v_0} be the universal covering graph of X constructed using some fixed vertex $v_0 \in V$ as a base point. We define the *covering map* $\phi_{v_0} : U_{v_0} \to X$, of U_{v_0} as follows:

1. For a vertex $(v_0, e_0, v_1, \ldots, v_{n-1}, e_{n-1}, v_n)$ of T_{v_0}, $\phi_{v_0}(v_0, e_0, v_1, \ldots, v_{n-1}, e_{n-1}, v_n) = v_n$.

2. Let e be the edge of T that is incident to $(v_0, e_0, v_1, \ldots, v_{n-1}, e_{n-1}, v_n)$

and $(v_0, e_0, v_1, \ldots, v_n, e_n, v_{n+1})$, $\phi_{v_0}(e) = e_n$.

Lemma 3.22. *Let $X, v_0, U_{v_0}, \phi_{v_0}$ be as described above. The number of closed walks of length $2d$ in X beginning and ending at v_0 is greater than or equal to the number of closed walks of length $2d$ in U_{v_0} beginning and ending at (v_0).*

3.3.3 A combinatorial proof of the Alon-Boppana theorem

Definition 3.23. Let X be a k-regular graph with n vertices. We define

$$\lambda(X) = \begin{cases} \max\{|\lambda_1(X)|, |\lambda_{n-1}(X)|\}, & \text{if } X \text{ is non bipartite, and} \\ \max\{|\lambda_1(X)|, |\lambda_{n-2}(X)|\}, & \text{if } X \text{ is bipartite.} \end{cases}$$

Lemma 3.24. $\lim_{d\to\infty} \begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix}^{1/2d} = 2.$

Theorem 3.25. *If (X_n) is a sequence of connected k-regular graphs with $|X_n| \to \infty$ as $n \to \infty$, then*

$$\liminf_{x\to\infty} \lambda(X_n) \geq 2\sqrt{k-1}.$$

Proof. For $n \geq 3$, let X be a k-regular graph of order n and $A = A(X)$. Let $w(2d)$ be the number of walks of length $2d$ in X then by [10, Lemma A.60], we have

$$\sum_{i=0}^{n-1} \lambda_i(X)^{2d} = \operatorname{tr}\left(A^{2d}\right) = \sum_{i=1}^{n} \left(A^{2d}\right)_{i,i} = w(2d).$$

Given a vertex $v \in V$, let $\rho_\nu(2d)$ be the number of walks of length $2d$ beginning and ending at (v) in the covering graph U_v. By Lemma 3.22, we have

$$\sum_{i=0}^{n-1} \lambda_i(X)^{2d} = w(2d) \geq \sum_{i=1}^{n} \rho_{v_i}(2d). \tag{3.6}$$

Let $\rho'_v(2d)$ denote the number of unfactorable closed walks of length $2d$ in the covering graph U_v beginning and ending at (v). Then, $\rho_v(2d) \geq \rho'_v(2d)$.

By Lemma 3.20, we have

$$\rho'_v(2d\,) = \frac{1}{d}\begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix} k(k-1\,)^{d-1}. \tag{3.7}$$

Hence, $\rho'_v(2d)$ is independent of the choice of v. Henceforth, we denote $\rho'_v(2d)$ by $\rho'(2d)$. So, from (2.6) , we have

$$\sum_{i=0}^{n-1} \lambda_i(X,)^{2d} \geq \sum_{i=1}^{n} \rho'(2d) = n\rho'(2d).$$

Now, we consider the following cases:

- If X is bipartite, then by Proposition 2.26 we have $\lambda_0(X) = k$ and $\lambda_{n-1}(X) = -k$, so

$$\begin{aligned} (n-2)\lambda(X)^{2d} &\geq \sum_{i=1}^{n-2} \lambda_i(X)^{2d} \geq n\rho'(2d) - 2k^{2d} \\ \implies \quad \lambda(X)^{2d} &\geq \frac{n}{n-2}\rho'(2d) - \frac{2d^{2d}}{n-2} \geq \rho'(2d) - \frac{2k^{2d}}{n-2}. \end{aligned}$$

- If X is not bipartite, then $\lambda_0(X) = k$, so

$$\begin{aligned} (n-1)\lambda(X)^{2d} &\geq \sum_{i=1}^{n-1} \lambda_i(X)^{2d} \geq n\rho'(2d) - k^{2d} \\ \implies \quad \lambda(X)^{2d} &\geq \frac{n}{n-1}\rho'(2d) - \frac{k^{2d}}{n-1} \geq \rho'(2d) - \frac{2k^{2d}}{n-2}. \end{aligned}$$

In either case, we have

$$\lambda(X)^{2d} \geq \rho'(2d) - \frac{2k^{2d}}{n-2}.$$

Letting $d \to \infty$ and applying Lemma 3.24, we get the desired result.

□

Thus, the Alon-Boppana theorem allows us to infer that for a large, k-regular graph X, the strongest upper bound for $\lambda(X)$ is $2\sqrt{k-1}$. Since

$(k - \lambda(X)) \leq (k - \lambda_1(X))$, for large k-regular graphs the best spectral gap is $k - 2\sqrt{k-1}$. This motivates the study of Ramanujan graphs [14, 16], which are regular graphs with spectral gaps as large as possible.

Definition 3.26. Let X be a k-regular graph. We say that X is *Ramanujan* if $\lambda(X) \leq 2\sqrt{k-1}$.

Example 3.27. The complete graph K_n is a Ramanujan graph for $n \geq 3$. Its spectrum is given by

$$Spec\,(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

Since K_n is $n-1$ regular, $(n-1)$ is the trivial eigenvalue of K_n. Therefore, $\lambda\,(K_n) = 1 < 2\sqrt{n-2}$ for $n \geq 3$.

4. QUADRATIC UNITARY CAYLEY GRAPHS

In this chapter, we will look at a special family of Cayley graphs called quadratic unitary Cayley graphs. Furthermore, we also calculate their spectrum for a few cases. Finally, we will study the conditions under which these graphs are Ramanujan. This chapter is based on the results in [13].

4.1 Introduction

Definition 4.1. Let R be a finite commutative ring. The *quadratic unitary Cayley graph (QUCG)*, denoted by $\mathcal{G}_R$, is the Cayley graph $Cay(R, T_R)$. Here, $T_R = Q_R \cup (-Q_R)$ where $Q_R = \{u^2 \ : \ u \in R^\times\}$.

In the above definition while defining $Cay(R, T_R)$ we consider R as group under addition. Thus, $V(\mathcal{G}_R) = R$ and $E(\mathcal{G}_R) = \{\{x, y\} \ : \ x, y \in R \ and \ x - y \in T_R\}$.

Remark 4.2. Notice that $Cay(R, T_R)$ is well defined as $T_R = Q_R \cup (-Q_R)$ is a symmetric subset of R under the addition operation.

Example 4.3. Let $R = \mathbb{Z}_5$ then $\mathbb{Z}_5^\times = \mathbb{Z}_5 \backslash \{\bar{0}\}$. Now $Q_{\mathbb{Z}_5} = \{\bar{1}, \bar{4}\}$ and $-Q_{\mathbb{Z}_5} = \{-\bar{1}, -\bar{4}\} = Q_{\mathbb{Z}_5}$. Hence, $T_{\mathbb{Z}_5} = \{\bar{1}, \bar{4}\}$ i.e. $x, y \in \mathbb{Z}_5$ are adjacent iff $x - y = \bar{1}$ or $x - y = \bar{4}$. The graph $\mathcal{G}_{\mathbb{Z}_5}$ is depicted in Figure 4.1 below..

From Proposition 2.49 dicussed in Chapter 2 we know that G_R is connected if and only if the set T_R generates R as a group under addition. Clearly in the above example we see that $T_{\mathbb{Z}_5}$ generates $\mathbb{Z}_5$ hence the graph $\mathcal{G}_{\mathbb{Z}_5}$ is connected, but this might not be true for every QUCG.

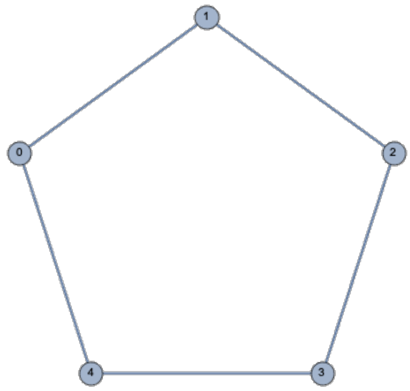


Fig. 4.1: The QUCG $\mathcal{G}_{\mathbb{Z}_5}$.

Example 4.4. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$, so $R^\times = \{(1,1),(1,3),(3,1),(3,3)\}$. We get $Q_R = \{(1,1)\}$ and $-Q_R = \{(3,3)\}$, so $T_{Q_R} = \{(1,1),(3,3)\}$. It is clear that the set T_R does not generate R as a group under addition. Figure 4.2 below shows how the graph $\mathcal{G}_R$ is disconnected with four components.

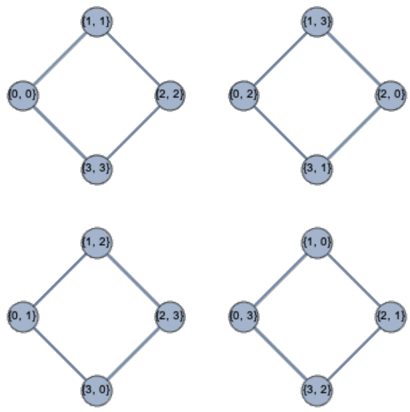


Fig. 4.2: A disconnected QUCG.

Example 4.5. Let P be a Paley graph of order $q = p^n$. Then

$$V(P) = \mathbb{F}_q \text{ and } E(P) = \left\{\{x,y\} : x,y \in \mathbb{F}_q, x - y \in \left(\mathbb{F}_q^*\right)^2\right\}.$$

We can observe that Paley graph is a QUCG for $\mathbb{F}_q$. This makes sense because $\mathbb{F}_q$ is a finite field, which is also a finite commutative ring.

Proposition 4.6. *Let $\mathbb{F}_q$ be a finite field of order $q = p^n$ for odd prime p and some non negative integer n. Let $\mathcal{G}_{\mathbb{F}_q}$ be corresponding QUCG. Then we have the following.*

1. $-1 \in Q_{\mathbb{F}_q}$ *if and only if* $q \equiv 1 \pmod 4$.

2. $|T_{\mathbb{F}_q}| = |Q_{\mathbb{F}_q}| = \frac{q-1}{2}$ *if and only if* $q \equiv 1 \pmod 4$.

Proof. For a finite field $\mathbb{F}_q$ we have ${\mathbb{F}_q}^{\times} = \mathbb{F}_q \backslash \{0\}$.

1. If $-1 \in Q_{\mathbb{F}_q}$ then there exists $x \in \mathbb{F}_q$ such that $x^2 = -1$ i.e. $x^4 = 1$. Therefore, subgroup generated by x has order 4. It follows from Lagrange's theorem $4 \mid |\mathbb{F}_q| = q - 1$, hence we get that $q \equiv 1 \pmod 4$. Conversely, suppose that $q \equiv 1 \pmod 4$. Let ${\mathbb{F}_q}^{\times} = \langle g \rangle$ for some $g \in \mathbb{F}_q$, since ${\mathbb{F}_q}^{\times}$ forms a cyclic group under multiplication. Then for some positive integer m, $g^m = -1$ and $g^{q-1} = 1$. We then have $g^2m = 1$, and by Lagrange's theorem we get $q-1|2m$ i.e. m is an even integer. If $m = 2b$, then $-1 = g^m = g^{2b} = (g^b)^2$. Hence, we get that $-1 \in Q_{\mathbb{F}_q}$.

2. Note that $|Q_{\mathbb{F}_q}| = \left|\mathbb{F}_q^{\times}\right| = \frac{q-1}{2}$. By part 1., we know that $-1 \in Q_{\mathbb{F}_q}$ if and only if $q \equiv 1 \pmod 4$. Thus, if $-1 \in Q_{\mathbb{F}_q}$, then $Q_{\mathbb{F}_q} = -Q_{\mathbb{F}_q}$, implying that $Q_{\mathbb{F}_q} \cup -Q_{\mathbb{F}_q} = Q_{\mathbb{F}_q} = T_{\mathbb{F}_q}$. Hence, we get that $|T_{\mathbb{F}_q}| = |Q_{\mathbb{F}_q}| = \frac{q-1}{2}$ if and only if $q \equiv 1 \pmod 4$.

□

Corollary 4.7. *Let $\mathbb{F}_q$ be a finite field of order $q = p^n$ for odd prime p and some non-negative integer n. Let $\mathcal{G}_{\mathbb{F}_q}$ be corresponding QUCG. Then we have the following.*

1. $-1 \notin Q_{\mathbb{F}_q}$ *if and only if* $q \equiv 3 \pmod 4$.

2. $|T_{\mathbb{F}_q}| = 2|Q_{\mathbb{F}_q}| = q - 1$ *if and only if* $q \equiv 3 \pmod 4$.

4.2 Spectra of quadratic unitary Cayley graphs

Let R be a finite commutative ring. From [3, Theorem 3.1.4] we know that $R = R_1 \times R_2 \times \cdots \times R_s$ where R_i, $1 \leq i \leq s$ are local rings with maximal ideals M_i of order m_i such that

$$\frac{|R_1|}{m_1} \leq \frac{|R_2|}{m_2} \leq \cdots \leq \frac{|R_s|}{m_s}.$$

Also, note that

$$|R^\times| = \prod_{i=1}^{s} |R_i^\times| = \prod_{i=1}^{s} |R_i| - m_i = \prod_{i=1}^{s} m_i \left(\frac{|R_i|}{m_i} - 1 \right). \tag{4.1}$$

Let R_0 be a local ring with maximal ideal M_0 with order m_0 such that $\frac{|R_0|}{m_0} \equiv 3 \pmod 4$. In this section, we determine the spectra of $\mathcal{G}_R$ and $\mathcal{G}_{R_0 \times R}$ under the condition that $\frac{|R_i|}{m_i} \equiv 1 \pmod 4$ for $1 \leq i \leq s$.

We begin with studying the spectra of QUCG for a finite local ring R. First, we recall the following well-known from ring theory.

Lemma 4.8. *[1, Proposition 2.1] Let R be a finite local ring and m the order of its unique maximal ideal. Then there exists a prime p such that $|R|$, m and $|R|/m$ are all powers of p.*

Lemma 4.9. *Let R be a finite local ring with maximal ideal M. If $|R|/|M|$ is odd, then $Q_R \cong Q_{R/M} \times M$.*

Proof. Define $\rho : R^\times \to (R/M)^\times$by $\rho(r) = r + M$ for $r \in R^\times$. Then ρ is a well-defined surjective homomorphism from the multiplicative group $R^\times$ to the multiplicative group $(R/M)^\times$ with kernel $\ker(\rho) = 1 + M$. Thus, $R^\times/(1+M) \cong (R/M)^\times$ and the corresponding isomorphism from $R^\times/(1+M)$ to $(R/M)^\times$is given by

$$r(1 + M) \mapsto r + M \text{ for } r \in R^\times.$$

Since $|1+M| = p^t$ (as $|M| = p^t$) and $|R^\times|/|1+M| = (|R|-|M|)/|M| = p^s - 1$, $|1+M|$ and $|R^\times|/|1+M|$ are coprime, and so $1+M$ is a Sylow p-subgroup of $R^\times$. Thus, $R^\times \cong R^\times/(1+M) \times (1+M)$, say, with isomorphism given by

$$r \mapsto (\hat{r}(1+M), 1+m_r),$$

where $\hat{r} \in R$ and $m_r \in M$ are determined by $r \in R^\times$.

Since $R^\times/(1+M) \cong (R/M)^\times$, it follows that $R^\times \cong (R/M)^\times \times (1+M)$ and the corresponding isomorphism is given by $\psi(r) = (\hat{r}+M, 1+m_r)$ for $r \in R^\times$. Since $1+M$ is a Sylow p-subgroup of $R^\times$and p is odd, we have $(1+M)^2 = 1+M$. This together with $R^\times \cong (R/M)^\times \times (1+M)$ implies that $Q_R \cong Q_{R/M} \times (1+M)$ as groups with the corresponding isomorphism giving by

$$\psi\left(r^2\right) = \left(\hat{r}^2 + M, (1+m_r)^2\right) \tag{4.2}$$

for $r \in R^\times$. Thus, by choosing appropriate isomorphism from $(1+M)$ to M we get that $Q_R \cong Q_{R/M} \times M$. $\square$

Theorem 4.10. *Let R be a finite local ring with maximal ideal M. If $|R|/|M|$ is odd, then $\mathcal{G}_R \cong \mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}$.*

Proof. By Definition 2.19, to show that $\mathcal{G}_R \cong \mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}$ it suffices to show that there exists a bijective map, $\tau : V(\mathcal{G}_R) \to V(\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|})$ such that $\{x,y\} \in E(\mathcal{G}_R)$ if and only if $\{\tau(x), \tau(y)\} \in E(\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|})$. Note that $V(\mathcal{G}_R) = R$ and $V(\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}) = R/M \times M$.

Since R is a local ring and M is its maximal ideal, we have $R^\times = R \backslash M$ and R/M is a finite field. By Lemma 4.8 we have $|R|/|M| = p^s$ and $|M| = p^t$ for a prime p and some integers $s \geq 1, t \geq 0$. Write $R/M = \{r_1 + M, r_2 + M, \ldots, r_{p^s} + M\}$. Then for each $r \in R$ there is a unique i and $n_r \in M$ such that $r = r_i + n_r$. Let $\tau : R \to R/M \times M$ be defined by $\tau(r) = (r_i + M, n_r) = (r + M, n_r)$. Since τ is clearly surjective, it is a bijection from R to $R/M \times M$ as the two sets have the same size.

We now show that $\{x,y\} \in E(\mathcal{G}_R) \iff \{\tau(x), \tau(y)\} \in E(\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|})$. Note that, $\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}$ is defined on $R/M \times M$ such that $(x+M, a), (y+M, b)$

are adjacent if and only if $x+M, y+M$ are adjacent in $\mathcal{G}_{R/M}$ (since any pair of vertices are adjacent in $\otimes \overset{\circ}{K}_{|M|}$). Suppose that $x, y \in R$ are adjacent in $\mathcal{G}_R$, that is, by the definition 4.1, $x - y = \pm r^2$ for some $r \in R^\times$. Without loss of generality, we may assume $x - y = r^2$ so that $(x+M) - (y+M) = r^2 + M = (r+M)^2$. Since $r \notin M$ and M is the zero-element of the field R/M, it follows that $(x+M) - (y+M) \in Q_{R/M}$. Therefore, $\tau(x)$ and $\tau(y)$ are adjacent in $\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}$. So we have proved that $\mathcal{G}_R$ is embedded into $\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}$ via τ as a spanning subgraph since the two graphs have the same number of vertices.

Conversely, if $x+M, y+M$ are adjacent in $\mathcal{G}_{R/M}$ then we have following cases.

- If $|R|/m \equiv 1 \pmod 4$, then by Proposition 4.6, we have $-1 \in Q_{R/M}$ and $-1 \in Q_R$. By Lemma 4.8, the degree of $\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}$ is equal to $\left|Q_{R/M}\right| |M| = |Q_R|$, which is the same as the degree of $\mathcal{G}_R$.
- If $|R|/m \equiv 3 \pmod 4$, then by Corollary 4.7, we have $-1 \notin Q_{R/M}$ and $-1 \notin Q_R$, and the degree of $\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}$ is equal to $2\left|Q_{R/M}\right| |M| = 2|Q_R|$, which is also the same as the degree of $\mathcal{G}_R$.

In either case $\mathcal{G}_R$ and $\mathcal{G}_{R/M} \otimes \overset{\circ}{K}_{|M|}$ must be isomorphic to each other because they have the same degree and one is a spanning subgraph of the other. □

Theorem 4.11. *Let R be a local ring with maximal ideal M of order m.*

1. *If $|R|/m \equiv 1 \pmod 4$, then*

$$Spec\,(\mathcal{G}_R) = \begin{pmatrix} \frac{|R|-m}{2} & \frac{m(-1+\sqrt{|R|/m})}{2} & \frac{m(-1-\sqrt{|R|/m})}{2} & 0 \\ 1 & (|R|/m-1)/2 & (|R|/m-1)/2 & |R|-|R|/m \end{pmatrix}.$$

2. *If $|R|/m \equiv 3 \pmod 4$, then*

$$Spec\,(\mathcal{G}_R) = \begin{pmatrix} |R|-m & -m & 0 \\ 1 & |R|/m-1 & |R|-|R|/m \end{pmatrix}.$$

Proof. Let R be a local ring with maximal ideal M of order m.

1. If $|R|/m \equiv 1 \pmod 4$, then $|R|/m$ is an odd prime power and $\mathcal{G}_{R/M}$ coincides with the Paley graph of order $|R|/m$ as seen in the Example 4.5. As we calculated in Example 2.44, the spectrum of Paley graph P of order q is,

$$Spec(P) = \begin{pmatrix} q & \frac{-1+\sqrt{q}}{2} & \frac{-1-\sqrt{q}}{2} \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}.$$

Also, recall from Example 2.25 that

$$Spec(\mathring{K}_n) = \begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}.$$

Thus, by Proposition 2.37 we get the required result.

2. If $|R|/m \equiv 3 \pmod 4$, then from Corollary 4.7 and Proposition 2.49, we get that $\mathcal{G}_{R/M}$ is a complete graph of order $|R|/m$. We know that $A(K_{|R|/m}) = J_n - I_n$. Thus, if λ is as eigenvalue of J_n, $\lambda - 1$ is an eigenvalue of $J_n - I_n$. Hence, the spectrum of $\mathcal{G}_{R/M}$ given by

$$Spec(K_{|R|/m}) = \begin{pmatrix} \frac{|R|}{m} - 1 & -1 \\ 1 & |R|/m - 1 \end{pmatrix}.$$

Thus, by Proposition 2.37 we get the required result.

□

The next theorem gives us the condition under which $\mathcal{G}_R$ is equal to graph tensor product of $\mathcal{G}_{R_i}$, where R_i are finite local rings.

Theorem 4.12. *Let A and B finite commutative rings then $\mathcal{G}_{A\times B} = \mathcal{G}_A \otimes \mathcal{G}_B$ if and only if -1 lies in at least one one of Q_A or Q_B.*

Proof. Note that $\mathcal{G}_{A\times B} = Cay(A \times B, T_{A\times B})$. Since A and B are finite commutative rings thus $A \times B$ is also a finite commutative ring with $|A \times B| = |A||B|$ with ring addition and multiplication as element wise-operations respectively. Since

$$(A \times B)^{\times} = \{(a, b) \mid a \in A^{\times} \text{ and } b \in B^{\times}\},$$

we get

$$T_{A\times B} = \{(a^2, b^2), (-a^2, -b^2) \mid a \in A^\times \text{ and } b \in B^\times\}\}.$$

Given $\mathcal{G}_A = Cay(A, T_A)$ and $\mathcal{G}_B = Cay(B, T_B)$, we get the graph $\mathcal{G}_A \otimes \mathcal{G}_B$ defined by

$$V(\mathcal{G}_A \otimes \mathcal{G}_B) = \{(a, b) \mid a \in A \text{ and } b \in B\} = A \times B = V(\mathcal{G}_{A\times B})$$

$$\begin{aligned} E(\mathcal{G}_A \otimes \mathcal{G}_B) =& \{\{(a_1, b_1), (a_2, b_2)\} \mid \{a_1, a_2\} \in E(\mathcal{G}_A) \text{ and } \{b_1, b_2\} \in E(\mathcal{G}_B)\} \\ =& \{\{(a_1, b_1), (a_2, b_2)\} \mid a_1 - a_2 \in T_A \text{ and } b_1 - b_2 \in T_B\} \\ =& \{\{(a_1, b_1), (a_2, b_2)\} \mid (a_1 - a_2, b_1 - b_2) \in T_A \times T_B\} \end{aligned}$$

Therefore, $\mathcal{G}_A \otimes \mathcal{G}_B = Cay(A \times B, T_A \times T_B)$.

Since $T_A \times T_B = \{(a^2, b^2), (-a^2, b^2), (a^2, -b^2), (-a^2, -b^2) \mid a \in A^\times \text{ and } b \in B^\times\}\}$, it follows that $T_{A\times B} \subseteq T_A \times T_B$. As $V(\mathcal{G}_{A\times B}) = V(\mathcal{G}_A \otimes \mathcal{G}_B) = A \times B$, we get that $\mathcal{G}_{A\times B} = \mathcal{G}_A \otimes \mathcal{G}_B$ if and only if $T_{A\times B} = T_A \times T_B$. Hence, it now suffices to show that $T_{A\times B} = T_A \times T_B$ if and only if -1 lies in at least one one of Q_A or Q_B.

If $-1 \notin Q_A$ and $-1 \notin Q_B$, then both $(-1, 1)$ and $(1, -1)$ are elements of $T_A \times T_B$ but neither of them is an element of $T_{A\times B}$. Thus, $T_{A\times B} \neq T_A \times T_B$ and so $\mathcal{G}_{A\times B} \neq \mathcal{G}_A \otimes \mathcal{G}_B$. On the other hand, suppose that at least one of Q_A and Q_B contains -1, then we have following cases.

- If $-1 \in Q_A$ and $-1 \in Q_B$ then it is easy to see that $T_{A\times B} = T_A \times T_B$.
- If exactly one of Q_A and Q_B has -1. Without loss of generality we may suppose $-1 \in Q_A$ so that $i^2 = -1$ for some $i \in A^\times$. Then for any $(a, b) \in (A \times B)^\times$and $s, t \in \{0, 1\}$ we have

$$\left((-1)^s a^2, (-1)^t b^2\right) = (-1)^t \left((-1)^{(s-t)} a^2, b^2\right) = (-1)^t \left(\left(i^{(s-t)} a\right)^2, b^2\right).$$

Hence $T_A \times T_B \subseteq T_{A\times B}$. Therefore, we proved the required result. □

Theorem 4.13. *Let R be a finite commutative ring such that $R = R_1 \times R_2 \times \cdots \times R_s$ where R_i, $1 \leq i \leq s$ are local rings with maximal ideals M_i of order m_i. Then $\mathcal{G}_R = \mathcal{G}_{R_1} \otimes \mathcal{G}_{R_2} \otimes \cdots \otimes \mathcal{G}_{R_s}$ if and only if there exists at most one R_j such that $-1 \in Q_{R_j}$.*

Proof. We prove the above using induction. The induction hypothesis states that for some natural number s, $\mathcal{G}_R = \mathcal{G}_{R_1} \otimes \mathcal{G}_{R_2} \otimes \cdots \otimes \mathcal{G}_{R_s}$ if and only if there exists at most one R_j such that $-1 \in Q_{R_j}$.

In the case when $s = 1$, the induction hypothesis holds trivially. The case when $s = 2$ follows from Theorem 4.12. Using the induction on s suppose that for the induction hypothesis holds for some integer k such that $2 \leq k \leq s$ i.e., $\mathcal{G}_R = \mathcal{G}_{R_1} \otimes \mathcal{G}_{R_2} \otimes \cdots \otimes \mathcal{G}_{R_k}$ if and only if there exists at most one R_j such that $-1 \in Q_{R_j}$. Hence, we now show that the above hypothesis also holds when $s = k + 1$.

Assume that there is at most one j between 1 and $k + 1$ such that $-1 \notin Q_{R_j}$. Then $(-1, -1, \ldots, -1) \in Q_{R_1 \times R_2 \times \cdots \times R_k}$ or $-1 \in Q_{R_{k+1}}$ no matter whether $1 \leq j \leq k$ or $j = k + 1$, and hence $\mathcal{G}_R = \mathcal{G}_{R_1 \times R_2 \times \cdots \times R_k} \otimes \mathcal{G}_{R_{k+1}}$ by Theorem 4.12 . On the other hand, by the induction hypothesis, we have $\mathcal{G}_{R_1 \times R_2 \times \cdots \times R_k} = \mathcal{G}_{R_1} \otimes \mathcal{G}_{R_2} \otimes \cdots \otimes \mathcal{G}_{R_k}$. Therefore, $\mathcal{G}_R = \mathcal{G}_{R_1} \otimes \mathcal{G}_{R_2} \otimes \cdots \otimes \mathcal{G}_{R_k} \otimes \mathcal{G}_{R_{k+1}}$.

Conversely, assume that $\mathcal{G}_R = \mathcal{G}_{R_1} \otimes \mathcal{G}_{R_2} \otimes \cdots \otimes \mathcal{G}_{R_k} \otimes \mathcal{G}_{R_{k+1}}$. We aim to prove that there exists at most one j between 1 and $k + 1$ such that $-1 \notin Q_{R_j}$. Suppose we assume otherwise. Without loss of generality we may assume that, for some integer t with $2 \leq t \leq k + 1$, we have $-1 \notin Q_{R_j}$ for $1 \leq j \leq t$ and $-1 \in Q_{R_j}$ for $t < j \leq k + 1$. If $t < k + 1$, then by Theorem 4.12, since $-1 \in Q_{R_{k+1}}$ we have $\mathcal{G}_R = \mathcal{G}_{R_1 \times R_2 \times \cdots \times R_k} \otimes \mathcal{G}_{R_{k+1}}$. Similarly, if $t < k$, then by Theorem 4.12, $\mathcal{G}_{R_1 \times R_2 \times \cdots \times R_k} = \mathcal{G}_{R_1 \times R_2 \times \cdots \times R_{k-1}} \otimes \mathcal{G}_{R_k}$, and hence $\mathcal{G}_R = \left(\mathcal{G}_{R_1 \times R_2 \times \cdots \times R_{k-1}} \otimes \mathcal{G}_{R_k}\right) \otimes \mathcal{G}_{R_{k+1}} = \mathcal{G}_{R_1 \times R_2 \times \cdots \times R_{k-1}} \otimes \mathcal{G}_{R_k} \otimes \mathcal{G}_{R_{k+1}}$. Continuing in this manner, we obtain $\mathcal{G}_R = \mathcal{G}_{R_1 \times R_2 \times \cdots \times R_t} \otimes \mathcal{G}_{R_{t+1}} \otimes \cdots \otimes \mathcal{G}_{R_{k+1}}$. Comparing this with the assumption $\mathcal{G}_R = \mathcal{G}_{R_1} \otimes \mathcal{G}_{R_2} \otimes \cdots \otimes \mathcal{G}_{R_k} \otimes \mathcal{G}_{R_{k+1}}$, we obtain that $\mathcal{G}_{R_1 \times R_2 \times \cdots \times R_t} = \mathcal{G}_{R_1} \otimes \mathcal{G}_{R_2} \otimes \cdots \otimes \mathcal{G}_{R_t}$. □

Our aim is to find spectrum of the QUCG $\mathcal{G}_R$, where R is a finite commutative ring. Note that by Theorem 3.14, we can get the spectrum of $\mathcal{G}_R$

since the spectrum of component QUCG for local rings is already know to us from Theorem 4.11.

Define

$$\lambda_{A,B} = (-1)^{|B|} \frac{|R^\times|}{2^s \prod_{i\in A}\left(\sqrt{|R_i|/m_i}+1\right)\prod_{j\in B}\left(\sqrt{|R_j|/m_j}-1\right)}$$

for disjoint subsets A, B of $\{1,2,\ldots,s\}$. In particular, $\lambda_{\emptyset,\emptyset} = |R^\times|/2^s$.

Theorem 4.14. *Let R be a finite commutative ring such that $R = R_1 \times R_2 \times \cdots \times R_s$ where R_i is a local rings with maximal ideal M_i of order m_i and $|R_i|/m_i \equiv 1 \pmod 4$ for $1 \le i \le s$. Then the eigenvalues of $\mathcal{G}_R$ are*

1. *$\lambda_{A,B}$, repeated $\frac{1}{2^{|A|+|B|}} \prod_{k\in A\cup B} (|R_k|/m_k - 1)$ times, for all pairs (A,B) of subsets of $\{1,2,\ldots,s\}$ such that $A \cap B = \emptyset$; and*
2. *0 with multiplicity $|R| - \sum_{\substack{A,B\subseteq\{1,\ldots,s\}\\ A\cap B=\emptyset}} \left(\frac{1}{2^{|A|+|B|}} \prod_{k\in A\cup B} (|R_k|/m_k - 1)\right)$.*

Proof. Let R be as in our hypothesis such that $|R_i|/m_i \equiv 1 \pmod 4$ for $1 \le i \le s$. From Theorem 4.11 (1), we get the spectrum of $\mathcal{G}_{R_i}$ for $1 \le i \le s$ to be

$$Spec\,(\mathcal{G}_{R_i}) = \begin{pmatrix} \frac{|R_i|-m_i}{2} & \frac{m_i(-1+\sqrt{|R_i|/m_i})}{2} & \frac{m_i(-1-\sqrt{|R_i|/m_i})}{2} & 0 \\ 1 & (|R_i|/m_i-1)/2 & (|R_i|/m_i-1)/2 & |R_i|-|R_i|/m_i \end{pmatrix}.$$

Let λ be any eigenvalue of $\mathcal{G}_R$. By Theorem 2.37, we know that $\lambda = \prod_{i=1}^{s}\mu_i$ where μ_i is some eigenvalue of $\mathcal{G}_{R_i}$. If at least one of the $\mu_i = 0$, then $\lambda = 0$. Suppose that $\mu_i \neq 0$ for each $1 \le i \le s$, then each μ_i can take any of the three values; $\frac{|R_i|-m_i}{2}, \frac{|R_i|-m_i}{2(\sqrt{|R_i|/m_i}+1}, \frac{-|R_i|+m_i}{2(\sqrt{|R_i|/m_i}-1}$. Then each of the non-zero eigenvalues λ can be written as

$$\lambda_{A,B} = (-1)^{|B|} \frac{|R^\times|}{2^s \prod_{i\in A}\left(\sqrt{|R_i|/m_i}+1\right)\prod_{j\in B}\left(\sqrt{|R_j|/m_j}-1\right)}$$

for disjoint subsets A, B of $\{1, 2, \ldots, s\}$. Here A is set of all μ_i which take value $\frac{|R_i|-m_i}{2(\sqrt{|R_i|/m_i}+1)}$ and B is set of all μ_i which take value $\frac{-|R_i|+m_i}{2(\sqrt{|R_i|/m_i}-1)}$. The multiplicities of eigenvalues $\lambda_{A,B}$ and 0 can be found as a direct consequence of the Theorem 2.37. □

Theorem 4.15. *Let R be as in Theorem 4.14 such that $|R_i|/m_i \equiv 1 \pmod 4$ for $1 \leq i \leq s$, and let R_0 be a local ring with maximal ideal M_0 of order m_0 such that $|R_0|/m_0 \equiv 3 \pmod 4$. Then the eigenvalues of $\mathcal{G}_{R_0 \times R}$ are*

1. *$\left|R_0^\times\right| \cdot \lambda_{A,B}$, repeated $\frac{1}{2^{|A|+|B|}} \prod_{k \in A \cup B} (|R_k|/m_k - 1)$ times, for all pairs (A, B) of subsets of $\{1, 2, \ldots, s\}$ such that $A \cap B = \emptyset$;*
2. *$-\frac{|R_0^\times|}{|R_0|/m_0 - 1} \cdot \lambda_{A,B}$, repeated $\frac{|R_0|/m_0 - 1}{2^{|A|+|B|}} \prod_{k \in A \cup B} (|R_k|/m_k - 1)$ times, for all pairs (A, B) of subsets of $\{1, 2, \ldots, s\}$ such that $A \cap B = \emptyset$; and*
3. *0 with multiplicity $|R| - \sum_{\substack{A,B \subseteq \{1,\ldots,s\} \\ A \cap B = \emptyset}} \left(\frac{|R_0|/m_0}{2^{|A|+|B|}} \prod_{k \in A \cup B} (|R_k|/m_k - 1) \right)$.*

The proof of this theorem is similar to that of Theorem 4.14 above.

4.3 Ramanujan quadratic unitary Cayley graphs

In the past few decades, the expander families of Ramanujan graphs, particularly those arising from families of Cayley graphs have been extensively studied (see [11, 14, 15, 16] and the references therein). In this section, we see the conditions under which a QUCG is Ramanujan.

Lemma 4.16. *[6] The only finite commutative local rings whose maximal ideal has prime order p and $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p[x]/(x^2)$.*

Theorem 4.17. *Let R be as in Theorem 4.14 such that $|R_i|/m_i \equiv 1 \pmod 4$ for $1 \leq i \leq s$, and let R_0 be a local ring with maximal ideal M_0 of order m_0 such that $|R_0|/m_0 \equiv 3 \pmod 4$. Then the following hold:*

1. *$\mathcal{G}_{R_0}$ is Ramanujan if and only if* $|R_0| \geq (m_0+2)^2/4$;

2. *$\mathcal{G}_R$ is Ramanujan if and only if R is isomorphic to $\mathbb{F}_5 \times \mathbb{F}_5$ or $\mathbb{F}_q$ for a prime power* $q \equiv 1 \pmod 4$.

Proof. 1. From Theorem 4.11.2, we know that,

$$Spec(\mathcal{G}_{R_0}) = \begin{pmatrix} |R_0| - m_0 & -m_0 & 0 \\ 1 & |R_0|/m_0 - 1 & |R_0| - |R_0|/m_0 \end{pmatrix}.$$

Thus, we get that $\mathcal{G}_{R_0}$ is Ramanujan if and only if $m_0 \leq 2\sqrt{|R_0| - m_0 - 1}$, which is equivalent to $|R_0| \geq (m_0+2)^2/4$.
2. From Theorem 4.14, we have eigenvalues of $\mathcal{G}_R$ to be $\lambda_{A,B}$ and 0 where

$$\lambda_{A,B} = (-1)^{|B|} \frac{|R^\times|}{2^s \prod_{i\in A}\left(\sqrt{|R_i|/m_i}+1\right)\prod_{j\in B}\left(\sqrt{|R_j|/m_j}-1\right)}$$

for disjoint subsets A, B of $\{1, 2, \ldots, s\}$. Note that, $\lambda_{A,B} \leq \frac{|R|^\times}{2^s}$ and $\lambda_{\phi,\phi} = \frac{|R|^\times}{2^s} = \lambda_0$. Therefore, we get that $\mathcal{G}_R$ is Ramanujan if and only if $|\lambda_{A,B}| \leq 2\sqrt{|R^\times|/2^s - 1}$ for all eigenvalues $\lambda_{A,B} \neq \pm|R^\times|/2^s$.

Note that $|\lambda_{A,B}| < |R^\times|/2^s$ is maximized if and only if $\prod_{i\in A}\left(\sqrt{|R_i|/m_i}+1\right)$ $\prod_{j\in B}\left(\sqrt{|R_j|/m_j}-1\right)$ is minimized. Since $|\lambda_{A,B}| \leq \left|\lambda_{\emptyset,\{1\}}\right| \neq |R^\times|/2^s, \mathcal{G}_R$ is Ramanujan if and only if

$$\left|\lambda_{\emptyset,\{1\}}\right| = \frac{|R^\times|}{2^s\left(\sqrt{|R_1|/m_1}-1\right)} \leq 2\sqrt{|R^\times|/2^s - 1}. \tag{4.3}$$

Since $2\sqrt{|R^\times|/2^s - 1} < 2\sqrt{|R^\times|/2^s}$, this condition is not satisfied unless

$$\left|R^\times\right|/2^s < 4\left(\sqrt{|R_1|/m_1}-1\right)^2. \tag{4.4}$$

In particular, if $s \geq 4$, then since

$$\left(\sqrt{\frac{|R_1|}{m_1}} - 1\right)^2 = \frac{|R_1|}{m_1} - 2\sqrt{\frac{|R_1|}{m_1}} + 1 < \frac{|R_1|}{m_1} - 1,$$

by (4.1), we have

$$\frac{|R^\times|}{2^s} \geq \frac{1}{2^s}\prod_{i=1}^{s}\left(\frac{|R_i|}{m_i} - 1\right) \geq 4\left(\frac{|R_1|}{m_1} - 1\right) > 4\left(\sqrt{\frac{|R_1|}{m_1}} - 1\right)^2.$$

Hence $\mathcal{G}_R$ is not Ramanujan.

It remains to consider the case when $1 \leq s \leq 3$.

- Case 1: $s = 3$.

  By (4.1), we see that (4.4) takes the form

  $$\frac{1}{8}\prod_{i=1}^{3} m_i\left(\frac{|R_i|}{m_i} - 1\right) < 4\left(\sqrt{\frac{|R_1|}{m_1}} - 1\right)^2.$$

  If $\prod_{i=1}^{3} m_i \geq 2$ or $\frac{|R_3|}{m_3} \geq 9$, then $\mathcal{G}_R$ does not satisfy the conditon above, and so it is not Ramanujan. Now assume $\prod_{i=1}^{3} m_i = 1$ and $\frac{|R_3|}{m_3} \leq 8$. Since $\frac{|R_i|}{m_i} \equiv 1 \pmod 4$, we get that $R_1 \cong R_2 \cong R_3 \cong \mathbb{F}_5$. As $\mathcal{G}_R$ does not satisfy the 4.3, thus it is not Ramanujan.

- Case 2: $s = 2$.

  In this case, (4.4) takes the form

  $$\frac{1}{4}\prod_{i=1}^{2} m_i\left(\frac{|R_i|}{m_i} - 1\right) < 4\left(\sqrt{\frac{|R_1|}{m_1}} - 1\right)^2.$$

  Thus, if $m_1 m_2 \geq 4$ or $|R_2|/m_2 \geq 17$, then $\mathcal{G}_R$ is not Ramanujan.

  Assume $m_1 m_2 \leq 3$ and $|R_2|/m_2 \leq 16$. Since $|R_i|/m_i \equiv 1 \pmod 4$ for $i = 1$, by Lemma 4.8, we have $m_1 m_2 = 1$ or $m_1 m_2 = 3$. From Lemma 4.16, we get that $\mathbb{Z}_9$ and $\mathbb{Z}_3[X]/(X^2)$ are the only local rings whose unique maximal ideal has exactly three elements. But their residue fields are $\mathbb{Z}_3$, which is a contradiction to $|R_i|/m_i \equiv 1 \pmod 4$ for $i = 1, 2$. So $m_1 m_2 = 3$ cannot occur. Thus, $m_1 m_2 = 1$ and one of the following occurs:

1. $R_1 \cong R_2 \cong \mathbb{F}_5$;
2. $R_1 \cong R_2 \cong \mathbb{F}_9$;
3. $R_1 \cong R_2 \cong \mathbb{F}_{13}$;
4. $R_1 \cong \mathbb{F}_5$ and $R_2 \cong \mathbb{F}_9$;
5. $R_1 \cong \mathbb{F}_5$ and $R_2 \cong \mathbb{F}_{13}$;
6. $R_1 \cong \mathbb{F}_9$ and $R_2 \cong \mathbb{F}_{13}$.

In Case 1., (4.3) is satisfied, and so $\mathcal{G}_R$ is Ramanujan. In Cases 2-6, (4.3) is not satisfied and so $\mathcal{G}_R$ is not Ramanujan.

- Case 3: $s = 1$.
  In this case, (4.4) takes the form

$$m_1 \left(\frac{|R_1|}{m_1} - 1 \right) < 8 \left(\sqrt{\frac{|R_1|}{m_1}} - 1 \right)^2 .$$

Thus, if $m_1 \geq 8$, then $\mathcal{G}_R$ is not Ramanujan.
Assume $m_1 \leq 7$. Since $\frac{|R_1|}{m_1} \equiv 1 \pmod 4$, by Lemma 4.8 and Lemma 4.16, we have $m_1 = 1$ or 5. In the former case, $R_1 \cong \mathbb{F}_q$, where $q \equiv 1 \pmod 4$ is a prime power. Since (4.3) is satisfied, $\mathcal{G}_{\mathbb{F}_q}$ is Ramanujan. In the latter case, $R_1 \cong \mathbb{Z}_{25}$ or $\mathbb{Z}_5[X]/(X^2)$, and so (4.3) would show that $\mathcal{G}_R$ is not Ramanujan.

□

Theorem 4.18. *Let R be as in Theorem 4.14 such that $|R_i|/m_i \equiv 1 \pmod 4$ for $1 \leq i \leq s$, and let R_0 be a local ring with maximal ideal M_0 of order m_0 such that $|R_0|/m_0 \equiv 3 \pmod 4$. Then $\mathcal{G}_{R_0 \times R}$ is Ramanujan if and only if $R_0 \times R$ is isomorphic to $\mathbb{F}_3 \times \mathbb{F}_5, \mathbb{F}_3 \times \mathbb{F}_9$ or $\mathbb{F}_3 \times \mathbb{F}_{13}$.*

Proof. Define

$$|\lambda| = \max\left\{ \left|R_0^\times\right| |\lambda_{A,B}|, \frac{\left|R_0^\times\right|}{|R_0|/m_0 - 1} |\lambda_{A,B}| \right\}.$$

By Theorem 4.15, $\mathcal{G}_{R_0\times R}$ is Ramanujan if and only if

$$|\lambda| \leq 2\sqrt{\frac{|R_0^\times||R^\times|}{2^s - 1}} \quad \text{for} \quad \lambda \neq \frac{|R_0^\times||R^\times|}{2^s}.$$

Let $\mu_{A,B} = \prod_{i\in A}\left(\sqrt{|R_i|/m_i} + 1\right)\prod_{j\in B}\left(\sqrt{|R_j|/m_j} - 1\right)$. Note that $|\lambda| < \left|R_0^\times\right| |R^\times|/2^s$ is maximized if and only if $\min\{\mu_{A,B}, (|R_0|/m_0 - 1)\,\mu_{A,B}\}$ is minimized. Then we have the following cases.

- Case 1: $\sqrt{|R_1|/m_1} < |R_0|/m_0$.
  In this case $\mathcal{G}_{R_0\times R}$ is Ramanujan if and only if

  $$\frac{\left|R_0^\times\right| |R^\times|}{2^s\left(\sqrt{|R_1|/m_1} - 1\right)} \leq 2\sqrt{\left|R_0^\times\right| |R^\times|/2^s - 1}.$$

  Since $2\sqrt{\left|R_0^\times\right| |R^\times|/2^s - 1} < 2\sqrt{\left|R_0^\times\right| |R^\times|/2^s}$, this condition is not satisfied unless

  $$\frac{|R_0^\times||R^\times|}{2^s} < 4\left(\sqrt{\frac{|R_1}{m_1}} - 1\right)^2.$$

  If $s \geq 3$, then by (4.1) we have

  $$\begin{aligned}\frac{|R_0^\times||R^\times|}{2^s} &\geq \frac{1}{2^s}(|R_0|/m_0 - 1) \times \prod_{i=1}^{s}((|R_i|/m_i) - 1)\\ &\geq 4((|R_1||/m_1) - 1)\\ &> 4\left(\sqrt{|R_1|/m_1} - 1\right)^2,\end{aligned}$$

  and hence $\mathcal{G}_{R_0\times R}$ is not Ramanujan.

  It remains to consider the case where $1 \leq s \leq 2$. We follow similar approach as we did in the proof of Theorem 4.18 and get that in the case of $s = 1$, $\mathcal{G}_{R_0 \times R}$ is Ramanujan when $R_0 \times R \cong \mathbb{F}_3 \times \mathbb{F}_5$ and for $s = 2$, $\mathcal{G}_{R_0 \times R}$ is not Ramanujan.

- Case 2: $\sqrt{|R_1| / m_1} \geq |R_0| / m_0$.
  In this case $\mathcal{G}_{R_0 \times R}$ is Ramanujan if and only if

$$\frac{\left|R_0^\times\right| |R^\times|}{2^s \left(|R_0| / m_0 - 1\right)} \leq 2\sqrt{\left|R_0^\times\right| |R^\times| / 2^s - 1}$$

Since $2\sqrt{\left|R_0^\times\right|\left|R^\times\right| / 2^s - 1} < 2\sqrt{\left|R_0^\times\right|\left|R^\times\right| / 2^s}$, this condition is not satisfied unless

$$\frac{|R_0^\times||R^\times|}{2^s} < 4\left(\frac{|R_0|}{m_0} - 1\right)^2.$$

In particular, if $s \geq 3$, then by 4.1 we have

$$\begin{aligned} \frac{\left|R_0^\times\right| |R^\times|}{2^s} &\geq \frac{1}{2^s}\left(\frac{|R_0|}{m_0} - 1\right) \times \left(\sqrt{\frac{|R_1|}{m_1}} - 1\right)\left(\sqrt{\frac{|R_1|}{m_1}} + 1\right)\prod_{i=2}^{s}\left(\frac{|R_i|}{m_i} - 1\right) \\ &> 4\left(\frac{|R_0|}{m_0} - 1\right)^2, \end{aligned}$$

and hence $\mathcal{G}_{R_0 \times R}$ is not Ramanujan. It remains to consider the case where $1 \leq s \leq 2$. We follow similar approach as we did in the proof of Theorem 4.18 and get that in the case of $s = 1$, $\mathcal{G}_{R_0 \times R}$ is Ramanujan when $R_0 \times R \cong \mathbb{F}_3 \times \mathbb{F}_9$ or $R_0 \times R \cong \mathbb{F}_3 \times \mathbb{F}_{13}$ and for $s = 2$, $\mathcal{G}_{R_0 \times R}$ is not Ramanujan.

□

BIBLIOGRAPHY

[1] Reza Akhtar, Megan Boggess, Tiffany Jackson-Henderson, Isidora Jiménez, Rachel Karpman, Amanda Kinzel, and Dan Pritikin. On the unitary cayley graph of a finite ring. *the electronic journal of combinatorics*, pages R117–R117, 2009.

[2] R. B. Bapat. *Graphs and matrices.* Universitext. Springer, London; Hindustan Book Agency, New Delhi, 2010.

[3] Gilberto Bini and Flaminio Flamini. *Finite commutative rings and their applications*, volume 680 of *The Kluwer International Series in Engineering and Computer Science.* Kluwer Academic Publishers, Boston, MA, 2002. With a foreword by Dieter Jungnickel.

[4] Andries E. Brouwer and Willem H. Haemers. *Spectra of graphs.* Universitext. Springer, New York, 2012.

[5] Fan R. K. Chung. *Spectral graph theory*, volume 92 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997.

[6] N. Ganesan. Properties of rings with a finite number of zero divisors. *Math. Ann.*, 157:215–218, 1964.

[7] Chris Godsil and Gordon Royle. *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2001.

[8] Richard H Hammack, Wilfried Imrich, Sandi Klavžar, Wilfried Imrich,

and Sandi Klavžar. *Handbook of product graphs*, volume 2. CRC press Boca Raton, 2011.

[9] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561, 2006.

[10] Mike Krebs and Anthony Shaheen. *Expander families and Cayley graphs.* Oxford University Press, Oxford, 2011. A beginner's guide.

[11] Wen-Ch'ing Winnie Li. A survey of Ramanujan graphs. In *Arithmetic, geometry and coding theory (Luminy, 1993)*, pages 127–143. de Gruyter, Berlin, 1996.

[12] Xueliang Li, Yongtang Shi, and Ivan Gutman. *Graph energy.* Springer, New York, 2012.

[13] Xiaogang Liu and Sanming Zhou. Quadratic unitary Cayley graphs of finite commutative rings. *Linear Algebra Appl.*, 479:73–90, 2015.

[14] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[15] Alexander Lubotzky. Expander graphs in pure and applied mathematics. *Bull. Amer. Math. Soc. (N.S.)*, 49(1):113–162, 2012.

[16] M. Ram Murty. Ramanujan graphs. *J. Ramanujan Math. Soc.*, 18(1):33–52, 2003.