BIRMAN-HILDEN THEORY

A THESIS

submitted in partial fulfillment of the requirements

for the award of the degree of

Master of Science

in

MATHEMATICS

by

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April 2021

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान भोपाल
Indian Institute of Science Education and Research Bhopal
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CERTIFICATE

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ABSTRACT

Let $S = S^b_{g,n}$ be the closed orientable surface of genus g with b boundary components and n punctures. Let $\text{Homeo}^+(S, \partial S)$ be the group of all the orientation-preserving homeomorphisms of S that fix the boundary of S pointwise and preserves the set of punctures. The mapping class group of S is defined as the set of connected components of $\text{Homeo}^+(S, \partial S)$. In the 1970s, Birman and Hilden contributed several significant results to the theory of 3-manifolds through a series of seminal papers that exploited the relationship between mapping class groups and covering spaces. In the first paper of the series, they derived a presentation for the mapping class group of S_2, the closed surface of genus two. In this thesis, we will study the main results in this paper. Additionally, we will explore an application of the Birman-Hilden theory to a purely algebraic problem concerning the Artin braid Group. Finally, we will also study a recent result by Ghaswala-Winarski on the liftability of mapping classes under cyclic covers of the sphere.

ACKNOWLEDGEMENT

I am indebted to my thesis supervisor Dr. Kashyap Rajeevsarathy for offering me this project and introducing me to this wonderful area of mathematics. I feel lucky to have such a spirited personality guiding me during my final year project.
I would like to express my gratitude towards Dr. Neeraj K Dhanwani for being approachable at all times and generously sharing his knowledge. I would like to thank Pankaj, Apeksha and Rajesh for making the weekly group meetings lively and informative. I would also like to thank my batchmates Sagar, Buddhabhushan, Shubham, Mandar and Aditya for making my life at IISER fun and eventful.
I am eternally indebted to my parents for their love and unwavering support.

Harsh Patil

CONTENTS

1. PRELIMINARIES

The aim of this chapter is to give a brief introduction to the theory of surfaces and mapping class groups. We also give a brief summary of results pertaining to hyperbolic structures on surfaces. We define the mapping class group and explicitly compute the mapping class groups of some surfaces. This chapter is based on Chapters 1, 2, and 4 from [2].

1.1 Geometric structures on surfaces

1.1.1 Topological surfaces

Definition 1.1.1. A *surface* is a two-dimensional manifold, that is, a second countable, Hausdorff topological space such that around every point x, there exists a neighborhood U that is homeomorphic to $\mathbb{R}^2$.

We state the following classical result without proof.

Theorem 1.1.2 (Classification theorem for closed surfaces)**.** *Any compact, connected, orientable surface is homeomorphic to S^2, or the connected sum of g tori for some $g > 0$.*

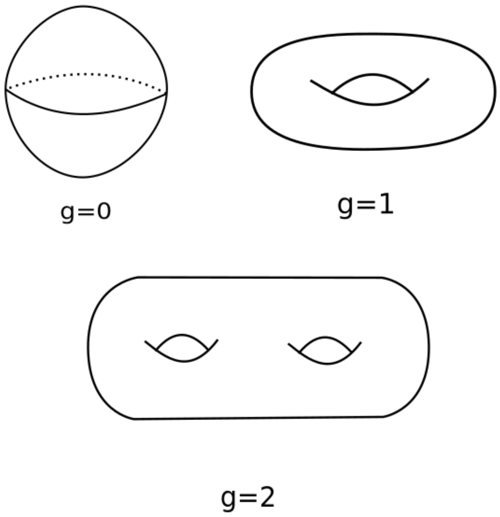

Fig. 1.1: Examples of closed orientable surfaces.

1.1.2 Hyperbolic structures on surfaces

The upper half-plane model is defined as the space

$$\mathbb{H} = \{x + iy : y > 0\}$$

together with the Riemmanian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The space $(\mathbb{H}, ds^2)$ has constant sectional curvature -1. The orientation-preserving isometries of $\mathbb{H}$ are given by

$$\mathrm{Isom}^+(\mathbb{H}) = \left\{ T(z) = \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

Proposition 1.1.3. $\mathrm{Isom}^+(\mathbb{H}) \simeq \mathrm{PSL}(2, \mathbb{R})$.

Proof. The map $\mathrm{Isom}^+(\mathbb{H}) \to \mathrm{PSL}(2, \mathbb{R})$ given by

$$\frac{az+b}{cz+d} \to \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

gives us the required isomorphism. □

Denote by ∞ the point at infinity in the Riemann sphere $\widehat{\mathbb{C}}$. The set $\widehat{\mathbb{R}} = \mathbb{R} \cup \infty$ is called the *boundary of* $\mathbb{H}$ *at infinity*. The action of any element of $\mathrm{Isom}^+(\mathbb{H})$ can be extended continuously to the set $\overline{\mathbb{H}} = \mathbb{H} \cup \widehat{\mathbb{R}}$. Elements of $\mathrm{Isom}^+(\mathbb{H})$ are classified into three types depending on the number of fixed points.

Definition 1.1.4. If an element of $\mathrm{Isom}^+(\mathbb{H})$ has one fixed point in $\mathbb{H}$, then it is said to be *elliptic*.

Example 1.1.5. Example of an elliptic isometry is

$$T(z) = \frac{cos(\theta)z + sin(\theta)}{-sin(\theta)z + cos(\theta)}.$$

Definition 1.1.6. If $T \in \mathrm{Isom}^+(\mathbb{H})$ has two fixed points in $\widehat{\mathbb{R}}$, then it is said to be *loxodromic*.

Example 1.1.7. Example of an loxodromic isometry is

$$T(z) = \lambda z, \text{ for some } \lambda > 0.$$

Definition 1.1.8. If $T \in \mathrm{Isom}^+(\mathbb{H})$ has one fixed point in $\widehat{\mathbb{R}}$, then it is said to be *parabolic*.

Example 1.1.9. Example of an parabolic isometry is

$$T(z) = z + 1$$

.

Remark 1.1.10. Any element of $\mathrm{Isom}^+(\mathbb{H})$ with 3 fixed points has to be the identity.

Definition 1.1.11. A *hyperbolic structure* on a topological surface S is a maximal collection of coordinate charts, that is, a collection of open sets $U_i \subset S$ and maps $\Phi_i : U_i \to \mathbb{H}$ such that:

(i) $\Phi_i : U_i \to \Phi_i(U_i)$ is a homeomorphism,

(ii) the collection $\{U_i\}$ cover S, and

(iii) whenever $U_i \cap U_j \neq \varnothing$, the map

$$\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)$$

is (a restriction of) a hyperbolic isometry.

Remark 1.1.12. Note that a hyperbolic structure induces a Riemannian metric of constant negative curvature on S.

Definition 1.1.13. Let X be a topological space and let $n \in \mathbb{N}$. An *orbifold chart of dimension* n is a triple $(\widetilde{U}, H, \phi)$ consisting of a connected open subset $\widetilde{U} \subset \mathbb{R}^n$, a finite group H acting on $\widetilde{U}$ and a H-invariant map $\phi : \widetilde{U} \to X$ that induces a homeomorphism between $\widetilde{U}/\Gamma$ and $\phi(\widetilde{U})$.
An *embedding* between two orbifold charts $(\widetilde{U}_i, H_i, \phi_i)$ and $(\widetilde{U}_j, H_j, \phi_j)$ is a topological embedding λ of $\widetilde{U}_i$ into $\widetilde{U}_j$ such that $\phi_j \circ \lambda = \phi_i$.

Definition 1.1.14. An *orbifold atlas* is a collection $A = \{(\widetilde{U}_i, H_i, \phi_i)\}_{i \in I}$ of charts that cover X and are compatible in the following sense: for any two charts $(\widetilde{U}_i, H_i, \phi_i)$ and $(\widetilde{U}_j, H_j, \phi_j)$ if $x \in \phi(\widetilde{U}_i) \cap \phi(\widetilde{U}_j)$, then there exists a open neighborhood $U \subset \phi(\widetilde{U}_i) \cap \phi(\widetilde{U}_j)$, and a chart $(\widetilde{U}_k, H_k, \phi_k)$ for U which embeds (in the sense defined above) into both $(\widetilde{U}_i, H_i, \phi_i)$ and $(\widetilde{U}_j, H_j, \phi_j)$.

Definition 1.1.15. An *n-dimensional orbifold* is a Hausdorff space X together with an orbifold atlas A.

The atlas A is said to be an *orbifold structure* on X.

Definition 1.1.16. A *covering* of a smooth orbifold O is a pair $(\widehat{O}, \rho)$, where $\widehat{O}$ is an orbifold and ρ is a surjective map which satisfies the following conditions:

(i) for each $x \in O$, there is a chart $(\widetilde{U}, H, \phi)$ such that $x \in U = \phi(\widetilde{U})$ and $\phi^{-1}(U)$ is a disjoint union of open sets V_i, and

(ii) each V_i admits an orbifold chart $(\widetilde{U}, H_i, \phi_i)$, where $H_i < H$ and the map $\rho_i = \rho \circ \phi_i$.

Note that the set $\widetilde{U}$ taken in conditions (i)-(ii) are the same.

Definition 1.1.17. A *universal covering* of an orbifold is a covering $p : \widehat{O} \to O$ such that given any other covering $p' : P \to O$, there exists a covering $q : \widehat{O} \to P$ such that $p = q \circ p'$.

Theorem 1.1.18. *Any connected O orbifold admits a universal covering $\rho : \widehat{O} \to O$. The universal covering is unique up to covering space isomorphisms.*

Definition 1.1.19. A surface is a *hyperbolic orbifold* if it satisfies the same conditions as that of a hyperbolic structure except that at finitely many points called *cone points*, the charts map to the quotient $U/\langle T\rangle$, T being a finite order element of $\mathrm{PSL}(2,\mathbb{R})$, and U is a neighborhood of a fixed point p of T.

For a cone point p as in definition 1.1.19, the order of the associated elliptic element T is called the *order* of that cone point.

Definition 1.1.20. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

A Fuchsian group is said to have *signature* $(g; m_1, m_2, \ldots, m_r)$ if the qoutient orbifold $\mathbb{H}/\Gamma$ has genus g and has exactly r cone points $p_1, p_2, \ldots p_r$ of orders $m_1, m_2, \ldots, m_r$, respectively.
The quantity

$$(2-2g) - \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)$$

is defined to be the *genus* of the hyperbolic orbifold $\mathbb{H}/\Gamma$.

Theorem 1.1.21. *[5, Theorem 4.3.2] If $g > 0$, $r \geq 0$ and $m_i \geq 2$, for $1 \leq i \leq r$, are integers such that*

$$(2g-2) + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) > 0,$$

then there exists a Fuchsian group Γ with signature $(g; m_1, \ldots, m_r)$.

Remark 1.1.22. Note that in the case $r = 0$ the Theorem 1.1.21 guarantees the existence of a Fuchsian group Γ such that $\mathbb{H}/\Gamma$ is homeomorphic to S_g.

Definition 1.1.23. A map $p : \widetilde{S} \to S$ is called a *branched cover* if there is a finite set $M \subset S$ such that $q : \widetilde{S}^\circ \to S^\circ$ is a covering map where $S^\circ = S - M$, $\widetilde{S}^\circ = \widetilde{S} - p^{-1}(M)$ and $q = p|_{\widetilde{S}^\circ}$.

Now let us consider a d-fold branched cover $(p, \widetilde{S}, S)$. This induces an orbifold structure on S. Let us denote this orbifold by O. Let $\widetilde{S}$ be a surface of negative Euler characteristic. The quantities $\chi(\widetilde{S})$ and $\chi(O)$ are related by the Riemann-Hurwitz equation

$$\chi(\widetilde{S}) = d\chi(O).$$

Since $\chi(\widetilde{S})$ is negative so is $\chi(O)$. By Theorem 1.1.21, there exists, a Fuchsian group Γ such that $\mathbb{H}/\Gamma \approx O$. Let $s : \mathbb{H} \xrightarrow{/\Gamma} O$ denote the orbifold covering. By Theorem 1.1.18, there exists a cover $q : \mathbb{H} \to \widetilde{S}$ such that $s = p \circ q$. It follows that the fundamental group $\pi_1(\widetilde{S})$ (seen as the group of deck transformations) is a subgroup of Γ. The deck transformations for the cover $p : \widetilde{S} \to S$ are precisely the projections of all deck transformations of the cover $\mathbb{H} \to S$ to $\widetilde{S}$. Thus, in conclusion, for a given regular covering $(p, \widetilde{S}, S)$ such that $\chi(\widetilde{S}) < 0$, one can assume without loss of generality that all the covering transformations are isometries of $\widetilde{S}$ with respect to some fixed hyperbolic metric and as a consequence, the covering map is analytic.

1.2 Mapping Class Groups

Let S be a surface of genus g with b boundary components and n marked points. $\mathrm{Homeo}^+(S, \partial S)$ denotes the set of all orientation-preserving homeomorphisms that fix the boundary pointwise and fix the set of marked points. Note that $\mathrm{Homeo}^+(S, \partial S)$ is a group under composition.

Definition 1.2.1. We say two homeomorphisms $f, g \in \mathrm{Homeo}^+(S, \partial S)$ are *isotopic* if there exists a continuous map $F : S \times [0, 1] \to S$ such that $F|_{S\times\{0\}} = f$, $F|_{S\times\{1\}} = g$, and $F|_{S\times\{t\}} \in \mathrm{Homeo}^+(S, \partial S)$, for all $t \in [0, 1]$.

Note that isotopy defines an equivalence relation on $\mathrm{Homeo}^+(S, \partial S)$. Let $\mathrm{Mod}(S)$ be the set of equivalence classes of $\mathrm{Homeo}^+(S, \partial S)$ under the equivalence relation of isotopy.

Definition 1.2.2. The *mapping class group* of S is defined as the set $\mathrm{Mod}(S)$ along with the following operation:

$$[f] \cdot [g] = [f \circ g].$$

The elements of $\mathrm{Mod}(S)$ are sometimes called *mapping classes.*

1.2.1 Basic computations of mapping class groups

The first example of a mapping class group that we give is that of the closed disk D^2.

Lemma 1.2.3. $\mathrm{Mod}(D^2)$ *is trivial.*

Proof. Let $\phi \in \mathrm{Homeo}^+(D^2, \partial D^2)$ be a homeomorphism. Then the map

$$F(x,t) = \begin{cases} (1-t)\phi(\frac{x}{1-t}), & \text{if } 0 \leq |x| \leq 1-t, \text{ and} \\ x, & \text{if } 1-t \leq |x| \leq 1, \end{cases}$$

defines an isotopy of ϕ with the identity, which proves the assertion. □

Corollary 1.2.4. $\mathrm{Mod}(D^2 - \{point\})$ *is trivial.*

Proof. Take the marked point to be at the origin. Since F (as defined in the proof of Lemma 1.2.3) always maps the origin to itself, it can be used to show that each homeomorphism ϕ of $D^2 - \{point\}$ is isotopic to identity. □

Next we consider the annulus $A := S^1 \times [0,1]$. Define $T : A \to A$, $T(\theta, t) = (\theta + 2\pi t, t)$ (as shown if Figure 1.2). Note that T fixes both of the boundary components of A pointwise.

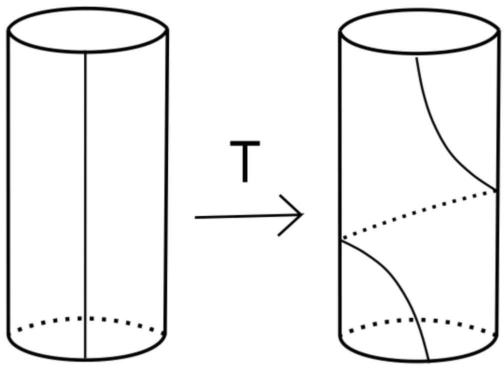

Fig. 1.2: Twist map.

Theorem 1.2.5. $\mathrm{Mod}(A) \cong \mathbb{Z} = \langle T \rangle$.

Proof. The universal cover of A is $\widetilde{A} := \mathbb{R} \times [0,1]$. Any homeomorphism $\phi : A \to A$ has a unique lift $\widetilde{\phi} : \widetilde{A} \to \widetilde{A}$ that fixes the origin. Let $\widetilde{\phi}_1$ be the restriction of ϕ to the boundary component $\mathbb{R} \times \{1\}$, which we identify canonically to $\mathbb{R}$. Since $\widetilde{\phi}_1$ is a lift of the identity map on $\mathbb{R} \times \{1\}$ of the boundary components of A. Thus, $\widetilde{\phi}_1$ translates $\mathbb{R} \times \{1\}$ by the integer $\widetilde{\phi}(0)$. We define a map $\rho : \mathrm{Mod}(A) \to \mathbb{Z} : [\phi] \overset{\rho}{\mapsto} \widetilde{\phi}_1(0)$. The map ρ is well-defined in the sense that it does not depend on the choice of the representative. Since an isotopy of homeomorphisms of A restricts to identity on the boundary, the two lifts must have identical action on $\mathbb{R} \times \{1\}$. Moreover, ρ is a homomorphism, as the composition of two homeomorphisms lift uniquely to the composition of their lifts. Thus, $\rho(f \circ g) = \widetilde{f} \circ \widetilde{g}(0) = \widetilde{f}(\widetilde{g}(0)) = \widetilde{f}(0) + \widetilde{g}(0)$.

ρ *is surjective*: It suffices to produce an element which maps to 1. We show T is one such element. Consider the path $\gamma(t) = (0,t)$. Under T, it maps to the path $T(\gamma) = (2\pi t, t)$, which lifts to the straight-line path joining the origin and $(0,1)$. Since $\widetilde{T}$ maps the lift of γ to the lift of $T(\gamma)$ and the lift of γ is the path given by $\widetilde{\gamma} = (0,t)$, it follows that $\rho(T) = 1$.

ρ *is injective:* Let f be such that $\rho(f) = 0$. Then $\widetilde{f}$ fixes the point $(0,1)$. Consequently $\widetilde{f}$ fixes both boundary components of $\widetilde{A}$. Let $\widetilde{H}(x,t)$ be the straight-line homotopy of $\widetilde{f}$ with identity. We show that $\widetilde{H}$ induces a homotopy of f with the identity. It suffices to show that $\widetilde{f}(x+n) = \widetilde{f}(x)+n$, for all $x \in \widetilde{A}$, and for all $t \in [0,1]$, since then $\widetilde{H}(x,t)$ will depend only on the image of x in A, that is,

$$\widetilde{H}(x+n,t) = (1-t)(x+n) + t\widetilde{f}(x+n) = (1-t)x + t\widetilde{f}(x) + n = \widetilde{H}(x,t) + n.$$

Now $x \in A$ and γ be an element of $\pi_1(A, x)$ corresponding to a covering transformation τ. Let $\widetilde{\gamma}$ be a lift of γ starting at some $\widetilde{x}$ in the fiber of x. It is evident that the endpoint of γ is $\tau(\widetilde{x})$. Thus, $\widetilde{f}(\widetilde{\gamma})$ is a path between $\widetilde{f}(\widetilde{x})$ and $\widetilde{f}\tau(\widetilde{x})$. Since $\widetilde{f}(\widetilde{\gamma})$ is a lift of $f(\gamma)$ and $\phi_*(\tau)$ is the unique covering transformation sending the starting point of the lift of $f(\gamma)$ to its end point, we have

$$\widetilde{f}\tau(\widetilde{x}) = f_*(\tau)\widetilde{f}(\widetilde{x}).$$

But f preserves ∂A pointwise, so we have that f_* is the identity automorphism of $\pi_1(A)$. Thus,

$$\widetilde{f}\tau(\widetilde{x}) = \tau\widetilde{f}(\widetilde{x}),$$

for any covering transformation. As every covering transformation is given by a translation by some integer, we are done. □

1.2.2 Dehn Twists and Half Twists

Definition 1.2.6. A *simple closed curve* on a surface S is a continuous injective map $f : S^1 \to S$.

Definition 1.2.7. Two simple closed curves a and b are said to be *isotopic* if there exists a continuous map $F : S^1 \times [0,1] \to S$ such that $F|_{S^1\times\{0\}} = a$, $F|_{S^1\times\{1\}} = b$, and the map $F|_{S^1\times\{t\}}$ is injective for each $t \in [0,1]$.

Definition 1.2.8. Let a and b be isotopy classes of simple closed curves on a surface. The *geometric intersection number* of a and b is defined to be the minimal number of intersection points between a representative curve in the class a and a representative curve in class b, that is,

$$i(a,b) = \min\{|\alpha \cup \beta| : \alpha \in a, \beta \in b\}.$$

Definition 1.2.9. Let α be a simple closed curve in S. Let N be a regular neighborhood of α and choose an orientation-preserving homeomorphism

$\phi : A \to N$. We define the *Dehn twist* about α to be the following homeomorphism:

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x), & \text{if } x \in N, \text{and} \\ x, & \text{if } x \in N/S, \end{cases}$$

where $T : A \to A$ is the homeomorphism as defined in Theorem 1.2.5 .

From here on, we will make no distinction between the homeomorphism T_α defined above and its isotopy class which is independent of the choice of annular neighborhood of α. Moreover, if a and b are two isotopic curves then T_a and T_b are isotopic homeomorphisms. Thus, for a isotopy class β of a simple closed curve, the mapping class T_β is well-defined.

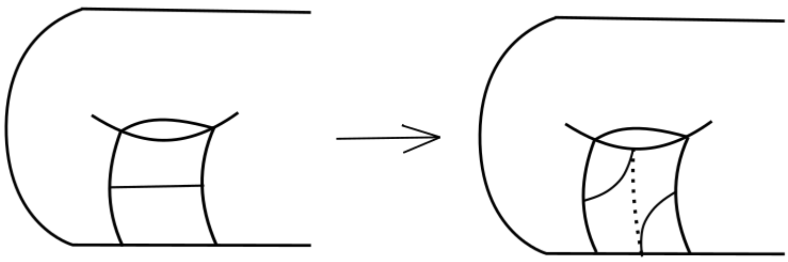

Fig. 1.3: A Dehn twist on a surface.

Definition 1.2.10. A proper *arc* in a surface S with a finite set of marked points P is a map $\alpha : [0,1] \to S$ such that $\alpha^{-1}(P \cup \partial S) = \{0,1\}$, that is, it intersects the boundary or the marked points only at the endpoints.

Definition 1.2.11. Let S be a surface with at least two marked points. Let α be an arc connecting two marked points. Let D be a closed disk containing the arc α. Parametrize the disk by radial coordinates (r, θ) such that the two marked points are situated at $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \pi)$ respectively. The homeomorphism H_α given by

$$\begin{cases} x = (r,\theta) \mapsto (r, \theta - 2\pi r), & \text{if } x \in D, \text{ and} \\ x \mapsto x, & \text{if } x \in S/D, \end{cases}$$

is called the *half-twist* about the arc α.

The effect of a half-twist is illustrated in Figure 1.4.

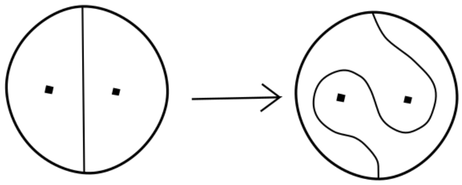

Fig. 1.4: Half twist.

Note that H_α interchanges the two punctures and leaves all other punctures fixed.

Lemma 1.2.12. *Let* $f \in \mathrm{Homeo}^+(S)$ *and let* a *be a simple closed curve. Then we have*

$$fT_a f^{-1} = T_{f(a)}.$$

Proposition 1.2.13. *If* a *and* b *are isotopy classes of simple closed curves in a surface* S *with* $i(a,b) = 1$*, then*

$$T_a T_b T_a = T_b T_a T_b.$$

Definition 1.2.14. A *chain* is a sequence of simple closed curves $c_1, c_2, \ldots, c_k$ in a surface S such that $i(c_i, c_{i+1}) = 1$ and $i(c_i, c_j) = 0$ when $|i - j| > 1$.

Theorem 1.2.15 (Chain relation)**.** *Let* $k \geq 0$ *and let* $c_1, \ldots, c_k$ *be a chain of curves in a surface* S*. If we take representatives for the* c_i *that are in minimal position and then take a closed regular neighborhood of their union, then the boundary of this neighborhood consists of one or two simple close curves, depending on whether* k *is even or odd. Denote the isotopy classes of these boundary curves by* d *in the even case and by* d_1 *and* d_2 *in the odd case. Then the following relations hold in* $\mathrm{Mod}(S)$*:*

$$(T_{c_1} T_{c_2} \ldots T_{c_k})^{2k+2} = T_d \text{ for even } k, \text{ and}$$

$$(T_{c_1} T_{c_2} \ldots T_{c_k})^{k+1} = T_{d_1} T_{d_2} \text{ for odd } k.$$

Example 1.2.16. Application of the chain relation to the chain of two curves indicated in Figure 1.5 gives us the relation $(T_a T_b)^6 = T_c$.

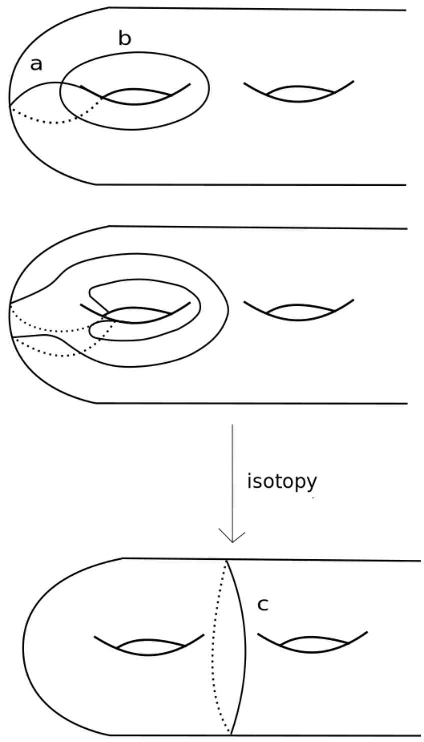

Fig. 1.5: Chain relation on a chain of size 2.

1.2.3 Finite generation of mapping class groups

We state the following theorems without proof:

Theorem 1.2.17 (Dehn-Lickorish[2]). *For $g \geq 1$, the Dehn Twists about the isotopy classes $a_1, \ldots, a_g, c_1, \ldots, c_{g-1}, m_1, \ldots, m_g$, shown in Figure 1.6, generate* $\mathrm{Mod}(S_g)$.

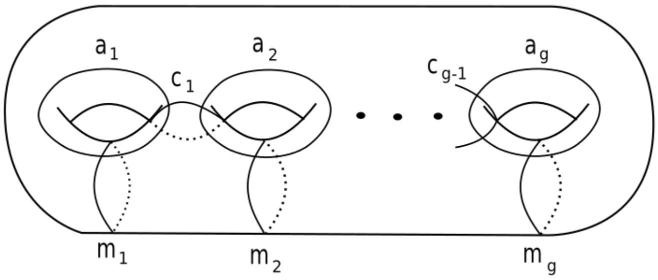

Fig. 1.6: Dehn twists about these $3g-1$ simple closed curves generate $Mod(S_g)$.

Theorem 1.2.18 (Humphries [2])**.** *For $g \geq 1$, the Dehn Twists about the isotopy classes $a_1, \ldots, a_g, c_1, \ldots, c_{g-1}, m_1$, and m_2 generate* $\mathrm{Mod}(S_g)$.

Definition 1.2.19. We say that a collection of curves $c_1, \ldots, c_n$ *fill* a surface S if the closure of $S - \cup_{i=1}^{n}\{c_i\}$ is homeomorphic to a union of disjoint disks.

Theorem 1.2.20. *Let S be a compact surface, possibly with marked points, and let $\phi \in \mathrm{Homeo}^+(S, \partial S)$. Let $c_1, \ldots, c_n$ be a collection of essential simple closed curves in S such that:*

(i) each pair c_i, c_j is in minimal position for $i \neq j$,

(ii) c_i is not isotopic to c_j, for $i \neq j$,

(iii) For distinct i, j, k atleast one of the $c_i \cap c_j$, $c_j \cap c_k$ and $c_i \cap c_k$ is empty, and

(iv) the collection $\{c_i\}$ fills S.

If $\phi(c_i)$ is isotopic to c_i for each i, then ϕ is isotopic to the identity.

Corollary 1.2.21. *If for a homeomorphism f of a surface S the induced isomorphism f_* of $\pi_1(S)$ is identity. Let δ be a fixed path between x and $f(x)$ in S and let f_* be the isomorphism of $\pi_1(S)$ given by $f_*([\gamma]) = [\delta][f(\gamma)][\delta^{-1}]$. If f_* is identity, then f is isotopic to identity.*

Proof. Pick a collection of simple closed curves c_i that fill S, and let γ_i be elements of the fundamental group such that c_i is homotopic to γ_i. One such collection of curves for the genus 2 surface is shown in Figure 1.2.3 below. Since f_* is identity we have that $\gamma_i \simeq \delta * f(\gamma_i) * \delta^{-1}$. When cosidered as maps from S^1 we have that $\delta * f(\gamma_i) * \delta^{-1}$ is homotopic to $f(\gamma_i)$. Thus, γ_i is isotopic to $f(\gamma_i)$, for each i, and so $f(c_i)$ is isotopic to c_i, for each i. The statement now follows from Theorem 1.2.20. □

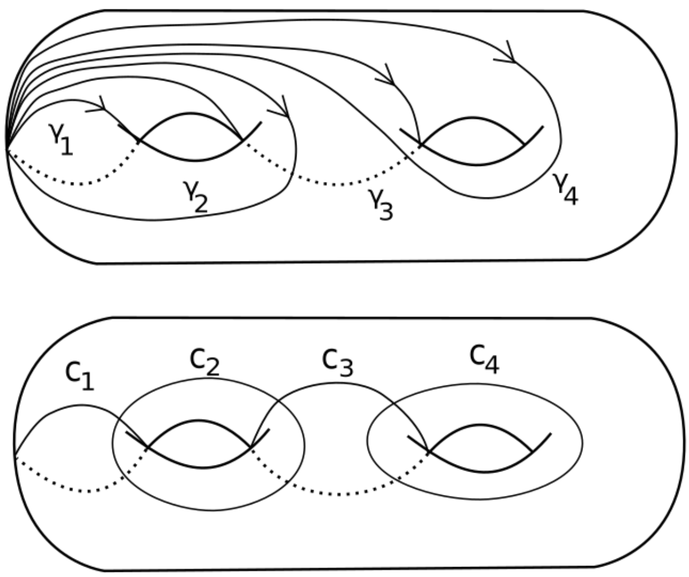

Fig. 1.7: A collection of curves that fill S_2.

1.2.4 Mapping class group of $S_{0,n}$

In this section, we derive a presentation for the mapping class group $\mathrm{Mod}(S_{0,n})$. We will need this in the next chapter to derive a presentation for $\mathrm{Mod}(S_2)$. We define D_n to be disk D^2 with n marked points. Let α_i be the arcs in D_n shown in Figure 1.8 below.

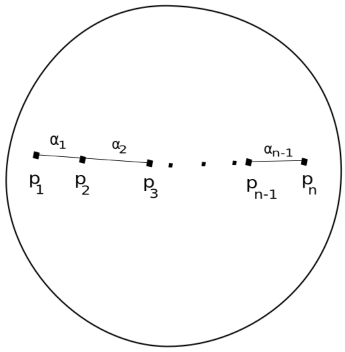

Fig. 1.8: Generators of $\mathrm{Mod}(D_n)$.

Denote by σ_i, the half-twist about the arc α_i.

Theorem 1.2.22. $\mathrm{Mod}(D_n)$ *is generated by* $n-1$ *half twists* $\{\sigma_i\}_{i=1}^{n-1}$ *and admits the following presentation:*

(i) $[\sigma_i, \sigma_j] = 1$, *for all* i, j *such that* $|i-j| \geq 2$, *and*

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, *for all* $1 \leq i \leq (n-2)$.

Capping Homomorphism

Let S be a surface with a non-empty boundary. Let S' be the surface obtained by gluing together the boundaries of a once-punctured disk and S with a homeomorphism. This is called *capping* a boundary component of S. Given a homeomorphism of S, one can extend it to a homeomorphism of S' by simply defining it to be identity on the one punctured disk. Thus, there is a well-defined homomorphism $Cap : \mathrm{Mod}(S) \to \mathrm{Mod}(S')$.

Theorem 1.2.23. *Let* S' *be the surface obtained from a surface* S *by capping the boundary component* β *with a once-marked disk, and denote the marked point in this disk as* x. *Let* $\mathrm{Mod}(S', x)$ *denote the subgroup of* $\mathrm{Mod}(S')$ *consisting of elements that fix the marked point* x, *and let* $Cap : \mathrm{Mod}(S) \to \mathrm{Mod}(S', x)$ *be the induced homomorphism. Then* Cap *is surjective and its kernel is generated by the Dehn Twist* T_α.

Birman exact sequence

Let S be a surface and denote by $\mathrm{Mod}(S,x)$ the subgroup of mapping class group $\mathrm{Mod}(S-x)$ that preserves the puncture coming from x. Since any homeomorphism of S fixing x is apriori a homeomorphism of S, one can define a homomorphism $\mathrm{Mod}(S,x) \to \mathrm{Mod}(S)$. The exact description of the kernel of this homomorphism is given by the *Birman exact sequence.*

Theorem 1.2.24. *The groups* $\pi_1(S,x)$, $\mathrm{Mod}(S,x)$ *and* $\mathrm{Mod}(S)$ *fit into an exact sequence:*

$$1 \to \pi_1(S,x) \xrightarrow{Push} \mathrm{Mod}(S,x) \xrightarrow{Forget} \mathrm{Mod}(S) \to 1.$$

For a simple loop, α the image $Push([\alpha])$ can be described as follows. Let a and b be the isotopy classes of the simple closed curves in (S,x) obtained by pushing α off itself to the left and right, respectively, as shown in Figure 1.9.

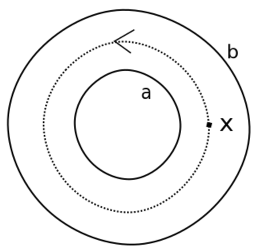

Fig. 1.9: Image of a simple loop under the *Push* map

Then $Push([\alpha]) = T_a T_b^{-1}$. Capping a n-times punctured disk gives us a sphere with $(n+1)$ punctures $S_{0,n+1}$ and $\mathrm{Mod}(D_n)$ surjects onto the subgroup $\mathrm{Mod}(S_{0,n+1},x)$ of $\mathrm{Mod}(S_{0,n+1})$. We can further compose the *Cap* homomorphism with the *Forget* homomorphism so that we get a surjective homomorphism from $\mathrm{Mod}(D_n)$ to $\mathrm{Mod}(S_{0,n})$ given by:

$$\mathrm{Mod}(D_n) \xrightarrow{Cap} \mathrm{Mod}(S_{0,n+1},x) \xrightarrow{Forget} \mathrm{Mod}(S_{0,n}).$$

Note each of the homomorphisms in the sequence above is sujective. The kernel of *Cap* is $\langle T_d \rangle$, where d is a curve isotopic to the boundary and the kernel of $Forget$ is $Push(\pi_1(S_{0,n}, x))$. Thus, the kernel of $Forget \circ Cap$ is $\langle T_d, Cap^{-1}(Push(\pi_1(S, x)))\rangle$. The fundamental group, $\pi_1(S_{0,n}, x)$ is generated by loops $\{\alpha_i\}_{i=1}^n$ such that α_i goes around the puncture P_i as shown in Figure 1.10.

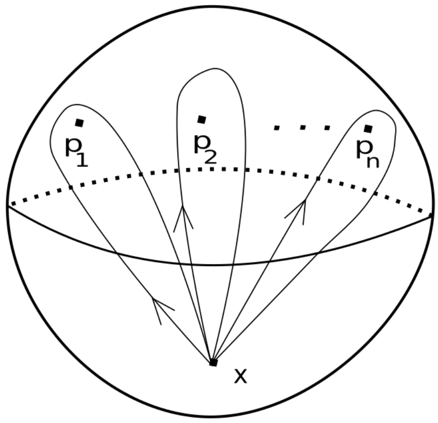

Fig. 1.10: Generators of $\pi_1(S_{0,n}, x)$.

Thus, $Push(\pi_1(S_{0,n}, x)) = \langle\{Push(\alpha_i)\}_{i=1}^n\rangle$. Also note that, we have $Push(\alpha_1) = T_a T_b^{-1}$, where a and b are the curves indicated in Figure 1.11.

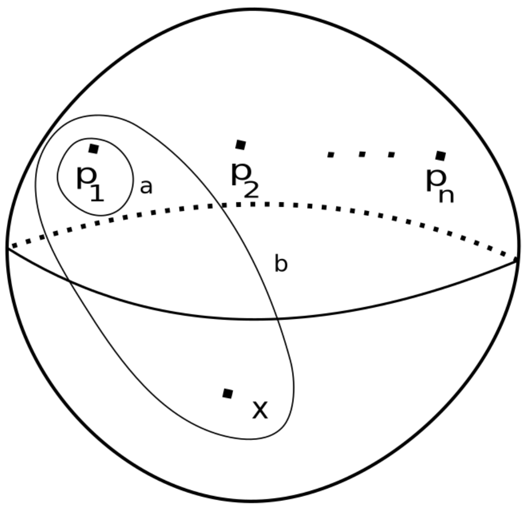

Fig. 1.11: The curves a and b.

But since T_a is supported on a disk with one marked point, it is trivial.

We need to find a preimage of T_b^{-1} under the Cap map. Since b is isotopic to the curve c shown in Figure 1.12 below, T_c^{-1} is a preimage of $Push(\alpha_1)$ under the Cap map.

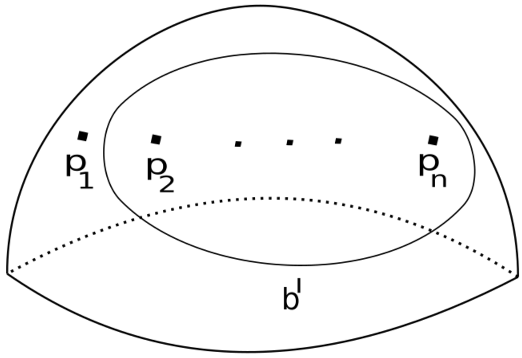

Fig. 1.12: pre-image of $Push(\alpha_i)$ under Cap.

Similarly for other α_i's, we can get an element of the preimage of $Push(\alpha_i)$ under the Cap map, which is a Dehn twist in $\mathrm{Mod}(D_n)$ about a curve that encloses $n-1$ points as shown in Figure 1.12.

Lemma 1.2.25. *$Push(\alpha_i)$ is a conjugate of $Push(\alpha_1)$, for $2 \leq i \leq n$.*

Proof. From the preceding discussion, each $Push(\alpha_i)$ is a Dehn Twist about a curve enclosing $n-1$ punctures. Since there always exists a homeomorphism that maps one curve enclosing $n-1$ points to another such curve, the statement follows from Lemma 1.2.12. □

Lemma 1.2.26. *For d and b' be as defined above, we have:*

(i) $T_d = (\sigma_1\sigma_2 \dots \sigma_{n-1})^n$, *and*

(ii) $T_{b'} = \sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \dots \sigma_2\sigma_1$.

Define $s_i := Forget \circ Cap(\sigma_i) = H_{\alpha_i}$.

Theorem 1.2.27. *$\mathrm{Mod}(S_{0,n})$ is generated by $\{s_1, \dots, s_{n-1}\}$ and admits the following presentation:*

(i) $[s_i, s_j] = 1$, *for all i, j such that $|i-j| \geq 2$,*

(ii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, *for* $1 \leq i \leq (n-2)$,

(iii) $(s_1 s_2 \ldots s_{n-1})^n = 1$, *and*

(iv) $s_1 s_2 \ldots s_{n-2} s_{n-1}^2 s_{n-2} \ldots s_2 s_1 = 1$.

Proof. Since $\{\sigma_1, \ldots, \sigma_{n-1}\}$ generates $\mathrm{Mod}(D_n)$ and $Forget \circ Cap$ is surjective, it follows that $\{s_1, \ldots, s_{n-1}\}$ generates $\mathrm{Mod}(S_{0,n})$. Moreover, as $ker(Forget \circ Cap)$ is given by $\langle\langle(\sigma_1 \sigma_2 \ldots \sigma_{n-1})^n, \sigma_1 \sigma_2 \ldots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \ldots \sigma_2 \sigma_1\rangle\rangle$, we get the desired presentation for $\mathrm{Mod}(S_{0,n})$. □

2. BIRMAN-HILDEN THEORY

In this chapter, we discuss the theory initiated by Birman-Hilden in their seminal paper [1].

2.1 Introduction

Let $\widetilde{S}$, S be orientable surfaces. Let $p : \widetilde{S} \to S$ be a regular covering either branched or unbranched. Let $\phi : S \to S$ be a homeomorphism. By a *lift* of ϕ to $\widetilde{S}$, we mean a homeomorphism $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{\phi}} & \widetilde{S} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{\phi} & S \end{array}$$

Note that for any two points on $\widetilde{S}$ say $\widetilde{x}, \widetilde{x}'$ lying in the fiber of $x \in S$, we have:

$$p\widetilde{\phi}(\widetilde{x}) = \phi p(\widetilde{x}) = \phi p(\widetilde{x}') = p\widetilde{\phi}(\widetilde{x}').$$

Thus, $\widetilde{\phi}(\widetilde{x}), \widetilde{\phi}(\widetilde{x}')$ belong to the fiber of $\phi(x)$. In other words, $\widetilde{\phi}$ sends fibers to fibers. This motivates the following definition:

Definition 2.1.1. Let $(p, \widetilde{S}, S)$ be a regular cover (possibly branched). A homeomorphism $\psi : \widetilde{S} \to \widetilde{S}$ is said to be *fiber-preserving* if for every pair of points x, x' with $p(x) = p(x')$, we have that $p\psi(x) = p\psi(x')$.

If a homeomorphism is a lift of some homeomorphism, then it is necessarily fiber-preserving. Conversely, any fiber-preserving homeomorphism of $\widetilde{S}$ induces a homeomorphism of S.

Theorem 2.1.2. *A homeomorphism $f : \widetilde{S} \to \widetilde{S}$ is fiber-preserving if and only if it lies in the normalizer of the group of covering transformations.*

Proof. Let x and x' be two points in the same fiber. Let τ be a covering transformation such that $\tau(x) = x'$. Then $f\tau f^{-1} = \tau'$, for some covering transformation τ', and so we have

$$f(x') = f\tau(x) = \tau' f(x).$$

Thus, it follows that $f(x)$ and $f(x')$ lie in the same fiber and f is fiber-preserving.
Conversely, suppose f is a fiber-preserving homeomorphism and τ is a covering transformation. Suppose that f maps the fiber of x to the fiber of y. Then f^{-1} will map the fiber of y to x. Since τ is a covering transformation it maps every fiber to itself. So $f\tau f^{-1}$ also maps every fiber to itself, and hence it is a covering transformation. Therefore, f lies in the normalizer of the group of covering transformations. □

Theorem 2.1.3. *A homeomorphism $f : S \to S$ lifts to a homeomorphism $\widetilde{f}$ if and only if for every $x \in S$ we have $f_* p_* \pi_1(\widetilde{S}, \widetilde{x}) = p_* \pi_1(\widetilde{S}, \widetilde{x}')$, where $\widetilde{x}$ and $\widetilde{x}'$ are points lying above x and $f(x)$.*

Proof. Let $f : S \to S$ be a homeomorphism that lifts to a homeomorphism $\widetilde{f}$. Let γ be a closed curve that lifts to a closed curve $\widetilde{\gamma}$. Then $f(\gamma) = f \circ p(\widetilde{\gamma}) = p \circ \widetilde{f}(\widetilde{\gamma})$. Thus, $f(\gamma)$ lifts to a closed curve $\widetilde{f}(\widetilde{\gamma})$ ie $f(\gamma) \in p_* \pi_1(\widetilde{S}, \widetilde{x}')$.
Conversely assume that $f_* p_* \pi_1(\widetilde{S}, \widetilde{x}) = p_* \pi_1(\widetilde{S}, \widetilde{x}')$. Let $x \in S$ and fix a $y_0 \in p^{-1}(x)$. Let $y_1 = y_0$. By the lifting criterion,[4, Proposition 1.33] there exists a unique map $\widetilde{f} : (\widetilde{S}, y_0) \to (\widetilde{S}, y_1)$ that is a lift of $p \circ f : (\widetilde{S}, y_0) \to (S, f(x))$. It remains to show that $\widetilde{f}$ is a homeomorphism. Similarly, by lifting f^{-1} we get a map $\widetilde{f}^{-1} : (\widetilde{S}, y_1) \to (\widetilde{S}, y_0)$ and we have the following commutative diagram:

$$\begin{array}{ccccc} (\widetilde{S}, y_0) & \xrightarrow{\widetilde{f}} & (\widetilde{S}, y_1) & \xrightarrow{\widetilde{f}^{-1}} & (\widetilde{S}, y_0) \\ \downarrow p & & \downarrow p & & \downarrow p \\ (S, x) & \xrightarrow{f} & (S, f(x)) & \xrightarrow{f^{-1}} & (S, x) \end{array}$$

Since $\widetilde{f} \circ \widetilde{f}^{-1}$ is a lift of the identity map and since $\widetilde{f} \circ \widetilde{f}^{-1}(y_0) = y_0$, it must be the identity map. Thus, $\widetilde{f}$ is a continuous map with a continuous inverse. □

Remark 2.1.4. It follows from Theorem 2.1.3 that every homeomorphism lifts to a homeomorphism of the universal cover of a surface.

If two fiber-preserving homeomorphisms g_1, g_2 are isotopic via an isotopy $G : S \times [0,1] \to S$, then this may not give us an isotopy between the corresponding induced homeomorphisms. The reason being that the intermediate homeomorphisms $G(s, \star)$ may not be fiber-preserving.

Definition 2.1.5. If two fiber-preserving homeomorphisms g_1 and g_2 are isotopic via some isotopy G such that each $G(s, \star)$ is a fiber-preserving homeomorphism then we say that they are *fiber-isotopic*.

Given two fiber-preserving homeomorphisms one can ask whether they are fiber-isotopic. Note that g_1 and g_2 being fiber-isotopic to each other is equivalent to $g_1 g_2^{-1}$ being fiber-isotopic to the identity.

Definition 2.1.6. We say a cover $(p, \widetilde{S}, S)$ has the *Birman-Hilden Property* if for every fiber-preserving homeomophism f of $\widetilde{S}$, we have that f is isotopic to identity if and only if it is fiber-isotopic to identity.

2.2 Main theorems

In this section, we state and prove the two main theorems from [1] which establish the Birman-Hilden property for certain covers.

Theorem 2.2.1. *Let $(p, \widetilde{S}, S)$ be a regular cover either branched or unbranched, with a finite group of covering transformations and a finite number of branch points. Let the covering transformations leave each branch point fixed. In the case of a branched covering, assume, $\widetilde{S}$ is not the closed sphere or the closed torus. Then $(p, \widetilde{S}, S)$ has the Birman-Hilden Property.*

Theorem 2.2.2. *Let $(p, \widetilde{S}, S)$ be a regular cover either branched or unbranched with at most finitely many branch points. Let the group of covering transformations be finite and solvable. Then $(p, \widetilde{S}, S)$ has the Birman-Hilden Property.*

Note that in the Theorem 2.2.2, we have removed the restriction that each covering transformation must fix all branch points and at the same time, imposed the condition that the group of covering transformations must be solvable.

2.2.1 Proof of theorem 2.2.1

We present the proof of Theorem 2.2.1 via a sequence of lemmas. Following the discussion in Section 1.1.2, we shall assume that all covering transformations are isometries and as a consequence the covering map p is analytic. We will also assume $\widetilde{S}$ is not homeomorphic to S^2, T^2, $S_{0,1}$, or $S_{0,2}$, that is, $\chi(\widetilde{S}) < 0$. This allows us to use the following two facts about $\widetilde{S}$:

(a) The universal cover of $\widetilde{S}$ is $\mathbb{H}$, and

(b) the center of $\pi_1(\widetilde{S})$ is trivial.

We will deal with the cases where $\widetilde{S}$ is $S_{0,1}$, or $S_{0,2}$, separately. The cases T^2 and S^2 have been excluded from the hypothesis of Theorem 2.2.1 as the statement does not hold in these cases.

Lemma 2.2.3. *Let f be a non-trivial homeomorphism of $\widetilde{S}$ with a fixed point P. Let f_* be the induced automorphism of $\pi_1(\widetilde{S}, P)$. Then f_* leaves no element of $\pi_1(\widetilde{S}, P)$ fixed except the identity.*

Proof. Let $[\gamma] \in \pi_1(\widetilde{S}), [\gamma] \neq 1$ such that $f(\gamma) \simeq \gamma$. Lift f to an element of $\text{Isom}^+(\mathbb{H})$ say $\widetilde{f}$. Let $\widetilde{P}$ be a point such that $p(\widetilde{P}) = P$. By pre-composing with a covering transformation, if necessary, we can assume that $\widetilde{f}(\widetilde{P}) = \widetilde{P}$. Since γ is a loop based at P, it lifts to a path starting at $\widetilde{P}$ and ending at some point $\widetilde{Q}$ in the pre-image of P. Since $f(\gamma) \simeq \gamma$, it follows that $\widetilde{f}(\widetilde{Q}) = \widetilde{Q}$. The only element of $\text{Isom}^+(\mathbb{H})$ that has two points fixed points (in $\mathbb{H}$) is the identity. Thus, $\widetilde{f}$ is the identity which implies that f is identity. □

Lemma 2.2.4. *Let g be a fiber-preserving homeomorphism of $\widetilde{S}$ which is isotopic to identity. Then g commutes with covering transformations.*

Proof. Let t be a covering transformation, and let $r = gtg^{-1}t^{-1}$. Since g preserves fibers, it follows the gtg^{-1} is a covering transformation. Therefore, r is also a covering transformation, and hence an analytic homeomorphism of $\widetilde{S}$. Since every branch point has exactly one pre-image under p, every covering transformation (and in particular r) must fix each branch point. Moreover, as g is isotopic to identity, tgt^{-1} is isotopic to identity, and so r is isotopic to identity. Thus, r induces an inner automorphism on $\pi_1(\widetilde{S})$ so that $r_*(x) = yxy^{-1}$, for some $y \in \pi_1(\widetilde{S})$. Since r_* leaves y fixed, by Lemma 2.2.3 we get that r must be the identity. □

Let $P_1, P_2, \ldots, P_n \in S$ be the branch points. By our hypothesis, the covering transformations fix each branch point. The orbit-stabilizer theorem would then imply that pre-image of each branch point is a single point. Let us denote the corresponding pre-images by $\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_n \in \widetilde{S}$.

Lemma 2.2.5. *Let g be a fiber-preserving homeomophism of $\widetilde{S}$ that is isotopic to identity via an isotopy $G(s, \star)$. Then:*

(i) $g(\widetilde{P}_i) = \widetilde{P}_i$, for all i, and

(ii) $G(s, \widetilde{P}_i)$ is nullhomotopic in $\pi_1(\widetilde{S}, \widetilde{P}_i)$.

Proof. We assume on the contrary that $g(\widetilde{P}_i) = \widetilde{P}_j$, for $i \neq j$. Let γ be a loop based at $\widetilde{P}_i$ and t be a non-trivial covering transformation. By Lemma 2.2.4, we have $gt(\gamma) = tg(\gamma)$. Let β be the path $G(s, \widetilde{P}_i)$. Then $\gamma \simeq \beta g(\gamma)\beta^{-1}$, and applying t to this homotopy, we get

$$t(\gamma) \simeq t(\beta)gt(\gamma)t(\beta^{-1}). \tag{2.1}$$

Since γ an arbitrary loop based at $\widetilde{P}$, taking $t(\gamma)$ in place of γ, we get

$$t(\gamma) \simeq \beta tg(\gamma)\beta^{-1}. \tag{2.2}$$

From equations (2.1) and (2.2) we get

$$\beta^{-1}t(\gamma)gt(\gamma)(\beta^{-1}t(\gamma))^{-1}) \simeq gt(\gamma). \tag{2.3}$$

Since γ was arbitrary and g and t are homeomorphisms, $\beta t(\beta)^{-1}$ commutes with all elements of $\pi_1(\widetilde{S}, \widetilde{P}_i)$. Moreover, as $\pi_1(\widetilde{S})$ has trivial center $[\beta^{-1}t(\beta)] = 0$, and thus, $t(\beta) \simeq \beta$, via a homotopy that keeps the endpoints $\widetilde{P}_i$ and $\widetilde{P}_j$ fixed throughout. Now let $\widetilde{t}$ be a lift of t to $\mathbb{H}^2$. By composing $\widetilde{t}$ with a covering transformation, if necessary, we may assume $\widetilde{t}(\widetilde{Q}_i) = \widetilde{Q}_i$, for some $\widetilde{Q}_i$ in the fiber of $\widetilde{P}_i$. Let $\widetilde{Q}_j$ be the endpoint of the lift of β starting at $\widetilde{Q}_i$. Since $\beta \simeq t(\beta)$, we have $\widetilde{t}(\widetilde{Q}_j) = \widetilde{Q}_j$. Thus, $\widetilde{t}$ has two fixed points, and so $\widetilde{t}$ is identity. This contradicts the fact that t is a non-trivial covering transformation.
Let t be a non-trivial covering transformation. The homotopy $t(\gamma) \simeq \beta gt(\gamma)\beta^{-1}$ is still valid. By *(i)*, β is closed loop based at $\widetilde{P}_i$, and it follows from equation2.3 that $\beta^{-1}t(\beta)$ commutes with $gt(\gamma)$. Thus, $\beta^{-1}t(\beta)$ lies in the center of $\pi_1(\widetilde{S}, \widetilde{P}_i)$ and so, $\beta \simeq t(\beta)$. Since t is a non-trivial covering transformation, by Lemma 2.2.3, we get $\beta \simeq 0$. □

We state the following lemma without proof. The statement follows from the simplicial approximation theorem [4, Theorem 2C.1] and the isotopy extension theorem.

Lemma 2.2.6. *Let P be a point in a piecewise-linear manifold X without boundary. Let $\beta(s)$ be a curve in X homotopic to 0 in $\pi_1(X, P)$. There is an isotopy $K(s, \star)$ of X such that $K(0, \star) = (K, \star) = id$, where $K(s, \star)$ has compact support and $K(s, P) = \beta(s)$.*

Lemma 2.2.7. *Let g be a fiber-preserving homeomorphism of $\widetilde{S}$ which is isotopic to the identity map via an isotopy $G(s, \star)$. Then there is another isotopy $\overline{G}(s, \star)$ of g with identity such that $\overline{G}(s, P_i) = P_i$, for all $0 \leq s \leq 1$.*

Proof. By Lemma 2.2.5, $g(\widetilde{P}_1) = \widetilde{P}_1$ and $\beta(s) = g_s(\widetilde{P}_1) \simeq 0$ in $\pi_1(\widetilde{S}, \widetilde{P}_1)$. By Lemma 2.2.6 there is an isotopy $K(s, \star)$ of $\widetilde{S}$ with $K(0, \star) = K(1, \star) = id$ such that $k_s(\widetilde{P}_1) = \beta(s)$. Let $H(s, \star) = K(s, \star)^{-1}G(s, \star)$ so that $H(s, \widetilde{P}_1) =$

$\widetilde{P}_1$. Now, consider the cover $(p, \widetilde{S} - \widetilde{P}_1, S - P_1)$ and the homeomorphism $g' = g|_{\widetilde{S}-\widetilde{P}_1}$. Thus, $H(s, \star)|_{\widetilde{S}-\widetilde{P}_1}$ is an isotopy of g' to identity. Note that the cover $(p, \widetilde{S} - \widetilde{P}_1, S - P_1)$ also satisfies the hypotheses of Lemma 2.2.6. By repeating the argument for $\widetilde{P}_2, \widetilde{P}_3, \ldots, \widetilde{P}_n$, one can achieve an isotopy with the desired property. □

Lemma 2.2.8. *Let $(q, \widetilde{Y}, Y)$ be a regular, unbranched covering space, where $\widetilde{Y}$ and Y are connected oriented surfaces. Let $\widetilde{g} : \widetilde{Y} \to \widetilde{Y}$ be a fiber-preserving homeomorphism which is isotopic to identity. Let the centralizer of $q_*(\pi_1(\widetilde{Y}))$ in $\pi_1(Y)$ be trivial. Then $\widetilde{g}$ is fiber-isotopic to identity.*

Proof. Since $\widetilde{g}$ is fiber-preserving, it projects to a homeomorphism g of S. Pick points $P \in Y$ and $\widetilde{P} \in \widetilde{Y}$ such that $q(\widetilde{P}) = P$. Let $G(s, \star)$ be an isotopy of $\widetilde{g}$ with the identity. Consider the path $\widetilde{\beta}(s) = G(s, \widetilde{P})$, and let β be a projection of $\widetilde{\beta}$ to Y. Let g_* be the automorphism of $\pi_1(Y, P)$ induced by g given by $g_*([\gamma]) = [\beta g(\gamma)\beta^{-1}]$. If $\gamma \in q_*(\pi_1(\widetilde{Y}, \widetilde{P}))$, then $\gamma = q(\widetilde{\gamma})$ for some $\gamma \in \pi_1(\widetilde{Y}, \widetilde{P})$. Now since $\widetilde{\gamma} \simeq \widetilde{\beta}\widetilde{g}(\widetilde{\gamma})\widetilde{\beta}^{-1}$, we have $g_*(\gamma) = \gamma$. Thus, g_* restricted to $q_*(\pi_1(\widetilde{Y}, \widetilde{P}))$ is the identity.
Now choose any $\alpha \in q_*(\pi_1\widetilde{Y}, \widetilde{P})$ and any $\delta \in \pi_1(Y, P)$. Since the covering is regular, we have $\delta\alpha\delta^{-1} \in q_*(\pi_1(\widetilde{Y}, \widetilde{P}))$. Thus, $\delta\alpha\delta^{-1} = g_*(\delta\alpha\delta^{-1}) = g_*(\delta)\alpha g_*(\delta)^{-1}$ and $(g_*(\delta)^{-1}\delta)\alpha(\delta^{-1}g_*(\delta)) = \alpha$. So, $g_*(\delta)^{-1}\delta$ lies in the centralizer of $\pi_1(Y)$ and is therefore trivial. Since the choice of δ was arbitrary, g_* is the identity. By Corollary 1.2.21, g is isotopic to the identity via an isotopy that can be lifted to a fiber-isotopy of $\widetilde{g}$ to identity. □

Proof of Theorem 2.2.1.
We divide our argument into three cases.
Case I: p is unbranched. We first consider the case where S is not homeomorphic to one of $\{S^2, S_{0,1}, S_{0,2}, T^2\}$. It suffices to check whether the hypothesis of the Lemma 2.2.8 are satisfied, that is, $p_*\pi_1(\widetilde{X})$ has trivial centralizer in $\pi_1(S)$. Since S is a surface, $\pi_1(S)$ is either a group of the form $\langle a_1 \ldots a_g, b_1 \ldots b_g | \Pi_{i=1}^{g}[a_i, b_i]\rangle$, for some $g > 2$, or a free group of finite rank (in case S has punctures). If it is a free group, it must have rank greater than 3 (since we have excluded $S^2, S_{0,1}, S_{0,2}$). Thus, every subgroup of $\pi_1(S)$ has a trivial centralizer.

Assume now that $\pi_1(S)$ is the 1-relator group given above. Suppose that α lies in the centralizer of $p_*\pi_1(\widetilde{S})$. Then as $p_*\pi_1(\widetilde{S})$ has trivial center, we have $\alpha \notin p_*\pi_1(\widetilde{S})$. Moreover, as $p_*\pi_1(\widetilde{S})$ has finite index in $\pi_1(S)$, we have $[\alpha]^\lambda = 1$ in the quotient $\pi_1(S)/p_*\pi_1(\widetilde{S})$, and thus α^λ lies in $p_*\pi_1(\widetilde{S})$. Moreover, since α^λ commutes with all elements of $p_*\pi_1(\widetilde{S})$ it lies in the center of $p_*\pi_1(\widetilde{S})$. Then $\alpha^\lambda = 1$ since $p_*\pi_1(\widetilde{S})$ has trivial center, and since $\pi_1(G)$ is torsion-free, it follows that $\alpha = 1$. The statement now follows from Lemma 2.2.5.

If S is homeomorphic to either S^2 or $S_{0,1}$ then the theorem is trivial since every orientation-preserving homeomorphism $g : S \to S$ is isotopic to identity and this isotopy can be lifted to an isotopy in $\widetilde{S}$. If $S \approx S_{0,2}$, then the only homeomorphism not isotopic to identity is the one that exchanges the branch points. Since its lift must also permute the two punctures, it cannot be isotopic to identity.

Finally, we are left with the case when $S \approx T^2$. As in Lemma 2.2.8, we can deduce that g_* restricted to $p_*\pi_1(\widetilde{S})$ is identity. Moreover, $\pi_1(T^2) \simeq \mathbb{Z}^2$, and any automorphism of $\mathbb{Z}^2$ which restricts to identity on a finite index subgroup must be identity. Thus, g_* is identity and so g is isotopic to identity.

Case II: p is branched and $\chi(\widetilde{S}) < 0$.

Lemma 3.3.5 guarantees the existence of an isotopy of $\widetilde{g}$ that fixes each branch point throughout the isotopy. This induces an isotopy of $h = g|_{\widetilde{S}-\{\widetilde{P_i}\}_{i=1}^n}$ with identity. Now we can apply Case I to the unbranched cover $(q, \widetilde{S}-\{\widetilde{P}_i\}_{i=1}^n, S-\{P_i\}_{i=1}^n)$, where $q = p|_{\widetilde{S}-\{\widetilde{P_i}\}_{i=1}^n}$. Thus, h is fiber-isotopic to identity which can be extended to a fiber-isotopy of g.

Case III: p is branched and $\widetilde{S}$ is either $S_{0,1}$ or $S_{0,2}$. Since $S_{0,2}$ ($\approx S^1 \times \mathbb{R}$) admits a flat metric, any covering transformation of $S_{0,2}$ must be a rotation. Since it fixes a branch point it must be the identity.

If there is a non-trivial covering transformation of $S_{0,1}$ ($\approx \mathbb{R}^2$) then it can have at most 1 fixed point (since it is an isometry of $\mathbb{R}^2$), which we may assume to be at the origin. Thus, the group of covering transformations is a finite group of rotations and is therefore cyclic. Thus, $S \approx S_{0,1}$. The statement now follows from the fact that any homeomorphism of $S_{0,1}$ is isotopic to the identity. $\square$

2.2.2 Proof of Theorem 2.2.2

Remark 2.2.9. Suppose we have a cover $(p, \widetilde{S}, S)$ with a group G as its group of covering transformations. Suppose H is a normal subgroup of G. We can factor p in the following way:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\pi_1} & \widetilde{S}/H \\ \downarrow p & \swarrow \pi_2 & \\ S = \widetilde{S}/G & & \end{array}$$

Here π_1 takes the quotient of $\widetilde{S}$ with the group H and π_2 takes the quotient of $\widetilde{S}/H$ with the group G/H. Suppose both the factor covering spaces have the Birman-Hilden property, then it follows that the original covering space also has the Birman-Hilden property.

Lemma 2.2.10. *Let $p : \widetilde{S} \to S$ be a regular, finite sheeted, branched covering of surfaces, with at least one branch point, and with $\widetilde{S}$ being either the torus or the sphere. Assume the group of covering transformation leaves the branch points fixed. If $\widetilde{g} : \widetilde{S} \to \widetilde{S}$ is a fiber-preserving homeomorphism which is isotopic to the identity, then the induced homeomorphism $g : S \to S$ is also isotopic to identity.*

Proof. When $\tilde{S} = S^2$, the assertion holds trivially, as $S = S^2$.
Now let $\widetilde{S} = T^2$. We shall think of T^2 as $\mathbb{C}$ modulo $\mathbb{Z}^2$. We can set 0 to be one of the branch points. The covering transformations of $\widetilde{S}$ can be lifted to a Möbius transformations of $\mathbb{C}$ that leave the lattice points (of $\mathbb{Z}^2$) invariant and fix 0. These transformations are of the form $T(z) = e^{i\theta}z$, where $\theta \in \{\pi/2, \pi, 3\pi/2\}$. Thus, S is homeomorphic to a sphere and the assertion follows from the fact that every homeomorphism of a sphere is isotopic to identity. □

Proof of Theorem 2.2.2. We first prove the theorem for the case when the group of covering transformations is cyclic of prime order p. Since the size of the orbit of any point divides the order of the group of covering transformations and since p is a prime, we have that each branch point must be a

fixed point. Thus, Theorem 2.2.1 applies when S is homeomorphic to neither the torus nor the sphere. If $S \approx S_1$ (or S^2), then the statement follows from Lemma 2.2.10.
We use induction on the order of the group of covering transformations G. Let $H \lhd G$ such that $G' = G/H$ is abelian. We can factor p as discussed in Remark 2.2.9. Let $K < G/H$ be of prime order (which always exists due to Cauchy's theorem). We can further factor $\widetilde{S}/H \xrightarrow{\pi_2} \widetilde{S}/G$ using K. The statement will hold for one of the factors by the special case proven above. The statement now follows via an inductive argument (as H and G'/K are solvable groups of order less than that of G). □

Definition 2.2.11. Let $(p, \widetilde{S}, S)$ be a regular cover (possibly branched). Let S° be the surface obtained after deleting the branch points from S. The *liftable mapping class group* $\mathrm{LMod}(S^\circ)$ of p is the subgroup of $\mathrm{Mod}(S^\circ)$ of mapping classes represented by homeomorphisms that lift under p.

Definition 2.2.12. The *symmetric mapping class group* $\mathrm{SMod}(\widetilde{S})$ for a cover $(p, \widetilde{S}, S)$ is the subgroup of $\mathrm{Mod}(\widetilde{S})$ of mapping classes represented by fiber-preserving homeomorphisms under p.

For a cover $(p, \widetilde{S}, S)$ satisfying the Birman-Hilden property, there is a natural homomorphism between $\mathrm{SMod}(\widetilde{S})$ and $\mathrm{LMod}(S^\circ)$ given as follows. Given a mapping class in $\mathrm{SMod}(\widetilde{S})$ choose a fiber-preserving representative f. Let $\overline{f}$ be the induced homeomorphism of S°. Define $\Theta_p([f]) = [\overline{f}]$. Suppose f and g are two fiber-preserving representatives in the same mapping class then the fiber-isotopy between them induces a isotopy between the two induced homeomorphisms $\overline{f}$ and $\overline{g}$.
Let D denote the subgroup of $\mathrm{Mod}(\widetilde{S})$ of mapping classes represented by the covering transformations.

Theorem 2.2.13. *For a cover $(p, \widetilde{S}, S)$ satisfying the Birman-Hilden property, there is an exact sequence:*

$$1 \to D \to \mathrm{SMod}(\widetilde{S}) \xrightarrow{\Theta_p} \mathrm{LMod}(S^\circ) \to 1.$$

Proof. It is apparent that D is contained in the kernel of Θ_p. Conversely, assume that f induces the identity homeomorphism of S°. Then f must send each fiber to itself, and hence it has to be a covering transformation. Thus, $\ker \Theta_p = D$, and our assertion follows. □

2.3 A presentation for $\mathrm{Mod}(S_2)$

In this section, we focus our attention on a particular branched cover (p, S_g, S^2) where p is induced by the $\mathbb{Z}_2$-action on S_g generated by the hyperelliptic involution i on S_g shown in Figure 2.1 below.

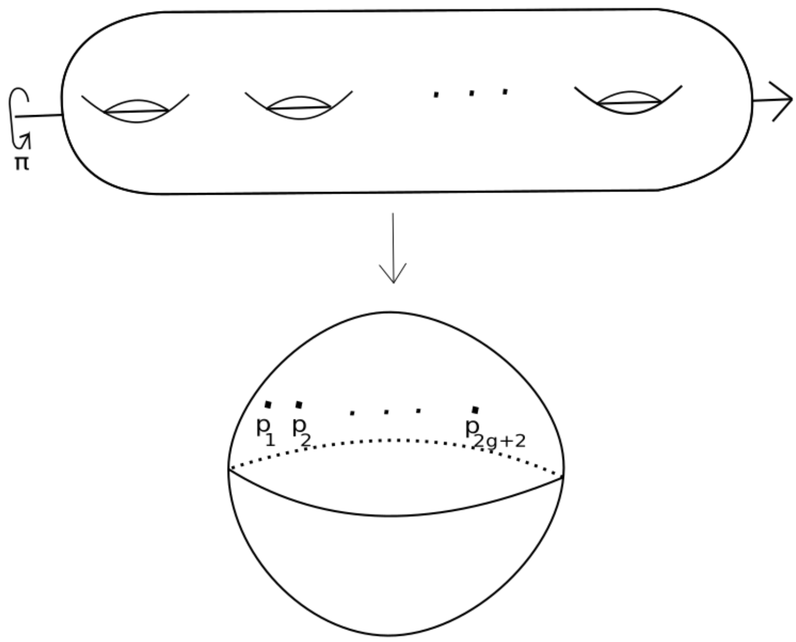

Fig. 2.1: The hyperelliptic cover.

Note that $S_g/\langle i \rangle \approx S^2$. The axis intersects S_g at $2g+2$ points. Since these points are fixed points of the rotation, their images lie in the branch set. Due to Theorem 2.2.1, this cover has the Birman-Hilden Property. Thus, we have the following exact sequence,

$$1 \to \langle [i] \rangle \to \mathrm{SMod}(S_g) \xrightarrow{\Theta_p} \mathrm{LMod}(S_{0,2g+2}) \to 1. \tag{2.4}$$

Later we show that $\mathrm{LMod}(S_{0,2g+2})$ coincides with $\mathrm{Mod}(S_{0,2g+2})$. It follows from Theorem 2.1.2 that a homeomorphism f is fiber-preserving if and only if it commutes with i. Let $c_1, c_2, \ldots c_{2g+1}$ be the curves shown in Figure 2.2. Let τ_i denote the Dehn twist about the i-th curve.

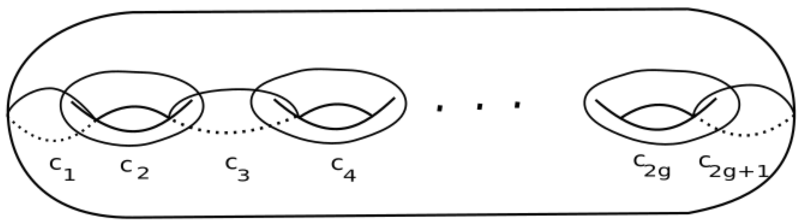

Fig. 2.2: Generators of $\mathrm{SMod}(S_g)$.

Theorem 2.3.1. *The half twist s_i (as defined in Section 1.2.4) lifts to τ_i, for each i and thus,* $\mathrm{SMod}(S_g)$ *is generated by $\{\tau_i\}_{i=1}^n \cup \{i\}$. Since the s_i generate* $\mathrm{Mod}(S_{0,2g+2})$*, every homeomorphism of $S_{0,2g+2}$ lifts.*

Proof. Choose a representative of the half twist τ_i such that the annulus that supports it lies symmetrically about the axis of i. Since each τ_i commutes with i, it is fiber-preserving. It remains to be shown that $\Theta_p(\tau_i) = s_i$.

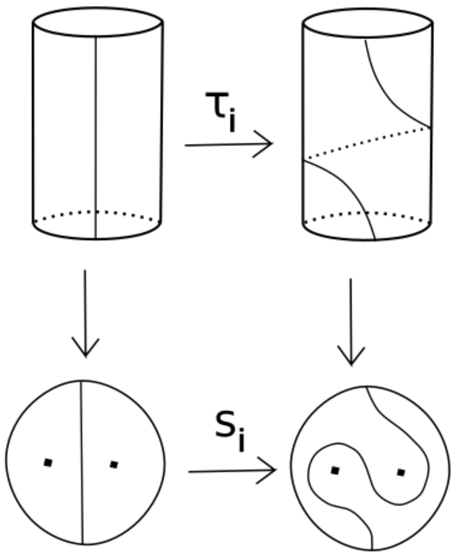

Fig. 2.3: Half-twist lifting to a Dehn twist.

For time being denote the homeomorphism induced by τ_i to be t_i. Consider the arc $c(t) = (t, \frac{\pi}{2}), t \in [-1, 1]$ shown above. It is clear that both the s_i and the t_i map c to arcs that are isotopic. Thus, $t_i^{-1}s_i$ maps c to an arc isotopic to c. We can modify $t_i^{-1}s_i$ so that it fixes c pointwise. Since the arc c divides the disk in two punctured disks and $\mathrm{Mod}(D^2 - \{point\})$ is trivial one can construct an isotopy of $t_i^{-1}s_i$ with the identity. □

The next result shows that adding i to the generating set is superfluous as it can be expressed in terms of the τ_i.

Theorem 2.3.2. *The hyperelliptic involution i is isotopic to $\tau_1\tau_2\ldots\tau_{2g}\tau_{2g+1}^2\tau_{2g}\ldots\tau_2\tau_1$.*

Proof. We note that $\tau_1\tau_2\ldots\tau_{2g}\tau_{2g+1}^2\tau_{2g}\ldots\tau_2\tau_1$ projects to $s_1s_2\ldots s_{2g}s_{2g+1}^2s_{2g}\ldots s_2s_1$, which is trivial. Thus, $\tau_1\tau_2\ldots\tau_{2g}\tau_{2g+1}^2\tau_{2g}\ldots\tau_2\tau_1$ can either be i or the identity. To show that it is not the identity consider the curve c shown in Figure 2.4.

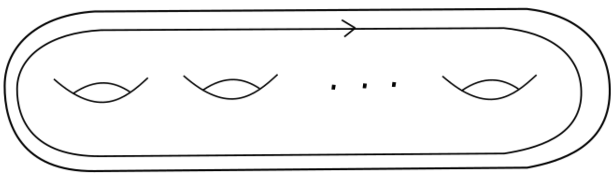

Fig. 2.4: the curve c.

Since $\tau_1\tau_2\ldots\tau_{2g}\tau_{2g+1}^2\tau_{2g}\ldots\tau_2\tau_1$ reverses the orientation of c, it cannot be isotopic to identity. □

In the case $g = 2$, $\mathrm{SMod}(S_2) = \mathrm{Mod}(S_2)$ and the relations (1-5) give a complete set of relations for $\mathrm{Mod}(S_2)$. However $\mathrm{SMod}(S_g)$ is a proper subgroup of $\mathrm{Mod}(S_g)$, for $g > 2$.

Theorem 2.3.3. *The following relations hold in* $\mathrm{SMod}(S_g)$ *and are complete with respect to the generating set* $\{\tau_i\}_{i=1}^{2g+1}$.

(i) $[\tau_i, \tau_j] = 1$, *for all* i, j *such that* $|i - j| \geq 2$,

(ii) $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$, *for* $i, 1 \leq i \leq 2g$,

(iii) $(\tau_1\tau_2\dots\tau_{2g+1})^{2g+2} = 1$,

(iv) $(\tau_1\tau_2\dots\tau_{2g}\tau_{2g+1}^2\tau_{2g}\dots\tau_2\tau_1)^2 = 1$, *and*

(v) $[\tau_1\tau_2\dots\tau_{2g}\tau_{2g+1}^2\tau_{2g}\dots\tau_2\tau_1, \tau_1] = 1$.

Proof. We first show that the above relations hold in $\mathrm{SMod}(S_g)$. Relation 1. follows that curves $i(c_i, c_j) = 0$ and Relation 2 follows from the fact that $i(c_i, c_{i+1}) = 1$ $i, 1 \leq i \leq 2g$. Relation 3 follows from the chain relation as the curves c_i's form a chain whose boundary components are non-essential curves. Relation 4 simply states that $i^2 = 1$, while Relation 5 states that i commutes with τ_1.
(Given a relation, one can obtain a word in the generating set that is trivial. For eg.

$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} \implies \tau_{i+1}^{-1}\tau_i^{-1}\tau_{i+1}^{-1}\tau_i\tau_{i+1}\tau_i = 1.$$

Such a word is called a *relator*.)
Let R denote the set of relators in the presentation of $\mathrm{Mod}(S_{0,2g+2})$ and S denote the presentation of $\mathrm{SMod}(S_g)$. Given a trivial word in $\mathrm{SMod}(S_g)$ we give a procedure for expressing it as a product of conjugates of relators.
If a word w in $\{\tau_i\}$ is trivial, then $\Theta_p(w) \in \mathrm{Mod}(S_{0,2g+2})$ is also trivial. Thus, $\Theta_p(w)$ can be expressed as $\Pi_{j=1}^N g_j r_j g_j^{-1}$, where $r_i \in R$. We can obtain one preimage of $\Theta_p(w)$ by simply replacing the s_i with the corresponding τ_i. Performing this operation on $\Theta_p(w)$, we get that w is either $\Pi_{j=1}^N h_j t_j h_j^{-1}$ or $i\Pi_{j=1}^N h_j t_j h_j^{-1}$, where t_i is a preimage of r_i under Θ_p. Observe that the preimages of the r_i constructed in this way are exactly the relators in $\mathrm{SMod}(S_g)$ except for $s_1s_2\dots s_{2g}s_{2g+1}^2s_{2g}\dots s_2s_1$ which gives $\tau_1\tau_2\dots\tau_{2g}\tau_{2g+1}^2\tau_{2g}\dots\tau_2\tau_1$. We can replace every occurence of $\tau_1\tau_2\dots\tau_{2g}\tau_{2g+1}^2\tau_{2g}\dots\tau_2\tau_1$ with i (Theorem 2.3.2). Now since i commutes with all elements in $\mathrm{SMod}(S_g)$ we can write $w = i^k\Pi_{j=1}^{N'} h_j t_{n_j} h_j^{-1}$. Since both w and $\Pi_{j=1}^{N'} h_j t_{n_j} h_j^{-1}$ are trivial, we have that $i^k = 0$. Thus, $w = \Pi_{j=1}^{N'} h_j t_{n_j} h_j^{-1}$, and we are done. □

2.4 A problem regarding the Artin braid group

The *braid group on n strands* B_n is defined as the group with $(n-1)$ generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and the following relations:

(i) $[\sigma_i, \sigma_j] = 1$ for all i, j such that $|i-j| \geq 2$, and

(ii) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, for $1 \leq i \leq (n-2)$

The braid group B_n affords a faithful representation as a group of automorphisms $\mathrm{Aut}(F_n)$ of the the free group of rank n as follows:

$$\zeta : B_n \to \mathrm{Aut}(F_n) :$$

$$\sigma_i \mapsto \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} & \\ x_{i+1} \mapsto x_i & \\ x_k \mapsto x_k & k \neq i, i+1 \end{cases}$$

The automorphisms of F_n in the image of ζ are called *Braid automorphisms*. They have two defining properties, namely:

(i) every generator x_i is mapped either to a conjugate of itself or some other x_j, and

(ii) the product $x_1 x_2 \dots x_n$ is mapped to itself.

Let N_k be the normal closure in F_n of the set of elements $\{x_1^k, x_2^k, \dots, x_n^k\}$. Each $\zeta(\sigma_i)$ maps N_k to itself. Thus, it induces an automorphism of $F_n/N_k \cong *_n \mathbb{Z}_k$. Let $\Psi_k : B_n \to \mathrm{Aut}(F_n/N_k)$ be the natural homomorphism that sends an element of the braid group to its induced automorphism of F_n/N_k.

Theorem 2.4.1. *$\Psi_k : B_n \to \mathrm{Aut}(F_n/N_k)$ is injective.*

We need the following two lemmas to prove the above Theorem 2.4.1.

Lemma 2.4.2. *If a homeomorphism f of a surface S which fixes a point x_0, is isotopic to identity, then the induced automorphism f_* of $\pi_1(S, x_0)$ is an inner automorphism.*

Proof. Let f be isotopic to identity via an isotopy $G(s, \star)$. Let $\beta : I \to S$ be a loop based at x_0. Then $f(\beta)$ is homotopic to β via the map $G(s, \beta(t))$. Let δ be the loop $G(s, x_0)$. Then $f(\beta) \simeq \delta\beta\delta^{-1}$ via a basepoint preserving homotopy. Thus, f_* is an inner automorphism. □

Let $S = \mathbb{C}$ and let $\{P_i\}_{i=1}^n$ be n distinct points in S. Let $P_0 \in S - \{P_1, P_2, \dots P_n\}$. Then F_n can be interpreted as the fundamental group $\pi_1(S - \{P_1, P_2 \dots, P_n\}, P_0)$ by identifying the generator x_i to the homotopy class of loops that go once around the point P_i (see Figure 2.5 below).

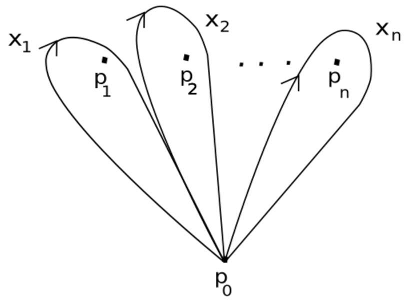

Fig. 2.5: Generators of $\pi_1(S - \{P_1, P_2 \dots, P_n\}, P_0)$.

Lemma 2.4.3. *Every braid automorphism can be induced by a homeomorphism of* S *which preserves the set* $\{P_1, P_2 \dots, P_n\}$.

Proof. It suffices to prove the statement for the generators σ_i. If α is an arc connecting P_i and P_{i+1}, then the half-twist H_α induces an automorphism on $\pi_1(S - \{P_1, P_2 \dots, P_n\}, P_0)$ that corresponds exactly with the braid automorphism σ_i, as shown in Figure 2.6.

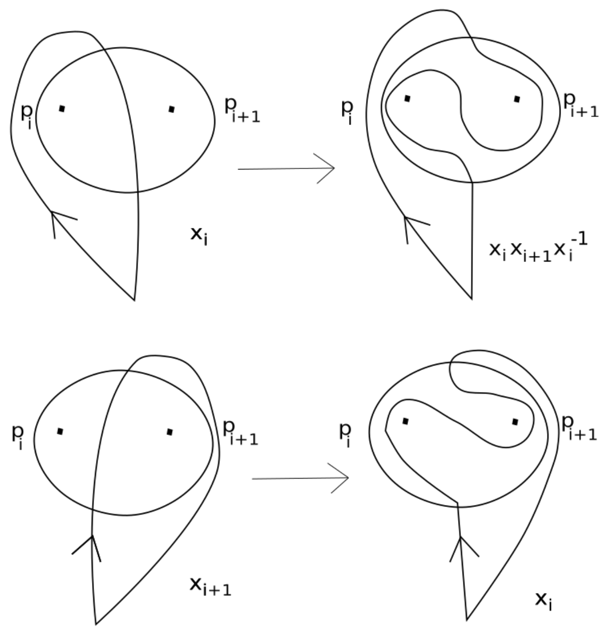

Fig. 2.6: Action of H_α on $\pi_1(S - \{P_1, P_2 \ldots, P_n\}, P_0)$.

□

Proof of Theorem 2.4.1. Assume first that $\beta \in \ker(\Psi_k) \cap Inn(F_n)$. Since $\beta \in Inn(F_n)$ we have $\beta(x_1x_2 \ldots x_n) = T(x_1x_2 \ldots x_n)T^{-1}$ for some $T \in F_n$. Thus, $\beta(x_1x_2 \ldots x_n) = x_1x_2 \ldots x_n$ implies $T = (x_1x_2 \ldots x_n)^\lambda$ for some integer λ. The braid automorphism $\sigma = \sigma_1\sigma_2 \ldots \sigma_{n-1}$ is given by

$$\sigma^\lambda : x_i \to (x_1x_2 \ldots x_n)^\lambda x_i (x_1x_2 \ldots x_n)^{-\lambda} \text{ for } 1 \leq i \leq n \tag{2.5}$$

Hence $B_n \cap Inn(F_n) = \langle\sigma\rangle$. Thus, $\beta = \sigma^\lambda$, for some integer λ. Since F_n/N_k is a free product of n copies of the cyclic group of order k, every element in F_n/N_k can be expressed uniquely as a product of elements in these factors (See appendix). If $\Psi_k(\sigma^\lambda) = 1$, then $\sigma^\lambda(x_i)$ contains only symbols of the form x_i^k. But it is apparent from equation 2.5 that $\sigma^\lambda(x_i)$ contains no kth powers. Thus, $\ker(\Psi_k)$ contains no non-trivial inner automorphisms.
We further prove that any $\beta \in \ker(\Psi_k)$ must be an inner automorphism. Let H be subgroup of F_n containing words with exponent sum 0 modulo k. Note that H is a index k normal subgroup of F_n and $F_n/H \cong$

$\mathbb{Z}_k$. Let $(p, \widetilde{S}, S)$ be the branched cover of S such that $\widetilde{S}$ has n branch points $\widetilde{P}_1, \widetilde{P}_2, \dots, \widetilde{P}_n$ that are pre-images of $P_1, P_2, \dots, P_n$, respectively, and $\widetilde{p}_*(\pi_1(\widetilde{S} - \{\widetilde{P}_1, \widetilde{P}_2, \dots, \widetilde{P}_n\}, \widetilde{P}_0)) = H$. Clearly $N_k < H$, and moreover N_k consists of precisely those curves in S which lift to closed curves that are nullhomotopic in $\widetilde{S}$, but not nullhomotopic in $\widetilde{S} - \{\widetilde{P}_1, \widetilde{P}_2, \dots, \widetilde{P}_n\}$. Thus, the group H/N_k can be identified with $\pi_1(\widetilde{S}, \widetilde{P}_0)$.

Let $\beta \in \ker(\Psi_k)$, and let B be a homeomorphism of S that induces the automorphism β on $\pi_1(S)$. Then a lift $\widetilde{B}$ of B to $\widetilde{S}$ induces an automorphism $\widetilde{B}_*$ on $\pi_1(\widetilde{S})$. This automorphism is the same as the restriction of $\Psi_k(\beta)$ to H/N_k. By our hypothesis, $\beta \in \ker(\Psi_k)$, so $\widetilde{B}_* = 1$. Thus, $\widetilde{B}$ is isotopic to identity and by Theorem 2.2.2 we get that, $\widetilde{B}$ is fiber-isotopic to identity. This fiber-isotopy projects to an isotopy of B with identity. Thus, by Lemma 6.1.1, $\beta = B_*$ must be an inner automorphism. □

2.5 Liftabilty criterion for cyclic covers of the sphere

In Section 2.3 we have seen that all homeomorphisms of $S_{0,2g+2}$ lift under the hyperelliptic cover. In this section, we answer the question ‘When do all homeomorphisms of a punctured sphere lift under a branched cover?’ In particular, we focus our attention on covers that have a cyclic group of covering transformations. For a topological space X, we denote by $H_1(X)$, the first homology group of X in integer coefficients. This chapter is based on recent work by Ghawala and Winarski [3].

2.5.1 Homological criterion for liftability

Let $(p, \widetilde{S}, S^2)$ be a regular cover (possibly branched) such that the group of covering transformations D is abelian. Let $(q, \widetilde{S}^\circ, S^\circ)$ be the corresponding unbranched cover. Let $q_\star : H_1(\widetilde{S}^\circ) \to H_1(S^\circ)$ denote the induced homomorphism between the first homology groups.

Theorem 2.5.1. *A homeomorphism f of S° lifts under q if and only if*

$f_\star(q_\star(H_1(\widetilde{S}^\circ))) = q_\star(H_1(\widetilde{S}^\circ))$.

Proof. First, let us assume that f lifts. Then for a point $x \in S^\circ$, we have $f_*(p_*(\pi_1(\widetilde{S}^\circ, \widetilde{x}))) = p_*(\pi_1(\widetilde{S}^\circ, \widetilde{x}'))$, where $\widetilde{x}_0$ and $\widetilde{x}_1$ are points lying above x and $x_1 = \phi(x)$. We have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(S^\circ, x_0) & \xrightarrow{f_*} & \pi_1(S^\circ, x_1) \\ \big\downarrow \Delta & & \big\downarrow \Delta \\ H_1(S^\circ) & \xrightarrow{f_\star} & H_1(S^\circ) \end{array}$$

The map Δ sends a loop γ to its homology class $[\gamma]$. Let x be an element of $q_\star(H_1(\widetilde{S}^\circ))$ and let γ denote an element of $\pi_1(S)$ such that $[\gamma] = x$. Then we have $f_\star(x) = f_\star\Delta(\gamma) = \Delta f_*(\gamma)$. Now $f_*(\gamma) \in p_*(\pi_1(\widetilde{S}^\circ, \widetilde{x}_1))$ so that $f_*(\gamma) = q_*(\gamma')$ for some $\gamma' \in \pi_1(\widetilde{S}^\circ, \widetilde{x}_1)$. Since the following diagram commutes

$$\begin{array}{ccc} \pi_1(\widetilde{S}^\circ, \widetilde{x}_1) & \xrightarrow{q_*} & \pi_1(S^\circ, x_1) \\ \big\downarrow \Delta & & \big\downarrow \Delta \\ H_1(\widetilde{S}^\circ) & \xrightarrow{q_\star} & H_1(S^\circ), \end{array}$$

we have $\Delta q_*(\gamma') = q_\star\Delta(\gamma')$. Thus, $f_\star(x) \in A$.
Conversely, let $f_\star(q_\star(H_1(\widetilde{S}^\circ))) = q_\star(H_1(\widetilde{S}^\circ))$. Then for any $\gamma \in p_*(\pi_1(\widetilde{S}^\circ))$ we have $\Delta f_*(\gamma) = f_\star\Delta(\gamma)$. Since $\Delta(\gamma) \in A$, we have $\Delta f_*(\gamma) \in q_\star H_1(\widetilde{S}^\circ)$. Since G/H is abelian, where $G = \pi_1(S^\circ, x_1)$ and $H = q_*(\pi_1(\widetilde{S}^\circ, \widetilde{x}_1))$, we have $[G, G] \subset H$. Thus $G/H \cong (G/[G, G])/(H/[G, G])$. Since $G/[G, G]$ is isomorphic to $H_1(S^\circ)$, we have that $\beta \in H \iff [\beta] \in q_\star(H_1(\widetilde{S}^\circ))$. Thus, it follows that $f_*(\gamma) \in q_*(\pi_1(\widetilde{S}^\circ, \widetilde{x}_1))$. □

Now the above criterion can be used to analyze covers for which the group of covering transformations is abelian. We focus on covers where the base space is S^2 and the group of covering transformations is cyclic.

Definition 2.5.2. Let A be an abelian group, an *admissible k-tuple* is a tuple $(a_1, a_2, \ldots, a_k)$ such that $a_i \in A/\{0\}$ and $\sum_{i=1}^{k} a_i = 0$ and the a_i generate A.

Remark 2.5.3. Every admissible k-tuple gives a surjective homomorphism $\phi : \pi_1(S_{0,k}) \to A$ as follows. Let x_i denote the homotopy class of loops that go around the ith puncture once in $S_{0,k}$. The kernel of the homomoprhism

defined by $\phi(x_i) = a_i$ corresponds to a regular cover of $\pi_1(S_{0,k})$. By filling in the punctures, we obtain a regular branched cover of the sphere.
Conversely, given a regular cover of $S_{0,k}$, we have a surjective homomorphism $\phi : \pi_1(S_{0,k}) \to A$, where A is the group of covering transformations. Taking a_i to be the images of the standard generators of $\pi_1(S_{0,k})$, we get a admissible k-tuple in A.

Theorem 2.5.4. *Let A be a finite cyclic group and let $(a_1, a_2 \dots, a_k)$ be an admissible k-tuple. Let (p, S, S^2) be the cover with A defined by this tuple. Every homeomorphism of $S_{0,k}$ lifts under p if and only if one of the following is true:*

(i) There is an isomorphism $\delta : A \to \mathbb{Z}_n$ with $k \equiv 0 \bmod(n)$ such that $\delta(a_i) = 1$ for all a_i.

(ii) $k = 2$ and there is an isomorphism $\delta : A \to \mathbb{Z}_n$, for some $n \geq 3$, such that $\delta(a_1) = 1$ and $\delta(a_2) = -1$.

Lemma 2.5.5. *Let $f, g : G \to A$ be surjective homomorphisms. Then $\ker(f) = \ker(g)$ if and only if $f = \zeta g$, for some $\zeta \in \operatorname{Aut}(A)$*

Proof. If $f = \zeta g$, then $f(x) = \zeta g(x) = 0 \implies g(x) = 0$, and so we have $\ker(f) \subseteq \ker(g)$. The other containment follows from the fact that $g = \zeta^{-1} f$.
Conversely assume $\ker(f) = \ker(g)$. Let $a \in A$ and $x \in G$ be such that $g(x) = a$. We define $\zeta(a) = f(x)$ and note that this does not depend on the choice of x. If $\zeta(a) = 0$ then $f(x) = 0$. Since $\ker(f) = \ker(g)$, we have $g(x) = a = 0$. This shows ζ is injective. Given $b \in A$, since f is surjective, there exists x such that $f(x) = b$. For $a = g(x)$ we have, $\zeta(a) = b$. Thus, $\zeta \in \operatorname{Aut}(A)$, and by the definition of ζ, we get $f = \zeta g$. □

Lemma 2.5.6. *Let A be a finte abelian group and let $(a_1, a_2, \dots, a_k)$ be an admissible k-tuple. Let $S^\circ \to S_{0,k}$ be the covering defined by this tuple. Let f be a homeomorphism of $S_{0.k}$ and let σ be the permutation of the punctures induced by f. The homeomorphism f lifts if and only if there exists an automorphism $\psi \in \operatorname{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i.*

Proof. Let $\phi : H_1(S_{0,k}) \to A$ be the homomorphism defined by setting $\phi(x_i) = a_i$. Since $q_\star(H_1(\widetilde{S}^\circ)) = \ker\phi$ we have that f lifts if and only if $f_\star(\ker\phi) = \ker\phi$. Moreover, $f_\star(\ker\phi) = \ker\phi$ if and only if $\ker(\phi f) = \ker(\phi)$. By Lemma 2.5.5, f lifts if and only if there exists $\psi \in \mathrm{Aut}(A)$ such that $\phi f_\star = \psi\phi$. Since $f_\star(x_i) = x_{\sigma(i)}$, we have $\psi(a_i) = \psi\phi(x_i) = \phi f_\star(x_{\sigma(i)}) = a_{\sigma(i)}$. □

Lemma 2.5.7. *Let A be a finite cyclic group, and let $(a_1, \dots, a_k)$ be an admissible tuple. For every permutation $\sigma \in S_k$, there exists $\psi \in \mathrm{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i, if and only if one of the following conditions hold:*

(i) *There is an isomorphism $\delta : A \to \mathbb{Z}_n$ with $k \equiv 0 \bmod(n)$ such that $\delta(a_i) = 1$ for all a_i.*

(ii) *$k = 2$ and there is an isomorphism $\delta : A \to \mathbb{Z}_n$ for some $n \geq 3$ such that $\delta(a_1) = 1$ and $\delta(a_2) = -1$.*

Proof. If all a_i are equal, then each a_i must be a generator of A and thus, there exists an isomorphism $\delta : A \to \mathbb{Z}_n$ such that $\delta(a_i) = 1$, for all i. Since $\sum_{i=0}^{k} a_i = 0$, we have that $k \equiv 0 \pmod n$.

Now consider the case when there are at least two distinict a_i. We show that a_i must be all distinct. Suppose we assume on the contrary, that there are three elements a_p, a_q, a_r such that $a_p = a_q \neq a_r$. Let σ be the transposition that switches q and r. By our hypothesis, there exists an automorphism ψ of A such that $\psi(a_i) = a_{\sigma(i)}$. Therefore, $a_p = \psi(a_p) = \psi(a_q) = a_r$, which is a contradiction. We must therefore assume all the a_i are distinct.

Since there is a subgroup of $\mathrm{Aut}(A)$ that permutes a_i we get that there is a subgroup of $\mathrm{Aut}(A)$ isomorphic of S_k. Since for any cyclic group A, $\mathrm{Aut}(A)$ is cyclic it must be that $k = 2$. It follows that, $a_1 = -a_2$ and each a_i is a generator of A. Thus, there exists an isomorphism δ such that $\delta(a_1) = 1$ and $\delta(a_2) = -1$.

Conversely, assume that either of the two conditions hold. Then it suffices to show that there always exists an automorphism ψ such that $\psi(a_i) = a_{\sigma(i)}$. For Condition *(i)* the identity automorphism suffices, and for Condition *(ii)*, the automorphism $a \mapsto -a$ gives us the required automorphism. □

Proof of Theorem 2.5.4: Let f be any homeomorphism of $S_{0,k}$. Lemma 2.5.7 gives the existence of an automorphism $\psi \in \mathrm{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i. So, by Lemma 2.5.6, f lifts.
Conversely, suppose that all homeomorphisms lift. Any permutation $\sigma \in S_k$ of the punctures can be induced by some homeomorphism f of $S_{0,k}$. Thus, for every permutation σ, there exists an automorphism $\psi \in \mathrm{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i. Thus, by Lemma 2.5.7, the assertion follows. $\square$

Appendices

Appendix A

I Generators and Relators

Let G be a group.

Definition I.1 (Generating Set)**.** A set $S \subset G$ is called a generating set of G if for every $g \in G$ can be expressed as $g = \Pi_{i=1}^{N} g_i$ where $g_i \in S$. Note that g_i may not be distinct.

For a fixed generating set S a relator r in G is a word in the alphabet S (i.e a product of elements of S) that is trivial as a group element.

Definition I.2. A set of relators $R = \{r_i\}_{i=I}$ is said to be complete if any word in S that is trivial can be expressed as a product of conjugates of r_i's. Equivalently G is isomorphism to the group $F_S/\langle\langle\{r_i\}\rangle\rangle$ where F_S is the free group on the set S and $\langle\langle\{r_i\}\rangle\rangle$ denotes the normal closure of the subgroup generated by the set $\{r_i\}$.

II A theorem concerning free products of groups

Let A and B two groups and let $A * B$ denote the free product of A and B. A sequence of elements in $A * B$ is a called a reduced sequence if $g_i \neq 1$, each g_i belongs to either A or B and g_i, g_{i+1} do not both belong to A or B i.e g_i's alternate between A and B.

Theorem II.1. *Every element of $A*B$ can be uniquely expressed as a product $g_1 g_2 \ldots g_n$ where each g_i is a reduced sequence.*

For a proof refer to [6].

III Some results on fundamental groups of Compact Surfaces

Theorem III.1. *For any surface with* $\chi(S) < 0$, $\pi_1(S)$ *has trivial center.*

Theorem III.2. *For any surface with* $\chi(S) < 0$, $\pi_1(S)$ *is torsion-free.*

Proof. If S has punctures or boundary then $\pi_1(S)$ is a free group. Hence there is nothing to prove in this case. Thus assume S is closed. By theorem 2.1.5 we can identify $\pi_1(S_g)$ with a Fuchsian group Γ that acts freely and properly discontinuously on $\mathbb{H}$ and such that $\mathbb{H}/\Gamma$ is homeomorphic to S_g. Any finite order element of $\mathrm{PSL}(2, R)$ is a rotation about some point. But then the action of Γ cannot be free as any finite order element always has a fixed point. Thus we have a contradiction. □

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