INTRODUCTION TO HYPERBOLIC GEOMETRY AND FUCHSIAN GROUPS

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CERTIFICATE

This is to certify that **Deepak Kumar**, BS-MS (Dual Degree) student in Department of Mathematics has completed bona fide work on the dissertation entitled **Introduction to hyperbolic geometry and Fuchsian groups** under my supervision and guidance.

April 2019 **Prof. Kashyap Rajeevsarathy**
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ABSTRACT

We will start by defining the *upper half plane model* $\mathbb{H}$ for two-dimensional hyperbolic space. We will discuss the geometry of the hyperbolic plane by considering *general Möbius group Möb*, which is generated by *Möbius* transformations and reflections in $\widehat{\mathbb{C}}$. We then derive a metric in $\mathbb{H}$, under which *Möb* forms a group of isometries. After introducing the notions of *convex sets* and *hyperbolic polygons*, we derive the *Gauss-Bonnet theorem*, which expresses the area of a hyperbolic polygon in terms of its angles. We then define *Fuchsian groups*, which are discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})(\cong \mathrm{Isom}^+(\mathbb{H}))$. We will show that a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is Fuchsian if, and only if, it has a properly discontinuous action on $\mathbb{H}$. To further understand the properties of Fuchsian groups, we consider the *fundamental domain* under their actions on $\mathbb{H}$. We derive some geometric properties of the fundamental domain, and provide a procedure to construct a special kind of fundamental domain known as the *Dirichlet region.*

CONTENTS

1. HYPERBOLIC GEOMETRY

1.1 Introduction

Hyperbolic geometry is an area of mathematics which has an interesting history. Let start with basic geometry we study in high school, know as Euclidean geometry. The postulates stated by Euclid are the foundation of this geometry, which we enlist here,

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight lines segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two Right Angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the Parallel Postulate.

In the mid 18^{th} century, Lobachevsky, Gauss, and some other mathematicians, in an attempt of eliminating Euclid's fifth postulate, realized that the first four axioms of Euclid could give rise to a separate geometry. Gauss claimed to have made the discovery of this new geometry in his unpublished work "Non- Euclidean Geometry", which he had mentioned in a letter he had sent to Franz, Taurinus, another mathematician who was then studying the same geometry. Finally, Nikolai Lobachevsky published the complete system of hyperbolic geometry around 1830, where he had altered the parallel postulate by stating the existence of infinitely many lines passing through a point which are parallel to a given line (see [5]). In the 19^{th} century, mathematicians started studying hyperbolic geometry extensively, and it is still being actively studied by researchers around the world. Hyperbolic geometry has close connections with a number of different fields, which include Abstract Algebra, Number theory, Differential geometry, and Low-dimensional Topology. In this chapter, which is based on [1, Chapters 1-5], we will explore upper half-plane model for hyperbolic geometry.

1.2 Upper half-Plane Model

In this section, we develop hyperbolic geometry for dimension 2. For that we use a model, known as *upper half-plane model.* The underlying space for this model is the upper half-plane $\mathbb{H}$ of the complex plane $\mathbb{C}$, defined as

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

Now, we are ready to define the geometry i.e. the lines and angles on this plane. First, we define straight line for this geometry as follows, which we called *hyperbolic lines.* We will prove the existence and uniqueness of these lines between any two points.

1.2.1 Hyperbolic lines

Definition 1.2.1. There are two different types of hyperbolic lines:

1. Intersection of Euclidean lines perpendicular to the $\mathbb{R}$ with $\mathbb{H}$ in $\mathbb{C}$.
2. Intersection of Euclidean circles in $\mathbb{C}$ centered at $\mathbb{R}$ with $\mathbb{H}$.

Proposition 1.2.1. *For each pair of p and q of distinct points in $\mathbb{H}$, there exist a unique hyperbolic line 'ℓ' in $\mathbb{H}$ passing through p and q.*

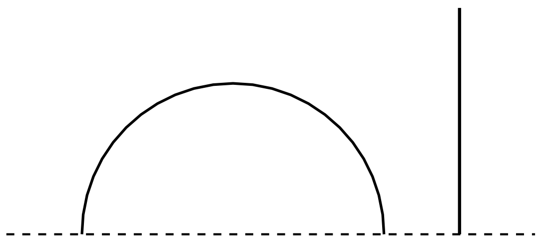

Fig. 1.1: Hyperbolic lines

Proof. To prove this proposition we give the way to construct a hyperbolic line for given any two points in $\mathbb{H}$, and the uniqueness of such line holds by our construction of these lines. Depending on the equalities of the real part of p and q we have the following two cases:
Case 1 : If $Re(p) = Re(q)$, we draw a Euclidean line passing through p and q. By construction this line is perpendicular to $\mathbb{R}$. Intersection of this line with $\mathbb{H}$ is the desired hyperbolic line $L_{p,q}$. In mathematical terms,

$$L_{p,q} = \{z \in \mathbb{H} \mid \mathrm{Re}(z) = \mathrm{Re}(p)\}.$$

Case 2 : If $Re(p) \neq Re(q)$. Let ℓ_{pq} be the Euclidean line segment joining p and q, and let ℓ be the perpendicular bisector of ℓ_{pq}. Then, every Euclidean circle that passes through p and q has its center on ℓ. As p and q have non-equal real parts, ℓ cannot be parallel to $\mathbb{R}$, so ℓ and $\mathbb{R}$ intersect at a unique point c. Let A be the Euclidean circle centered at this point of intersection point c with radius $\mid c - p \mid$. Note that $\mid x - p \mid = \mid x - q \mid$, for each point $x \in \ell$. So, A passes through q also. The intersection $\text{Ł}_{p,q} = \mathbb{H} \cap A$ is then the desired hyperbolic line passing through p and q. Equation of this line can be given by,

$$L_{c,r} = \{z \in \mathbb{H} \mid\mid z - c \mid^2 = r^2\},$$

where $r = \mid c - p \mid = \mid c - q \mid$. □

We define the *angle* between two hyperbolic lines as looking at them as curves in complex plane. Note that under this notion of angle, we can define the parallelism of two lines as follows.

Definition 1.2.2. Two hyperbolic lines are said to *parallel* if they are disjoint in the $\mathbb{H}$.

The following theorem shows the difference between the Euclidean and hyperbolic geometry. In Euclidean geometry, Euclid's fifth postulates tell us that "For given line ℓ and point x disjoint from the line, we have unique line passing through x and parallel to the given line ℓ." But in hyperbolic geometry, we can find infinitely many parallel lines with the same condition.

Theorem 1.1. *Let ℓ be a hyperbolic line, and let p be a point in $\mathbb{H}$ not on ℓ. Then, there exist infinitely many distinct hyperbolic lines through p that are parallel to ℓ.*

Proof. There are two cases to consider. First, suppose that ℓ is contained in a Euclidean line, L. As p is not on L, there exists a Euclidean line K through p that is parallel to L. As L is perpendicular to $\mathbb{R}$, we have that K is perpendicular to $\mathbb{R}$ as well. So, one hyperbolic line in $\mathbb{H}$ through p and parallel to ℓ is the intersection $\mathbb{H} \cap K$.

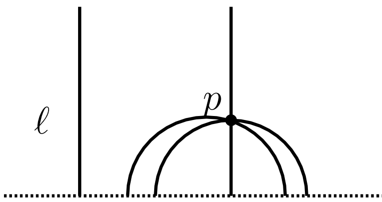

Fig. 1.2

To construct another hyperbolic line through p and parallel to ℓ, take a point x on $\mathbb{R}$ between K and L, and let A be the Euclidean circle centered on $\mathbb{R}$ that passes through x and p. We know that such a Euclidean circle A exists because $Re(x) \neq Re(p)$.

□

1.3 The General Möbius group

We begin with the group of homeomorphisms of $\overline{\mathbb{C}}$, and we denote it by $\text{Homeo}(\bar{\mathbb{C}})$, now we determine the group of homeomorphisms of $\overline{\mathbb{C}}$ taking circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$, and denote it by $\text{Homeo}^c(\bar{\mathbb{C}})$.

Note that $\text{Homeo}(\bar{\mathbb{C}})$ and $\text{Homeo}^c(\bar{\mathbb{C}})$ form a group under composition.

Example 1.3.1. The functions

$$F(z) = \begin{cases} az+b & \text{if } z \in \mathbb{C} \textit{ and } a \neq 0 \\ \infty & \text{if } z = \infty. \end{cases}$$

and

$$J(z) = \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C} - \{0\} \\ \infty & \text{if } z = 0 \\ 0 & \text{if } z = \infty. \end{cases}$$

belong to $\text{Homeo}^c(\bar{\mathbb{C}})$.

To show $F(z) \in \text{Homeo}^c(\bar{\mathbb{C}})$, set $w = az+b$, so that $z = \frac{1}{a}(w-b)$. Substituting this into the equation of a Euclidean circle, namely, $\alpha z\overline{z}+\beta z+\beta\overline{z}+\gamma = 0$, gives

$$\begin{aligned} \alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma &= \alpha\frac{1}{a}(w-b)\frac{1}{a}(w-b) + \beta\frac{1}{a}(w-b) + \overline{\beta}\frac{1}{a}(w-b) + \gamma \\ &= \frac{\alpha}{|a|^2}(w-b)\overline{(w-b)} + \frac{\beta}{a}(w-b) + \overline{\left(\frac{\beta}{a}\right)}\overline{w-b} + \gamma \\ &= \frac{\alpha}{|a|^2}\left|w + \frac{\overline{\beta}a}{\alpha} - b\right|^2 + \gamma - \frac{|\beta|^2}{\alpha} = 0, \end{aligned}$$

which is again the equation of a Euclidean circle in $\mathbb{C}$.

To show $J(z) \in \text{Homeo}^c(\bar{\mathbb{C}})$, set $w = \frac{1}{z}$, so that $z = \frac{1}{w}$. Substituting this back into the equation of Euclidean circle, $\alpha z\overline{z} + \beta z + \beta\overline{z} + \gamma = 0$, gives

$$\alpha\frac{1}{w}\frac{\overline{1}}{w} + \beta\frac{1}{w} + \overline{\beta}\frac{\overline{1}}{w} + \gamma = 0.$$

Multiplying through by $w\overline{w}$, we see that w satisfies the equation

$$\alpha + \beta\overline{w} + \overline{\beta}w + \gamma w\overline{w} = 0$$

As α and γ are real and as the coefficients of w and $\overline{w}$ are complex conjugates, this is again the equation of a circle in $\overline{\mathbb{C}}$.

Definition 1.3.1. A *Möbius transformation* is a function $m : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ of the form

$$m(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Let $M\ddot{o}b^+$ denote the set of all Möbius transformations.
The set of all Möbius transformations ($M\ddot{o}b^+$) on $\bar{\mathbb{C}}$ forms a group under composition.

Theorem 1.2. $M\ddot{o}b^+ \subset Homeo^c(\bar{\mathbb{C}})$

Proof. We prove this by writing a typical element of $M\ddot{o}b^+$ as a composition of elements of $\mathrm{Homeo}^c(\bar{\mathbb{C}})$, using Example 1.3.1. Consider the Möbius transformation

$$m(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.
If $c = 0$, then $m(z) = \frac{a}{d}z + \frac{b}{d}$. If $c \neq 0$, then $m(z) = f(J(g(z)))$, where $g(z) = c^2 z + cd$ and $f(z) = -(ad-bc)z + \frac{a}{c}$ for $z \in C$, and $f(\infty) = \infty = g(\infty)$ and $J(z) = \frac{1}{z}$ for $z \in \bar{\mathbb{C}} - \{0\}, J(0) = \infty$, and $J(\infty) = 0$. Hence, we can see that every element of $M\ddot{o}b^+$ can be written in terms of elements of $\mathrm{Homeo}^c(\bar{\mathbb{C}})$. □

Theorem 1.3.

$$M\ddot{o}b^+ \cong PGL_2(\mathbb{C}) = PSL_2(\mathbb{C}).$$

Proof. Consider the map $\mu : \mathrm{GL}_2(\mathbb{C}) \to M\ddot{o}b^+$ defined as

$$\mu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(m(z) = \frac{az+b}{cz+d} \right), \quad with\ a, b, c, d \in \mathbb{C}\ and\ ad - bc \neq 0.$$

Observe that kernel of μ is given by,

$$ker(\mu) = \{\lambda I_2 \ \mid \lambda \in \mathbb{C}\} \leq \mathrm{GL}_2(\mathbb{C}).$$

By the First Isomorphism Theorem, we get

$$M\ddot{o}b^{+} \cong \mathrm{GL}_2(\mathbb{C})/ker(\mu) = \mathrm{PGL}_2(\mathbb{C}).$$

$\square$

Now we talk about *uniquely triply transitive* action of $M\ddot{o}b^{+}$ on $\bar{\mathbb{C}}$. Which means for given two distinct triples (z_1, z_2, z_3) and (w_1, w_2, w_3) of points of $\bar{\mathbb{C}}$, there exists a unique element $m \in M\ddot{o}b^{+}$ so that $m(z_1) = w_1$, $m(z_2) = w_2$, and $m(z_3) = w_3$.

Definition 1.3.2. A group G acts on a set X if there is a homomorphism from G into the group $bij(X)$ of bijections of X.

Definition 1.3.3. Suppose that a group G acts on a set X, and let $x_0 \in X$. If for each point $y \in X$ there exists an element $g \in G$ so that $g(y) = x_0$, then, G is said to act transitively on X.

Theorem 1.4. *$M\ddot{o}b^{+}$ acts transitively on the set C of circles in $\bar{\mathbb{C}}$.*

Proof. We use the fact that any triple of distinct points in $\bar{\mathbb{C}}$ defines a unique circle in $\bar{\mathbb{C}}$. To check this, let (z_1, z_2, z_3) be a triple of distinct points of $\bar{\mathbb{C}}$ and are not collinear, then there exists a unique Euclidean circle passing through all three with center at the intersection of the perpendicular bisector of the line joining z_1, z_2 and the line joining z_2, z_3. If the points are collinear, then there exists a unique Euclidean line passing through all three. If one of the z_1, z_2, z_3 is ∞, then there is a unique Euclidean line passing through the other two.

Let C_1 and C_2 be two circles in $\bar{\mathbb{C}}$. Choose a triple of distinct points on C_1 and a triple of distinct points on C_2, and let m be the Möbius transformation taking the triple of distinct points determining C_1 to the triple of distinct points determining C_2. As $m(C_1)$ and C_2 are two circles in $\bar{\mathbb{C}}$ that pass through the same triple of distinct points and as *Möbius* transformations take circles in $\bar{\mathbb{C}}$ to circles in $\bar{\mathbb{C}}$, we have that $\mathrm{m}(C_1) = C_2$. $\square$

1.4 The group $M\ddot{o}b(\mathbb{H})$

We have seen that $M\ddot{o}b^+$ is contained in the set Homeo$^c(\bar{\mathbb{C}})$ of homeomorphisms of $\bar{\mathbb{C}}$ that takes circles in $\bar{\mathbb{C}}$ to circles in $\bar{\mathbb{C}}$. There is a natural extension of $M\ddot{o}b^+$ that also lies in Homeo$^c(\bar{\mathbb{C}})$. Consider the homeomorphism of $\bar{\mathbb{C}}$ not in $M\ddot{o}b^+$, namely, *complex conjugation*, which helps to extend $M\ddot{o}b^+$ to a larger group.
Note that C is its own inverse, that is $C^{-1}(z) = C(z)$, so C is a bijection. Further C is continuous, because, for any, point $z \in \overline{\mathbb{C}}$ and any $\epsilon > 0$, we have that $C\left(U_\varepsilon(z)\right) = U_\varepsilon(C(z))$.

Definition 1.4.1. The *general Möbius group* ($M\ddot{o}b$) is the group generated by $M\ddot{o}b^+$ and C. That means we can express every element $p \in M\ddot{o}b$ as a composition

$$p = C \circ m_k \circ ...C \circ m_1$$

for some $k \geq 1$, where each m_k is an element of $M\ddot{o}b^+$.

Theorem 1.5. *$M\ddot{o}b \subset Homeo^c(\bar{\mathbb{C}})$.*

Proof. By Definition 1.4.1, we can see that elements of the general $M\ddot{o}bius$ group are either from $M\ddot{o}b^+$ or a composition of elements of $M\ddot{o}b^+$ with the reflection $C(z) = \bar{z}$ for, $z \in \mathbb{C}$ and $C(\infty) = \infty$. Hence, by Theorem 1.2, we have that every element of $M\ddot{o}b^+$ lies in Homeo$^c(\bar{\mathbb{C}})$. So we only need to check that complex conjugation C lies in Homeo$^c(\bar{\mathbb{C}})$. Let A be the circle in $\overline{\mathbb{C}}$ given by the equation $\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$. Set $w = C(z) = \overline{z}$, so that $z = \overline{w}$ and note that w then satisfies the equation $\alpha w\overline{w} + \overline{\beta}w + \beta\overline{w} + \gamma = 0$, which is again the equation of a circle in $\overline{\mathbb{C}}$, as desired.

□

1.5 Isometries of $\mathbb{H}$

Definition 1.5.1. A homeomorphism of $\bar{\mathbb{C}}$ that preserves the absolute value of the angle between curves is said to be *conformal*.

Theorem 1.6. *The elements of $M\ddot{o}b$ are conformal homeomorphisms of $\bar{\mathbb{C}}$.*

Proof. As $M\ddot{o}b$ is generated by the transformations of the form $f(z) = az + b$ for $a, b \in \mathbb{C}$ and $a \neq 0$, and $J(z) = \frac{1}{z}$ and $C(z) = \bar{z}$. So, we only need to check for these transformations. Let X_1 and X_2 be two Euclidean lines in $\mathbb{C}$ that intersect at a point z_0, let z_k be a point on X_k not equal to z_0, and let s_k be the slope of X_k. These quantities are connected by the equation

$$s_k = \frac{\operatorname{Im}(z_k - z_0)}{\operatorname{Re}(z_k - z_0)}.$$

Let θ_k be the angle that X_k makes with the real axis $\mathbb{R}$, and note that $s_k = \tan(\theta_k)$. In particular, the angle $angle(X_1, X_2)$ between X_1 and X_2 is given by

$$\operatorname{angle}(X_1, X_2) = \theta_2 - \theta_1 = \arctan(s_2) - \arctan(s_1).$$

Consider $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$. Write $a = \rho e^{i\beta}$. As $f(\infty) = \infty$, both $f(X_1)$ and $f(X_2)$ are again Euclidean lines in $\mathbb{C}$. As $f(X_k)$ passes through the points $f(z_0)$ and $f(z_k)$, the slope t_k of the Euclidean line $f(X_k)$ is

$$\begin{aligned} t_k = \frac{\operatorname{Im}(f(z_k) - f(z_0))}{\operatorname{Re}(f(z_k) - f(z_0))} &= \frac{\operatorname{Im}(a(z_k - z_0))}{\operatorname{Re}(a(z_k - z_0))} \\ &= \frac{\operatorname{Im}\left(e^{i\beta}(z_k - z_0)\right)}{\operatorname{Re}\left(e^{i\beta}(z_k - z_0)\right)} = \tan(\beta + \theta_k). \end{aligned}$$

We see that,

$$\begin{aligned} \operatorname{angle}(f(X_1), f(X_2)) &= \arctan(t_2) - \arctan(t_1) \\ &= (\beta + \theta_2) - (\beta + \theta_1) \\ &= \theta_2 - \theta_1 = \operatorname{angle}(X_1, X_2), \end{aligned}$$

and so m is conformal.

For the case $J(z) = \frac{1}{z}$, $J(X_1)$ and $J(X_2)$ need not be necessarily Euclidean lines in $\mathbb{C}$, but instead they may be both Euclidean circles in $\mathbb{C}$ that intersect at 0, or one might be a Euclidean line and other a Euclidean circle. We give

the argument for the case. When both are Euclidean circles.
Let X_k is given as the solutions of the equation $\beta_k z + \overline{\beta_k}\overline{z} + 1 = 0$, where $\beta_k \in \mathbb{C}$. The slope of X_k is given by

$$s_k = \frac{\mathrm{Re}\,(\beta_k)}{\mathrm{Im}\,(\beta_k)}.$$

Given the form of the equation for X_k, we also know that $J(X_k)$ is the set of solutions to the equation

$$z\overline{z} + \overline{\beta_k} z + \beta_k \overline{z} = 0,$$

which we can rewrite as

$$|z + \beta_k|^2 = |\beta_k|^2\,,$$

so that $J(X_k)$ is the Euclidean circle with Euclidean center $-\beta_k$ and Euclidean radius $|\beta_k|$. The slope of the tangent line to $J(X_k)$ at 0 is then

$$-\frac{\mathrm{Re}\,(\beta_k)}{\mathrm{Im}\,(\beta_k)} = -\tan(\theta_k) = \tan(-\theta_k)\,,$$

and so $J(X_k)$ makes angle $-\theta_k$ with $\mathbb{R}$. The angle between $J(X_1)$ and $J(X_2)$ is then given by

$$\mathrm{angle}\,(J\,(X_1)\,, J\,(X_2)) = -\theta_2 - (-\theta_1) = -\,\mathrm{angle}\,(X_1, X_2)\,,$$

and so J is conformal.
Now consider $C(z) = \bar{z}$. As X_k passes through z_0 and z_k, we have that $C(X_k)$ passes through $C(z_0) = \overline{z_0}$ and $C(z_k) = \overline{z_k}$, and so $C(X_k)$ has slope

$$S_k = \frac{\mathrm{Im}\,(\overline{z_k} - \overline{z_0})}{\mathrm{Re}\,(\overline{z_k} - \overline{z_0})} = -\frac{\mathrm{Im}\,(z_k - z_0)}{\mathrm{Re}\,(z_k - z_0)} = -s_k.$$

Then

$$\begin{aligned}\mathrm{angle}\,(C\,(X_1)\,, C\,(X_2)) &= \arctan(S_2) - \arctan(S_1)\\ &= -\arctan(s_2) + \arctan(s_1) = -\,\mathrm{angle}\,(X_1, X_2)\,.\end{aligned}$$

Hence, C is conformal, as it preserves the absolute value of the angle between Euclidean lines.

□

Definition 1.5.2. The group $M\ddot{o}b(\mathbb{H}) := \{m \in M\ddot{o}b \mid m(\mathbb{H}) = \mathbb{H}\ \}$.

Theorem 1.7. *Every element of $M\ddot{o}b(\mathbb{H})$ takes hyperbolic lines in $\mathbb{H}$ to hyperbolic lines in $\mathbb{H}$.*

Proof. By Definition 1.2.1 of hyperbolic line, every hyperbolic line in $\mathbb{H}$ is the intersection of $\mathbb{H}$ with a circle in $\bar{\mathbb{C}}$ perpendicular to $\bar{\mathbb{R}}$ and that every element of $M\ddot{o}b$ takes circles in $\bar{\mathbb{C}}$ to circles in $\bar{\mathbb{C}}$. Now by Theorem 1.6, we know that every element of $M\ddot{o}b(\mathbb{H})$ preserves angle between circles in $\bar{\mathbb{C}}$, from which the assertion follows. □

As $\mathbb{H}$ is a disc in $\bar{\mathbb{C}}$ determined by the circle in $\bar{\mathbb{C}}$, $\bar{\mathbb{R}}$, let us first determine the explicit form of an element of

$$\text{Möb}(\overline{\mathbb{R}}) = \{m \in \text{Möb}\,|m(\overline{\mathbb{R}}) = \overline{\mathbb{R}}\}.$$

As we know that every element of $M\ddot{o}b$ can be written either as $m(z) = \frac{az+b}{cz+d}$ or as $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Now we determine some condition on a, b, c, d by requiring that $m(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$.
As m takes $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$, we have that the three points

$$m^{-1}(\infty) = -\frac{d}{c}, m(\infty) = \frac{a}{c},\ \text{and}\ m^{-1}(0) = -\frac{b}{a}.$$

all lie in $\overline{\mathbb{R}}$.
If $a \neq 0$ and $c \neq 0$, then all these three points lie in $\mathbb{R}$. We can express each coefficient of m as a multiple of c. We have that

$$a = m(\infty)c,\ b = -m^{-1}(0)a = -m^{-1}(0)m(\infty)c,\ \text{and}\ d = -m^{-1}(\infty)c.$$

So, we can rewrite m as

$$m(z) = \frac{az+b}{cz+d} = \frac{m(\infty)cz - m^{-1}(0)m(\infty)c}{cz - m^{-1}(\infty)c}.$$

As determinant of m is 1, we have

$$\begin{aligned} 1 = ad - bc &= c^2 \left[-m(\infty)m^{-1}(\infty) + m(\infty)m^{-1}(0) \right] \\ &= c^2 \left[m(\infty) \left(m^{-1}(0) - m^{-1}(\infty) \right) \right]. \end{aligned}$$

As $m(\infty), m^{-1}(0)$, and $m^{-1}(\infty)$ are all real, this implies that c is either real or purely imaginary. Hence the coefficients of m are either all real or all purely imaginary.

Theorem 1.8. *Every element of $M\ddot{o}b(\mathbb{H})$ has the form*

$$m(z) = \frac{az+b}{cz+d}, \textit{where } a,b,c,d \in \mathbb{R} \textit{ and } ad - bc = 1,$$

or the form

$$m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}, \textit{where } a,b,c,d \textit{ are purely imaginary and } ad - bc = 1.$$

Proof. Note that each element of $M\ddot{o}b(\overline{\mathbb{R}})$ either preserves each of the two discs in $\bar{\mathbb{C}}$ determined by $\bar{\mathbb{R}}$, namely the upper and lower half-planes, or interchanges them. We consider the image of a single point in one of the discs. An element m of $M\ddot{o}b(\overline{\mathbb{R}})$ is an element of $M\ddot{o}b(\overline{\mathbb{H}})$ if and only if the imaginary part of $m(i)$ is positive. So, it suffices to check the value of $Im(m(i))$ for $m \in M\ddot{o}b(\overline{\mathbb{R}})$.

If m is of the form $m(z) = \frac{az+b}{cz+d}$, where a,b,c,d are real and $ad - bc = 1$, then the imaginary part of $m(i)$ is,

$$\begin{aligned} \operatorname{Im}(m(i)) &= \operatorname{Im}\left(\frac{ai+b}{ci+d}\right) \\ &= \operatorname{Im}\left(\frac{(ai+b)(-ci+d)}{(ci+d)(-ci+d)}\right) = \frac{ad-bc}{c^2+d^2} = \frac{1}{c^2+d^2} > 0, \end{aligned}$$

and so m lies in $M\ddot{o}b(\mathbb{H})$.

If m is of form $m(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, where a,b,c,d are real and $ad - bc = 1$, then the imaginary part of $m(i)$ is,

$$\begin{aligned}\mathrm{Im}(m(i)) &= \mathrm{Im}\left(\frac{-ai+b}{-ci+d}\right) \\ &= \mathrm{Im}\left(\frac{(-ai+b)(ci+d)}{(-ci+d)(ci+d)}\right) = \frac{-ad+bc}{c^2+d^2} = \frac{-1}{c^2+d^2} < 0,\end{aligned}$$

and so m does not lie in $M\ddot{o}b(\mathbb{H})$.
If m is of the form $m(z) = \frac{az+b}{cz+d}$, where a, b, c, d are purely imaginary and $ad - bc = 1$, write $a = \alpha i$, $b = \beta i$, $c = \gamma i$ and $d = \delta i$, such that $\alpha\delta - \beta\gamma = -1$. then the imaginary part of $m(i)$ is,

$$\begin{aligned}\mathrm{Im}(m(i)) &= \mathrm{Im}\left(\frac{ai+b}{ci+d}\right) = \mathrm{Im}\left(\frac{-\alpha+\beta i}{-\gamma+\delta i}\right) \\ &= \mathrm{Im}\left(\frac{(-\alpha+\beta i)(-\gamma-\delta i)}{(-\gamma+\delta i)(-\gamma-\delta i)}\right) = \frac{\alpha\delta-\beta\gamma}{\gamma^2+\delta^2} = \frac{-1}{\gamma^2+\delta^2} < 0,\end{aligned}$$

and so m does not lie in $M\ddot{o}b(\mathbb{H})$.
If m has the form $m(z) = \frac{a\bar{z}+b}{c\bar{z}+b}$, where a, b, c, and d are purely imaginary and $ad - bc = 1$, write $a = \alpha i$, $b = \beta i$, $c = \gamma i$ and $d = \delta i$, such that $\alpha\delta - \beta\gamma = -1$. then the imaginary part of $m(i)$ is,

$$\begin{aligned}\mathrm{Im}(m(i)) &= \mathrm{Im}\left(\frac{-ai+b}{-ci+d}\right) = \mathrm{Im}\left(\frac{\alpha+\beta i}{\gamma+\delta i}\right) \\ &= \mathrm{Im}\left(\frac{(\alpha+\beta i)(\gamma-\delta i)}{(\gamma+\delta i)(\gamma-\delta i)}\right) = \frac{-\alpha\delta+\beta\gamma}{\gamma^2+\delta^2} = \frac{1}{\gamma^2+\delta^2} > 0,\end{aligned}$$

and so m lies in $M\ddot{o}b(\mathbb{H})$.

□

Theorem 1.9. *$M\ddot{o}b(\mathbb{H})$ acts transitively on $\mathbb{H}$.*

Proof. It suffices to show that for any point w of $\mathbb{H}$, there exists an element m of $M\ddot{o}b(\mathbb{H})$, such that $m(w) = i$. Let $w = a + ib$, where $a, b \in \mathbb{R}$ and $b > 0$. Now we construct an element of $M\ddot{o}b(\mathbb{H})$ which takes w to i. First, move w to the positive imaginary axis using $p(z) = z - a$, so that $p(w) = p(a + ib) = bi$. Now, we apply $q(z) = \frac{1}{b}z$ to $p(w)$, so that $q(p(w)) = q(bi) = i$. Observe that $q \circ p(z) = \frac{z-a}{b}$ lies in $M\ddot{o}b(\mathbb{H})$ for $-a \in \mathbb{R}$ and $\frac{1}{b} > 0$ as desired. □

Theorem 1.10. *$M\ddot{o}b(\mathbb{H})$ acts transitively on the set L of hyperbolic lines in $\mathbb{H}$.*

Proof. Let ℓ be a hyperbolic line. Using definition of transitive action it is enough to construct an element of $M\ddot{o}b(\mathbb{H})$ which takes ℓ to the positive imaginary axis. We can construct such element by taking the endpoints at infinity of ℓ to 0 and ∞.

□

1.5.1 Classification of elements of M$\ddot{o}b^+(\mathbb{H})$

Two Möbius transformations m_1, $m_2 \in M\ddot{o}b^+$ are said to be *conjugate* if there exists $p \in M\ddot{o}b^+$ such that $m_2 = p \circ m_1 \circ p^{-1}$. As $\frac{az+b}{cz+d} = z$ yields a quadratic equation

$$p(z) = cz^2 + (d-a)z - b = 0,$$

any $m \in M\ddot{o}b^+$ can have at most 2 fixed points in $\overline{\mathbb{C}}$.

Definition 1.5.3. Using the notion of fixed points we classify elements of $M\ddot{o}b^+$ into three categories:

1. If any element $m \in M\ddot{o}b^+$ has one fixed point in $\overline{\mathbb{R}}$ called *parabolic.*
2. If m has two fixed points in $\overline{\mathbb{R}}$ called *hyperbolic.*
3. If m has one fixed point in $\mathbb{H}$ called *elliptic.*

Theorem 1.11. *Let $m(z) = \frac{az+b}{cz+d} \in M\ddot{o}b^+(\mathbb{H})$; where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, then on the basis of fixed point of m, we have following classifications of the elements of $M\ddot{o}b^+(\mathbb{H})$,*

1. *If m is parabolic then it is conjugate in Möb$^+(\mathbb{H})$ to $q(z) = z+1$.*
2. *If m is hyperbolic then it is conjugate in Möb$^+(\mathbb{H})$ to $q(z) = \lambda z$ for some positive real number $\lambda \neq \{0, 1\}$.*
3. *If m is elliptic then it is conjugate in Möb$^+(\mathbb{H})$ to $q(z) = \frac{cos(\theta)z+sin(\theta)}{-sin(\theta)z+cos(\theta)}$ for some real number θ.*

Proof. 1. Let the fixed point of m in $\overline{\mathbb{R}}$ be x. Let y be any point in $\overline{\mathbb{C}} - \{x\}$, and observe that $(x, y, m(y))$ is a triple of distinct points. Let $p \in M\ddot{o}b^+(\mathbb{H})$ which maps $(x, y, m(y))$ to the triple $(\infty, 0, 1)$. Now by our construction, $p \circ m \circ p^{-1}(\infty) = p \circ m(x) = p(x) = \infty$. As $p \circ m \circ p^{-1}$ fixes ∞, we can write it as $p \circ m \circ p^{-1}(z) = az + b,$ with $a \neq 0$. As $p \circ m \circ p^{-1}$ has only one fixed point, and there is no other fixed point to the equation $p \circ m \circ p^{-1}(z) = z$, so it must be that $a = 1$. As $p \circ m \circ p^{-1}(0) = p \circ m(y) = 1$, we have $b = 1$, and so $p \circ m \circ p^{-1}(z) = z + 1$.
2. Let the fixed points of m in $\overline{\mathbb{R}}$ be x and y, and let $q \in M\ddot{o}b^+(\mathbb{H})$ such that $q(x) = 0$ and $q(y) = \infty$. Now by our construction $q \circ m \circ q^{-1}(\infty) = \infty$ and $q \circ m \circ q^{-1}(0) = 0$. Hence, we can write $q \circ m \circ q^{-1}(z) = \lambda z$ for $\lambda \in \mathbb{C} - \{0, 1\}$.
3. We may assume without loss of generality that $m(i) = i$. As m is an element of $M\ddot{o}b^+(\mathbb{H})$, has form $m(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$. As i is fixed, we have

$$\frac{ai + b}{ci + d} = i,$$

so we get

$$ai + b = -c + di.$$

By comparing real and imaginary parts, we get $a = d$ and $b = -c$, now substituting these in equation $ad - bc = 1$ we get $a^2 + b^2 = 1$. This shows that $a = d = \cos(\theta)$ and $b = -c = \sin(\theta)$ for some real θ. □

1.5.2 Length and distance in $\mathbb{H}$

We construct an invariant notion of distance on $\mathbb{H}$ and explore some of its basic properties.
The metric for the upper half plane model is defined as

$$ds^2 = \frac{|dz|^2}{Im(z)^2}.$$

For every piecewise C^1 path $f : [a, b] \to \mathbb{H}$, we define the hyperbolic length of f to be

$$length_{\mathbb{H}}(f) = \int_f \frac{1}{Im(z)}|dz| = \int_a^b \frac{1}{Im(f(t))}|f'(t)|dt.$$

Theorem 1.12. *Hyperbolic length of any piecewise continuous path γ in $\mathbb{H}$ is invariant under the action of $M\ddot{o}b(\mathbb{H})$.*

Proof. Let

$$T(z) = \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{R}, ad - bc = 1).$$

Then,

$$\frac{dT}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}.$$

Also, if $z = x + iy, T(z) = u + iv$, then

$$v = \frac{y}{|cz+d|^2},$$

and hence, we have

$$\left|\frac{dT}{dz}\right| = \frac{v}{y}.$$

Thus,

$$\begin{aligned} length_{\mathbb{H}}(T(\gamma)) &= \int_0^1 \frac{\left|\frac{dT}{dt}\right| dt}{v} = \int_0^1 \frac{\left|\frac{dT}{dz}\frac{dz}{dt}\right| dt}{v} = \int_0^1 \frac{v\left|\frac{dz}{dt}\right| dt}{y} \\ &= \int_0^1 \frac{\left|\frac{dz}{dt}\right| dt}{y} = length_{\mathbb{H}}(\gamma). \end{aligned}$$

□

Proposition 1.5.1. *For every element $\gamma \in M\ddot{o}b(\mathbb{H})$ and for every pair $x, y \in \mathbb{H}$, we have*

$$d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(\gamma(x), \gamma(y)).$$

Definition 1.5.4. An *isometry* of a metric space (X, d) is a homeomorphism f of X that preserve distance, that is,

$$d(x, y) = d(f(x), f(y)).$$

$\mathrm{Isom}(\mathbb{H}, d_{\mathbb{H}})$ denotes the *group of isometries* of $(\mathbb{H}, d_{\mathbb{H}})$.

Theorem 1.13. $M\ddot{o}b(\mathbb{H}) = Isom(\mathbb{H}, d_{\mathbb{H}})$.

Proof. We have already seen that $M\ddot{o}b(\mathbb{H}) \subset Isom(\mathbb{H}, d_{\mathbb{H}})$. It remains to show that

$$M\ddot{o}b(\mathbb{H}) \supset Isom(\mathbb{H}, d_{\mathbb{H}}).$$

Let f be a hyperbolic isometry. For any pair $p, q \in \mathbb{H}$, let ℓ_{pq} denote the hyperbolic line segment joining p *to* q. We have that $\ell_{f(p)f(q)} = f(\ell_{pq})$. Let $\ell = \{z \in \mathbb{H} \mid d_{\mathbb{H}}(p, z) = d_{\mathbb{H}}(q, z)\}$ (perpendicular bisector of ℓ_{pq}), so that $f(\ell)$ is the perpendicular bisector of $f(\ell_{pq})$.
Choose x and y on the positive imaginary axis I^+ in $\mathbb{H}$, and let H be one of the half-planes in $\mathbb{H}$ determined by I^+. We can find $\gamma \in M\ddot{o}b(\mathbb{H})$, which satisfies $\gamma(f(x)) = x$ and $\gamma(f(y)) = y$, because $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(f(x), f(y))$. We can see that $\gamma \circ f$ fixes both x and y, and so $\gamma \circ f$ takes I^+ to I^+ and H to H.
As we can determine every point $z \in I^+$ using $d_{\mathbb{H}}(y, z)$ and $d_{\mathbb{H}}(x, z)$, and both distances are preserved by $\gamma \circ f$, $\gamma \circ f$ fixes every point $z \in I^+$.

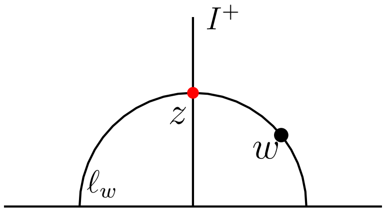

Fig. 1.3: Hyperbolic line passing through w and perpendicular to I^+

Let $w \in \mathbb{H} \backslash I^+$, and let ℓ_w be the hyperbolic line through w and perpendicular to I^+, Figure 1.3. Since

$$d_{\mathbb{H}}(z, w) = d_{\mathbb{H}}(\gamma \circ f(z), \gamma \circ f(w)) = d_{\mathbb{H}}(z, \gamma \circ f(w)),$$

$\gamma \circ f$ fixed z, which implies $\gamma \circ f$ fixes w, (using the fact that $\gamma \circ f$ preserves the two half-planes determined by I^+). Hence $\gamma \circ f$ fixes every point of $\mathbb{H}$, which implies that $\gamma \circ f$ is the identity.

$\square$

Definition 1.5.5. Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}$. Then their *cross-ratio* is defined as

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}.$$

Theorem 1.14. *Let z_1, z_2 be two distinct points in $\mathbb{H}$ and let z_1^*, z_2^* be the endpoints of geodesic joining z_1 and z_2 in $\overline{\mathbb{R}}$, chosen in such a way that z_1 lies between z_1^* and z_2. Then*

$$d_{\mathbb{H}}(z_1, z_2) = \ln[z_2, z_1^*; z_1, z_2^*].$$

Proof. Using transitive action of $M\ddot{o}b(\mathbb{H})$ on the set of hyperbolic lines, we can find an element $S \in M\ddot{o}b(\mathbb{H})$ which maps the geodesic joining z_1 and z_2 to the imaginary axis. We may assume that $S(z_1^*) = 0$, $S(z_2^*) = \infty$ and $S(z_1) = i$. Then $S(z_2) = ik(k > 1)$. Let $\gamma : [0, 1] \to \mathbb{H}$ is any piecewise differentiable path joining i and ik, with $\gamma(t) = (x(t), y(t))$, then

$$length_{\mathbb{H}}(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt}{y(t)} = \int_0^1 \frac{\frac{|dy|}{|dt|} dt}{y(t)} \geq \int_0^1 \frac{\frac{dy}{dt} dt}{y(t)} = \int_1^k \frac{dy}{y} = \ln k.$$

But $k = [ki, 0; i, \infty]$, and the assertion follows from the invariance of the cross-ratio under linear fractional transformations (see [4]). □

Theorem 1.15. *The geodesics in $\mathbb{H}$ are semicircles and straight lines orthogonal to the real axis $\mathbb{R}$.*

Proof. Let $z_1 = ia$ and $z_2 = ib$ in $\mathbb{H}$ $(b > a)$. Choose any piecewise C^1 path γ joining ia and ib, with $\gamma(t) = (x(t), y(t))$, then $length_{\mathbb{H}}(\gamma) = ln\frac{b}{a}$. For arbitrary z_1 and z_2 in $\mathbb{H}$, let L be the unique Euclidean line or circle orthogonal to the real axis passing through those points. Using the transitive action of $M\ddot{o}b(\mathbb{H})$ we can map L to the imaginary axis. From the isometric behavior of $M\ddot{o}b(\mathbb{H})$, it follows that the geodesic joining z_1 and z_2 is the line segment of L joining them. □

1.6 The Poincaré Disk Model

In this section, we explore a second model, the *Poincaré disk* model $\mathbb{D}$, of the hyperbolic plane. We construct this model using upper half-plane model. The underlying space for the Poincaré disk model of the hyperbolic plane is the open unit disk,

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

in the complex plane $\mathbb{C}$. Now we construct an element $m : \mathbb{D} \to \mathbb{H}$ of Möb to transport hyperbolic geometry from $\mathbb{H}$ to $\mathbb{D}$. There are so many ways to construct such an element, but we consider the unique Möbius transformation m mapping the triple $(i, 1, -i)$ to the triple $(0, 1, \infty)$, namely

$$m(z) = \frac{iz + 1}{z + i}.$$

A hyperbolic line in $\mathbb{D}$ is defined as the image of hyperbolic line in $\mathbb{H}$ under m^{-1}. For any piecewise differentiable path $f : [a, b] \to \mathbb{D}$, the composition $m \circ f : [a, b] \to \mathbb{H}$ will be a piecewise differentiable path in $\mathbb{H}$. So, hyperbolic length of f in $\mathbb{D}$ is defined as

$$length_{\mathbb{D}}(f) = length_{\mathbb{H}}(m \circ f).$$

Theorem 1.16. *The hyperbolic length of a piecewise differentiable path $f : [a, b] \to \mathbb{D}$ is given by*

$$length_{\mathbb{D}}(f) = \int_f \frac{2}{1 - |z|^2} |dz|.$$

Proof. We prove this by taking $m : \mathbb{D} \to \mathbb{H}$ which we have constructed above, but this is independent of the choice of the element of M*öb* taking $\mathbb{D}$ to $\mathbb{H}$.

$$\begin{aligned}
length_{\mathbb{D}}(f) = length_{\mathbb{H}}(m \circ f) &= \int_{m \circ f} \frac{1}{\mathrm{Im}(z)} |\mathrm{d}z| \\
&= \int_a^b \frac{1}{\mathrm{Im}((m \circ f)(t))} \left|(m \circ f)'(t)\right| \mathrm{d}t \\
&= \int_a^b \frac{1}{\mathrm{Im}(m(f(t)))} \left|m'(f(t))\right| \left|f'(t)\right| \mathrm{d}t \\
&= \int_f \frac{1}{\mathrm{Im}(m(z))} |m'(z)||\mathrm{d}z|.
\end{aligned}$$

We have

$$\mathrm{Im}(m(z)) = \mathrm{Im}\left(\frac{zi+1}{z+i}\right) = \frac{1-|z|^2}{|z+i|^2},$$

and that

$$|m'(z)| = \frac{2}{|z+i|^2},$$

and so

$$\frac{1}{\mathrm{Im}(m(z))}|m'(z)| = \frac{2}{1-|z|^2}.$$

Hence,

$$length_{\mathbb{D}}(f) = \int_f \frac{2}{1-|z|^2}|dz|.$$

To show this hyperbolic length is independent of the choice of m. Let n be any other element of Möb taking $\mathbb{D}$ to $\mathbb{H}$. As $n \circ m^{-1}$ takes $\mathbb{H}$ to $\mathbb{H}$, we have $q = n \circ m^{-1} \in M\ddot{o}b(\mathbb{H})$. By the invariance of hyperbolic length under Möb($\mathbb{H}$), we have

$$length_{\mathbb{H}}(m \circ f) = length_{\mathbb{H}}(q \circ m \circ f) = length_{\mathbb{H}}(n \circ f)$$

□

Theorem 1.17. *Let a and b be two distinct points inside Poincaré disk. Then hyperbolic distance between a and b is,*

$$d_{\mathbb{D}}(a,b) = \cosh^{-1}\left(1 + \frac{2|a-b|^2}{(1-|a|^2)(1-|b|^2)}\right).$$

Proof. Let choose these points are 0 and r, where $0 < r < 1$. Now parametrize the hyperbolic line segment between 0 and r by the path $f : [0, r] \to \mathbb{D}$ given by $f(t) = t$. As the image of f is the hyperbolic line segment in $\mathbb{D}$ joining 0 and r, we have that

$$d_{\mathbb{D}}(0, r) = length_{\mathbb{D}}(f),$$

$$\begin{aligned}
length_{\mathbb{D}}(f) &= \int_f \frac{2}{1 - |z|^2}|dz| \\
&= \int_0^r \frac{2}{1 - t^2}dt \\
&= \int_0^r \left[\frac{1}{1+t} + \frac{1}{1-t}\right] dt \\
&= \ln\left[\frac{1+r}{1-r}\right].
\end{aligned}$$

We can further solve for r, and we get

$$r = \tanh\left[\frac{1}{2}d_{\mathbb{D}}(0, r)\right].$$

Now choose an element $p(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \in M\ddot{o}b^+(\mathbb{D})$ (where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$) such that $p(a) = 0$ and $p(b)$ is real and positive. One way to do this is to set $\beta = -\alpha a$, so that

$$p(z) = \frac{\alpha(z - a)}{\overline{\alpha}(-\overline{a}z + 1)},$$

where $|\alpha|^2 (1 - |a|^2) = 1$. Now choose the argument of α so that $p(b) = r$ is real and positive. Now using the fact that for any pair $a, b \in \mathbb{D}$, we can write

$$\frac{|a - b|^2}{(1 - |a|^2)(1 - |b|^2)} = \frac{|p(a) - p(b)|^2}{(1 - |p(a)|^2)(1 - |p(b)|^2)}.$$

By substituting $p(a) = 0$ and $p(b) = r$, we get

$$\frac{|a - b|^2}{(1 - |a|^2)(1 - |b|^2)} = \frac{r^2}{1 - r^2}.$$

We have $r = \tanh\left(\frac{1}{2}\mathrm{d}_{\mathbb{D}}(0, r)\right)$, and so

$$\frac{r^2}{1-r^2} = \frac{1}{2}\left(\cosh\left(\mathrm{d}_{\mathbb{D}}(a,b)\right) - 1\right).$$

Therefore, we get

$$d_{\mathbb{D}}(a,b) = \cosh^{-1}\left(1 + \frac{2r^2}{1-r^2}\right) = \cosh^{-1}\left(1 + \frac{|a-b|^2}{(1-|a|^2)(1-|b|^2)}\right).$$

□

1.6.1 Convexity

Definition 1.6.1. A subset X of the hyperbolic plane is *convex* if for each pair of distinct points x and y in X, the closed hyperbolic line segment ℓ_{xy} joining x to y is contained in X.

Proposition 1.6.1. *Hyperbolic lines, hyperbolic rays, and hyperbolic line segments are convex.*

Proof. Let ℓ be a hyperbolic line, and let x and y be two points of ℓ. By Proposition 1.2.1, x and y determine a unique hyperbolic line, namely, ℓ, and so the closed hyperbolic line segment ℓ_{xy} joining x to y is necessarily contained in ℓ. Hence, ℓ is convex.
This same argument also shows that hyperbolic rays and hyperbolic line segments are convex. □

For a given hyperbolic line ℓ in the hyperbolic plane, the complement of ℓ in the hyperbolic plane has two components, which are the two *open half-planes determined by* ℓ. A *closed half-plane determined by* ℓ is the union of ℓ with one of the two open half-planes determined by ℓ .

1.7 Hyperbolic Polygons

As in Euclidean geometry, the *polygon* is one of the basic objects in hyperbolic geometry. In the Euclidean plane, a polygon is a closed convex set that is bounded by Euclidean line segments. We would like to mimic this definition as much as possible in the hyperbolic plane.

Definition 1.7.1. Any collection of half-planes $\mathcal{H}$ is *locally finite* if for each point z in the hyperbolic plane, there exists some $\epsilon > 0$ so that only finitely many bounding lines ℓ_α of the half-plane in $\mathcal{H}$ intersect the open hyperbolic disc $U_\epsilon(z)$ of hyperbolic radius ϵ and hyperbolic center z.

Definition 1.7.2. A *hyperbolic polygon* is a closed convex set in the hyperbolic plane that can be expressed as the intersection of a locally finite collection of closed half-planes.

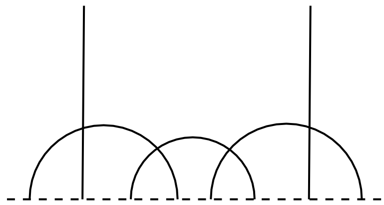

Fig. 1.4: A hyperbolic polygon in $\mathbb{H}$.

Definition 1.7.3. A hyperbolic polygon is *nondegenerate* if it has non-empty interior, and *degenerate* if has empty interior.

Definition 1.7.4. Let P be a hyperbolic polygon, and let v be a vertex of P that is the intersection of two sides s_1 and s_2 of P. Let ℓ_k be the hyperbolic line containing s_k. The union $\ell_1 \cup \ell_2$ divides the hyperbolic plane into four components, one of which contains P. The *interior angle* of P at v is the angle between ℓ_1 and ℓ_2, measured in the component of the complement of $\ell_1 \cup \ell_2$ containing P.

Definition 1.7.5. A hyperbolic polygon P in the hyperbolic plane has an *ideal vertex at* v if there are two adjacent sides of P that are either closed hyperbolic rays or hyperbolic lines and that share v as an endpoint at infinity.

1.7.1 Hyperbolic Area

In addition to those we have already mentioned, one of the nice properties of hyperbolic convex sets in general, and hyperbolic polygons, in particular, is

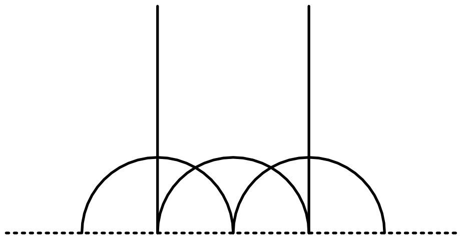

Fig. 1.5: A hyperbolic polygon with an ideal vertex at ∞.

that it is easy to calculate their hyperbolic area. But first, we need to define the hyperbolic area. For now, we work in the upper half-plane model $\mathbb{H}$.

Definition 1.7.6. The *hyperbolic area* $area_{\mathbb{H}}(X)$ of a set X in $\mathbb{H}$ is given by the integral

$$area_{\mathbb{H}}(X) = \int_X \frac{1}{Im(z)^2} dxdy = \int_X \frac{1}{y^2} dxdy,$$

where $z = x + iy$.

Theorem 1.18. *Hyperbolic area in $\mathbb{H}$ is invariant under the action of $M\ddot{o}b(\mathbb{H})$.*

Proof. Let $z = x + iy$,

$$T(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1$$

and $w = T(z) = u + iv$. Then using the Cauchy-Riemann equations we calculate the Jacobian

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{dT}{dx}\right|^2 = \left|\frac{dT}{dz}\right|^2 \\ &= \frac{1}{|cz+d|^4}.\end{aligned}$$

Thus, for a hyperbolic polygon in E

$$area_{\mathbb{H}}(T(E)) = \iint_{T(E)} \frac{dudv}{v^2} = \iint_E \frac{\partial(u,v)}{\partial(x,y)} \frac{dxdy}{v^2}$$
$$= \iint_E \frac{1}{|cz+d|^4} \frac{|cz+d|^4}{y^2} dxdy = area_{\mathbb{H}}(E).$$

□

1.7.2 Gauss-Bonnet formula

In this section, we see the hyperbolic area of any hyperbolic polygon can be written in the form of interior angles. First, we start with the hyperbolic triangle.

Proposition 1.7.1. *Let P be a hyperbolic triangle with one ideal vertex, and let α_2 and α_3 be the interior angles at the other two vertices, which might or might not be ideal vertices. Then,*

$$area_{\mathbb{H}}(P) = \pi - (\alpha_2 + \alpha_3).$$

Proof. Let a hyperbolic triangle P with one vertex is ideal, say v_1, and two other vertices(v_2,v_3) might or might not be ideal. Let ℓ_{jk} be the hyperbolic line determined by v_j and v_k. Using the transitivity properties of $M\ddot{o}b(\mathbb{H})$. We can find a $\gamma \in M\ddot{o}b(\mathbb{H})$ which takes v_1 to ∞ and takes ℓ_{23} to the hyperbolic line contained in the unit circle, so that $v_2 = e^{i\phi}$ and $v_3 = e^{i\theta}$, where $0 \leq \theta < \phi \leq \pi$.

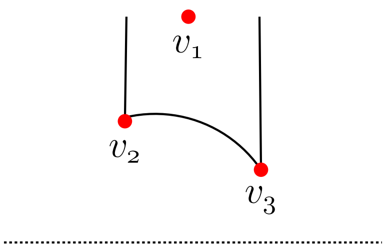

Fig. 1.6: Hyperbolic triangle with one ideal vertex (v_1).

As the hyperbolic area is invariant under the action of $M\ddot{o}b(\mathbb{H})$, we can assume that the hyperbolic area of P with new vertices remain unchanged.

So

$$area_{\mathbb{H}}(P) = \int_P \frac{1}{y^2} dxdy = \int_{cos(\phi)}^{cos(\theta)} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dydx = \int_{cos(\phi)}^{cos(\theta)} \frac{1}{\sqrt{1-x^2}} dx.$$

substituting,

$$x = cos(w), \Rightarrow dx = -sin(w)dw, \text{we get}$$

$area_{\mathbb{H}}(P) = \int_{\phi}^{\theta} -dw = \phi - \theta$. $\alpha_1 = 0, \alpha_2 = \theta, \alpha_3 = \pi - \theta$, and hence

$$area_{\mathbb{H}}(P) = \phi - \theta = \pi - (\alpha_2 - \alpha_3).$$

□

Theorem 1.19 (Gauss-Bonnet)**.** *Let P be a hyperbolic triangle with interior angles α, β, and γ. Then,*

$$area_{\mathbb{H}}(P) = \pi - (\alpha + \beta + \gamma).$$

Proof. Let P be hyperbolic triangle given by (v_1, v_2, v_3), and let $\ell_{i,j}$ be the hyperbolic line passing through (v_i, v_j). Now extend the hyperbolic line $\ell_{1,3}$ up to $\mathbb{R}$, and mark the point of intersection of $\ell_{1,3}$ with $\mathbb{R}$ by v_0 as in following diagram. Now draw a hyperbolic line passing through v_0, v_2.

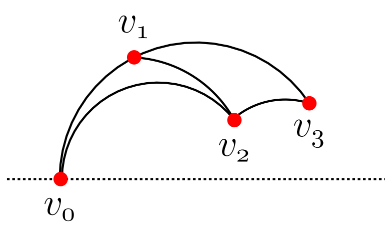

Fig. 1.7

As we can see in the Figure 1.1, we get two hyperbolic triangles (v_0, v_1, v_2) and (v_0, v_2, v_3) with one ideal vertex as v_0. Now by Proposition 1.7.1, we can calculate the the hyperbolic area of triangles (v_0, v_1, v_2) and (v_0, v_2, v_3), and the hyperbolic area of the triangle P is equal to the hyperbolic area of the triangle (v_0, v_2, v_3) minus the hyperbolic area of the triangle (v_0, v_1, v_2).

□

Theorem 1.20. *Let P be a hyperbolic with vertices and ideal vertices $v_1, ..., v_n$. Let α_k be the interior angle at v_k. Then,*

$$area_{\mathbb{H}}(P) = (n-2)\pi - \sum_{k=1}^{n} \alpha_k.$$

Proof. We prove this by decomposing P into hyperbolic triangles. Choose any point in the interior of hyperbolic polygon P, call that point x. Now draw hyperbolic line segment ℓ_i, joining x to each vertex v_i, and as hyperbolic polygons are convex, each hyperbolic lines ℓ_i is contained in P. The hyperbolic line segments $\ell_1, \ell_2, ..., \ell_n$ break P into n-triangles $T_1, T_2, ...T_n$. Now we use Gauss-Bonnet formula (Theorem 1.19) to get the hyperbolic area of each T_i, and add them all to get hyperbolic area of P. □

2. FUCHSIAN GROUPS

2.1 Introduction

When considering the geometry of the hyperbolic plane, the most important group that arises is the group of orientation-preserving isometries of the plane, which we have shown to be $\mathrm{PSL}(2, \mathbb{R})$. Fuchsian groups, which are the discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$ are particularly important to the areas of hyperbolic geometry of Riemann surfaces and algebraic curves. This chapter is based on [2] and [3].

2.2 The group PSL$(2, \mathbb{R})$

$\mathrm{PSL}(2, \mathbb{R})$, besides being a group, is also a topological space in that the transformation $z \mapsto (az + b)/(cz + d)$ can be identified with the point $(a, b, c, d) \in \mathbb{R}^4$. More precisely, as a topological space, $\mathrm{SL}(2, \mathbb{R})$ can be identified with the subset of $\mathbb{R}^4$,

$$X = \left\{(a, b, c, d) \in \mathbb{R}^4 : ad - bc = 1\right\},$$

and if we define $\delta(a, b, c, d) = (-a, -b, -c, -d)$ then $\delta : X \to X$ is a homeomorphism and δ together with the identity forms a cyclic group of order 2 acting on X. The group $\mathrm{PSL}(2, \mathbb{R})$ can be topologized as the quotient space.

Definition 2.2.1. A *discrete subgroup* of a topological group G is a subgroup of G which, as a topological space, inherits the discrete topology from G.

Definition 2.2.2. A discrete subgroup of $\mathrm{Isom}(\mathbb{H})$ is called a *Fuchsian group* if it consists of orientation-preserving transformations. In other words, a

Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

For any discrete group Γ of $\mathrm{Isom}(\mathbb{H})$, its subgroup Γ^+ of index ≤ 2 consisting of orientation-preserving transformations is a Fuchsian group. Thus, the main ingredient in the study of discrete subgroups of isometries of $\mathbb{H}$ is the study of Fuchsian groups.
Let X be a metric space and let G be the group of homeomorphisms of X.

Definition 2.2.3. A family $\{M_\alpha | \alpha \in A\}$ of subsets of X indexed by elements of a set A is called *locally finite* if for any compact subset $K \in X$, $M_\alpha \cap K \neq \Phi$ for only finitely many $\alpha \in A$

Definition 2.2.4. Let a group G acts on a set X. For $x \in X$, the set $G_x = \{g(x) | g \in G\}$ is called the $G-$ *orbit* of the point x.

Definition 2.2.5. Given a metric space X, a group G which is a subgroup of the isometries of X acts *properly discontinuously* on X if and only if any of the following three conditions hold:
1) If the G-orbit of any point $x \in X$ is locally finite.
2) The G-orbit of any point is discrete and the stabilizer of that point is finite.
3) If each point $x \in X$ has a neighborhood V such that

$$T(V) \cap V \neq \emptyset, \text{ for only finitely many } T \in G.$$

Lemma 1. *(i) A non-trivial discrete subgroup of $\mathbb{R}$, the additive group of real numbers is infinite cyclic.*

(ii) A discrete subgroup of S^1, the multiplicative group of complex numbers of modulus 1, is finite cyclic.

Proof. (i). Let Γ be a discrete subgroup of $\mathbb{R}$. There exist a smallest positive $x \in \Gamma$, otherwise, we get a sequence which converges to 0, which violates the discreteness. Then consider this $\{nx \mid n \in \mathbb{Z}\}$ discrete subgroup of Γ. Suppose $\exists\ y \in \Gamma$, such that $y \neq nx$. Then by the Archimedean Property there exists an integer $k \geq 0$ such that $kx < y < (k+1)x$, and $y - kx < x$, which contradicts our choice of x. $\square$

(ii). Let Γ now be a discrete subgroup of $S^1 = \{z \in \mathbb{C} \mid z = e^{i\phi}\}$. By discreteness, there exists $z = e^{i\phi} \in \Gamma$, with the smallest argument ϕ_0, and for some $m \in \mathbb{Z}$, $m\phi_0 = 2\pi$, otherwise, we get a contradiction with the choice of ϕ_0. $\square$

Example 2.2.1. Consider

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\{ z \to \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}.$$

It is clearly a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, because $\mathbb{Z}$ is discrete in $\mathbb{R}$. and hence a Fuchsian group.

Lemma 2. *Let $z_0 \in \mathbb{H}$ be given and let K be a compact subset of $\mathbb{H}$, then the set*

$$E = \{T \in PSL(2,\mathbb{R}) \mid T(z_0) \in K\}$$

is compact.

Proof. As $\mathrm{PSL}(2,\mathbb{R})$ is topologized as a quotient space of $\mathrm{SL}(2,\mathbb{R})$. We have continuous map

$$\psi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$$

such that,

$$\psi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = T, T(z) = \frac{az+b}{cz+d}.$$

So we only need to show that

$$E_1 = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R}) \mid \frac{az_0+b}{cz_0+d} \in K\}$$

is compact, as it would then follow that $E = \psi(E_1)$ is compact. We prove that E_1 is compact by showing that it is closed and bounded when regarded as a subset of $\mathbb{R}^4$. We have a continuous map $\beta : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{H}$ defined by $\beta(A) = \psi(A)\,(z_0)$. Since $E_1 = \beta^{-1}(K)$, it follows that E_1 is closed as the inverse image of the closed set K.

We now show that E_1 is bounded. As K is bounded there exist $M_1 > 0$ such that

$$\left|\frac{az_0+b}{cz_0+d}\right| < M_1,$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in E_1$.
Also, as K is compact in $\mathbb{H}$ there exist $M_2 > 0$ such that

$$\mathrm{Im}\left(\frac{az_0+b}{cz_0+d}\right) \geq M_2.$$

As the left-hand side of the inequality is $\mathrm{Im}\,(z_0)\,/\,|cz_0+d|^2$ so that

$$|cz_0+d| \leq \sqrt{\left(\frac{Im\,(z_0)}{M_2}\right)},$$

and thus

$$|az_0+b| \leq M_1\sqrt{\left(\frac{Im\,(z_0)}{M_2}\right)},$$

and we deduce that a, b, c, d are bounded.

□

Lemma 3. *Let Γ be a subgroup of $PSL(2,R)$ acting properly discontinuously on $\mathbb{H}$, and $p \in \mathbb{H}$ be fixed by some element of Γ. Then there is a neighborhood W of p such that no other point of W is fixed by an element of Γ other than the identity.*

Proof. Suppose $T(p) \in p$ for some $Id \neq T \in \Gamma$, and in any neighborhood of p there are fixed points of transformations in Γ. There is a sequence of points in $\mathbb{H}$, $p_n \to p$, such that for $T_n \in \Gamma$, $T_n(P_n) = P_n$. Let $\overline{B_{3\epsilon}(p)}$ be a closed hyperbolic disc, centered at p. $\overline{B_{3\epsilon}(p)}$ is compact. Since Γ acts properly discontinuously, the set $\{T \in \Gamma \mid T(p) \in \overline{B_{3\epsilon}(p)}\}$ is finite. For sufficiently large N, $n > N$ implies that $\rho(T_n(p), p) > 3\epsilon$, while $\rho(p_n, p) < \epsilon$. By triangle inequality and invariance of hyperbolic metric, we have

$$\rho(T_n(p), p) \leq \rho(T_n(p), T_n(p_n)) + \rho(T_n(p_n), p),$$

$$= \rho(p, p_n) + \rho(p_n, p) < 2\epsilon,$$

which is a contradiction.

□

Theorem 2.1. *Let* Γ *be a subgroup of* $PSL(2, R)$. *Then* Γ *is a Fuchsian group if and only if* Γ *acts properly discontinuously on* $\mathbb{H}$.

Proof. ($\Rightarrow$) Let $z \in \mathbb{H}$ and K be a compact subset of $\mathbb{H}$. Then

$$\{T \in \Gamma \mid T(z) \in K\} = \{T \in PSL(2, R) \mid T(z) \in K\} \cap \Gamma$$

is a finite set, and hence Γ acts properly discontinuously.
($\Leftarrow$) Suppose Γ acts properly discontinuously, but it is not discrete subgroup of PSL$(2, R)$. Choose a point $s \in \mathbb{H}$ not fixed by any non-identity element of Γ. As Γ is not discrete, there exists a sequence $\{T_k\}$ of distinct elements of Γ such that $T_k \to Id$ as $k \to \infty$. Hence $T_k(s) \to s$ as $k \to \infty$ and as s is not fixed by any non-identity element of Γ, $\{T_k(s)\}$ is a sequence of points distinct from s. Hence every closed hyperbolic disc centered at s contains infinitely many points of the Γ - orbit of s. Hence, Γ does not acts properly discontinuously.

□

2.3 Algebraic properties of Fuchsian groups

If G is any group and $g \in G$, then the *centralizer* of g in G is defined by

$$C_G(g) = \{h \in G | hg = gh\}.$$

Lemma 4. *If* $ST = TS$ *then* S *maps the fixed-point of* T *to itself.*

Proof. Let p be a fix point of T. Then

$$S(p) = ST(p) = TS(p),$$

so that $S(p)$ is also fixed by T. □

Theorem 2.2. *Two non-identity elements of $PSL(2, R)$ commute if and only if they have the same fixed-point set.*

Proof. Suppose the two elements, T and S, commute. Then T maps the fixed point set of S to itself injectively. Similarly, S maps the fixed point set of T to itself injectively. Hence, T and S must have the same number of fixed points. If they only have one fixed point, then T sends the fixed point of S to itself. This means T also fixes the fixed points of S, and vice-versa. Therefore, if S and T only have one fixed point, they must have the same fixed point.

The only remaining case is if S and T have two fixed points (i.e. are hyperbolic). Then we can choose a conjugator such that the conjugate of $T, C^{-1}TC$ fixes 0 and ∞. We do not know what S fixes, but we do know its conjugate has the following forms,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda^{-1} \\ c\lambda & d\lambda^{-1} \end{pmatrix}$$
$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda \\ c\lambda^{-1} & d\lambda^{-1} \end{pmatrix}$$

Since these two elements commute, both these lines must be equal. In particular, $b\lambda^{-1} = b\lambda$ and $c\lambda = c\lambda^{-1}$. Since T is hyperbolic, $\lambda \neq 1$, so the only way to satisfy these conditions is if $b = c = 0$. But, this means $C^{-1}SC$ fixes 0 and ∞. Since the conjugates of T and S fix the same points, T and S must fix the same points.

($\Leftarrow$) Suppose two elements have the same fixed point set. They are then of the same type. They are also mapped by the same conjugator to one of the following three forms:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Each of these forms commutes with the other forms of their type. Since the conjugates of the elements commute, the elements themselves must also commute.

□

Theorem 2.3. *Let Γ be a Fuchsian group all of whose non-identity elements have the same fixed-point set. Then Γ is cyclic.*

Proof. Types of elements of $\mathrm{PSL}(2, R)$ on the basis of fixed point set :
two points in $\mathbb{R} \cup \{\infty\}$, one point in $\mathbb{R} \cup \{\infty\}$, one point in $\mathbb{H}$. Hence all elements of Γ must be of the same type. Suppose all elements of Γ are hyperbolic.
Then by choosing a conjugate group, we may assume that each $S \in \Gamma$ fixes 0 and ∞. Thus, Γ is a discrete subgroup of H $=\{z \to \lambda z \mid \lambda > 0\}$. Note that H $\cong \mathbb{R}^*$, the multiplicative group of positive real numbers, and $\mathbb{R}^* \cong \mathbb{R}$ via isomorphism $x \to \ln(x)$. Hence, by Lemma 1, Γ is infinite cyclic.
Similarly, if Γ contains a parabolic element, then Γ is an infinite cyclic group containing only parabolic elements. Suppose Γ contains an elliptic element. In $\mathbb{D}$, Poincaré disk model, Γ is a discrete subgroup of orientation-preserving isometries of $\mathbb{D}$. By choosing a conjugate group we may assume that all elements of Γ have 0 as a fixed point, and therefore all elements of Γ are of the form $z \to e^{i\theta}z$. Thus, Γ is isomorphic to a subgroup of S^1, and it is discrete if and only if the corresponding subgroup of S^1 is discrete. Now by Lemma 1 the assertion follows.

□

Corollary 1. *Every abelian Fuchsian group is cyclic.*

Proof. By Theorem 2.2, all non-identity elements in an abelian Fuchsian group have the same fixed-point set. Now by Theorem 2.3, it is cyclic. □

Corollary 2. *No Fuchsian group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

Proof. Since $\mathbb{Z} \times \mathbb{Z}$ is abelian but not cyclic, the assertion follows from Corollary 1. □

2.4 Elementary groups

Definition 2.4.1. A subgroup Γ of $\mathrm{PSL}(2, R)$ is called *elementary* if there exists a finite Γ-orbit in $\overline{\mathbb{H}}$.

Theorem 2.4. *Let* Γ *be a subgroup of* $PSL(2, R)$ *containing besides the identity only elliptic elements. Then all elements of* Γ *have the same fixed point, and hence* Γ *is a cyclic group, abelian and elementary.*

Proof. We want to prove that all elliptic elements in Γ must have the same fixed point. And we prove this for Poincaré disk model. Hence Γ used in the proof is a subgroup of orientation-preserving isometries of Poincaré disk model.
We can conjugate Γ in such a way that an element $Id \neq g \in \Gamma$ fixes 0 :

$$g = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$$

and let

$$h = \begin{bmatrix} a & \bar{c} \\ c & \bar{a} \end{bmatrix} \in \Gamma, h \neq g.$$

$tr[g, h] = 2 + 4 \mid c \mid^2 (Im(u))^2$. As Γ does not contain hyperbolic elements, $\mid tr[g, h] \mid \leq 2$, which gives either $Im(u) = 0$ or $c = 0$. If $Im(u) = 0$ then $u = \bar{u} \in \mathbb{R}$ and hence $g = Id$, a contradiction.
Hence $c = 0$, and so

$$h = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$$

also fixes 0. Hence, Γ is finite cyclic and abelian. Since 0 is a Γ-orbit, Γ is elementary.

□

Theorem 2.5. *A non-elementary subgroup* Γ *of* $PSL(2, \mathbb{R})$ *must contain a hyperbolic element.*

Proof. Suppose Γ does not contain hyperbolic elements.

1. If Γ contains only elliptic elements, then by previous theorem it is elementary.

2. Hence Γ contains a parabolic element fixing, say ∞ : $f(z) = z + 1$.

Let $g(z) = \frac{az+b}{cz+d}$ be any element in Γ. Then

$$f^n \circ g(z) = \frac{(a+nc)z + (b+nd)}{cz+d}$$

and

$$tr^2(f^n \circ g) = (a+d+nc)^2.$$

Since all elements in the group are either elliptic or parabolic, we have $0 \leq (a+d+nc)^2 \leq 4$, for all n, which implies $c = 0$, because n is arbitrary. But then g also fixes ∞, so that ∞ is fixed by all elements in Γ. Hence, Γ is elementary, which is a contradiction.

$\square$

2.5 Fundamental region

Definition 2.5.1. A closed region $F \subset X$ is defined to be a *fundamental region* for G if

1. $\cup_{T \in G} T(F) = X$, and
2. $\mathring{F} \bigcap T(\mathring{F}) = \phi$ for all $T \in G - \{Id\}$.

In Figure 2.1, the fundamental region for $\Gamma = \{T^n \mid T(z) = 2z\}$ is indicated. The set $\{T(F) \mid T \in G\}$ is called the *tessellation* of X.

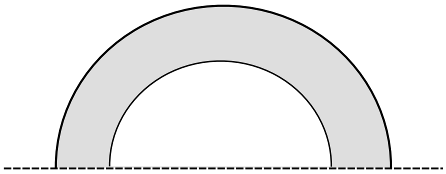

Fig. 2.1: Fundamental region for $\Gamma = \{T^n \mid T(z) = 2z\}$.

Theorem 2.6. *Let F_1 and F_2 be two fundamental regions for a Fuchsian group Γ, and $\mu(F_1) < \infty$. Suppose that the boundaries of F_1 and F_2 have zero hyperbolic area. Then $\mu(F_1) = \mu(F_2)$.*

Proof. We have $\mu(\mathring{F}_i) = \mu(F_i)$, $i = 1, 2$. So,

$$F_1 \supseteq F_1 \cap (\cup_{T\in\Gamma} T(\mathring{F}_2)) = \cup_{T\in\Gamma}(F_1 \cap T(\mathring{F}_2)).$$

Since $\mathring{F}_2$ is the interior of a fundamental region, the sets $F_1 \cap T(\mathring{F}_2)$ are disjoint, and hence

$$\mu(F_1) \geq \sum_{T\in\Gamma} \mu(F_1 \cap T(\mathring{F}_2)) = \sum_{T\in\Gamma} \mu(T^{-1}(F_1) \cap \mathring{F}_2) = \sum_{T\in\Gamma} \mu(T(F_1) \cap \mathring{F}_2).$$

Since F_1 is a fundamental region, we have

$$\cup_{T\in\Gamma} T(F_1) = \mathbb{H}.$$

Therefore,

$$\cup_{T\in\Gamma}(T(F_1) \cap \mathring{F}_2) = \mathring{F}_2.$$

Hence

$$\sum_{T\in\Gamma} \mu(T(F_1) \cap \mathring{F}_2) \geq \mu(\cup_{T\in\Gamma} T(F_1) \cap \mathring{F}_2) = \mu(\mathring{F}_2) = \mu(F_2).$$

Interchanging F_1 and F_2, we obtain $\mu(F_2) \geq \mu(F_1)$, and the result follows. □

2.5.1 The Dirichlet region

Definition 2.5.2. Let Γ be a Fuchsian group. A *Dirichlet region for* Γ *centered at* p is the set

$$D_p(\Gamma) = \{z \in \mathbb{H} \mid \rho(z,p) \leq \rho(z, T(p)) \ for\ all\ T \in \Gamma\}.$$

By the invariance of the hyperbolic metric under $\mathrm{PSL}(2,\mathbb{R})$, this region can also be defined as

$$D_p(\Gamma) = \{z \in \mathbb{H} \mid \rho(z,p) \leq \rho(T(z), p) \ for\ all\ T \in \Gamma\}.$$

Definition 2.5.3. A line given by the equation

$$\rho(z, z_1) = \rho(z, z_2)$$

is the *perpendicular bisector* of the geodesic segment $[z_1, z_2]$.

We shall denote the perpendicular bisector of the geodesic segment $[p, T_1(p)]$ by $L_p(T_1)$, and the hyperbolic half-plane containing p by $H_p(T_1)$.

Example 2.5.1. Consider $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. As for $k > 1$, point ki is not fixed by any non-identity element of group Γ, choose $p = ki$, where $k > 1$. We show that the region in following Figure 2.2 is the Dirichlet region for Γ centered at p.
First, note that the isometries $T(z) = z+1$, $S(z) = \frac{-1}{z}$ are in Γ, and, three geodesic sides of F are $L_p(T)$, $L_p(T^{-1})$ and $L_p(S)$. This shows that $D_p(\Gamma) \subset F$. If $D_p(\Gamma) \neq F$, there exist $z \in \mathring{F}$ and $h \in \Gamma$ such that $h(z) \in \mathring{F}$. We shall now show that this cannot happen. Suppose that

$$h(z) = \frac{az+b}{cz+d}, \quad (a,b,c,d \in \mathbb{Z}, ad-bc = 1).$$

Then

$$|cz+d|^2 = c^2|z|^2 + 2Re(z)cd + d^2 > c^2 + d^2 - |cd| = (|c|-|d|)^2 + |cd|,$$

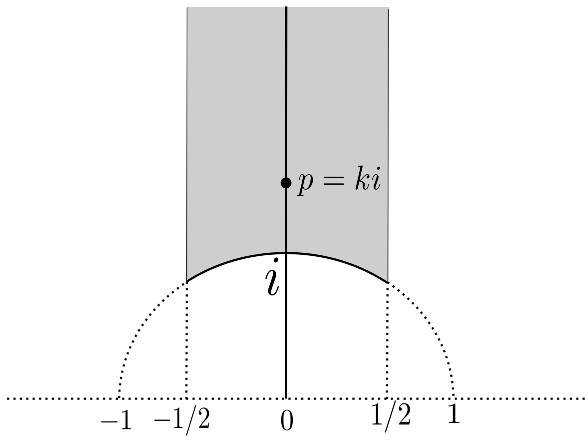

Fig. 2.2: $F = \{z \in \mathbb{H} \mid |z| \geq 1, |Re(z)| \leq \frac{1}{2}\}$.

as $|z| > 1$ and $Re(z) > \frac{-1}{2}$. This lower bound is an integer: (it is non-negative and is not zero if and only if $c = d = 0$, which contradicts $ad - bc = 1$). We deduce that $|cz + d| > 1$, and so

$$\operatorname{Im} h(z) = \frac{\operatorname{Im}(z)}{|cz + d|^2} < \operatorname{Im}(z).$$

Exactly the same argument holds with z, h replaced by $h(z)$, h^{-1} and a contradiction is reached. Thus we have $D_p(\Gamma) = F$.

Definition 2.5.4. Let

$$T(z) = \frac{az + b}{cz + d} \in \mathrm{PSL}(2, R) \text{ with } c \neq 0.$$

The circle

$$I(T) = \{z \in \mathbb{C} \mid |cz + d| = 1\},$$

which is a complete locus of points where the transformation T acts as a Euclidean isometry is called the *isometric circle* of the transformation T.

We shall denote the set of points inside of the isometric circle $I(T)$ by Ǐ(T), and the set of points outside of $I(T)$ by Î(T).

Theorem 2.7. *Isometric circles are geodesics in* $\mathbb{H}$.

Proof. Let T be of the form

$$T(z) = \frac{az+b}{cz+d} \in \mathrm{PSL}(2,\mathbb{R}).$$

Then the center of $I(T)$ is $\frac{-d}{c} \in \mathbb{R}$. Therefore, $I(T)$ is orthogonal to the real axis. □

Definition 2.5.5. A fundamental region F for a Fuchsian group Γ is called *locally finite* if the tessellation $\{T(F) \mid T \in \Gamma\}$ is locally finite.

Theorem 2.8. *A Dirichlet region is locally finite.*

Proof. Let $F = D_p(\Gamma)$, where p is not fixed by any element of $\Gamma - \{Id\}$. Let $a \in F$, and let $K \subset \mathbb{H}$ be a compact neighborhood of a. Suppose that $K \cap T_i(F) \neq 0$ for some infinite sequence $T_1, T_2, ...$ of distinct elements of Γ. Let $\sigma = sup_{z \in K}\rho(p,z)$. Since $\sigma \leq \rho(p,a) + \rho(a,z), \forall z \in K$ and K is bounded, σ is finite. Let $w_j \in K \cap T_j(F)$. Then $w_j = T_j(z_j)$ for $z_j \in F$, and by triangle inequality, we have

$$\begin{aligned}
\rho(p, T_j(p)) &\leq \rho(p, w_j) + \rho(w_j, T_j(p)) \\
&= \rho(p, w_j) + \rho(z_j, p) \\
&\leq \rho(p, w_j) + \rho(w_j, p)(as z_j \in D_p(\Gamma)) \\
&\leq 2\sigma.
\end{aligned}$$

Thus, the infinite set of points $T_1(p), T_2(p), ...$ belongs to the compact hyperbolic ball with center p and radius 2σ. But this contradicts the properly discontinuous action of Γ.

□

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