AN INTRODUCTION TO EXPANDER GRAPHS

A REPORT

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by

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Ashwin K

ABSTRACT

We will begin by discussing some of the basic facts about graph theory in view towards understanding this thesis. We then will derive bounds on the spectrum of a k-regular graph, and understand the relation between the spectrum, and the bipartiteness and connectivity of such a graph. Following a brief introduction to group actions on graphs, we will establish that all Cayley graphs are vertex-transitive.We will then introduce Ramanujan Graphs, and establish that the undirected n-cycle is Ramanujan. After describing about Ramanujan graphs we then discuss some of the basic properties of expander families of graphs,begin by defining some operators associated with the graph such as adjacency operator,laplacian operators.Then we will define the Isoperimetric constant associated with the graph and from that we will describe the definition of expander families of graphs.We will also understand the proof of Rayleigh-Ritz theorem which will help us in understanding the relationship between spectral gap and isoperimetric constant.We will see on what bounds of the diameter's of families of graphs will make those families to expander.Then we will see why Abelian groups do not yield expander families of graph.Finally, we will state Alon-Boppana theorem and will understand the proof,and will show why Ramanujan graphs with regularity greater than three turns out to be expander graphs.

LIST OF SYMBOLS OR ABBREVIATIONS

- $\mathrm{Diam}(X)$ represents the diameter of graph X.
- $\lambda(X)$ denotes the eigenvalue of graph X.
- $\|g\|_\Gamma$ represents the word norm of g wrt subset Γ.
- $\mathrm{dist}(x, y)$ denotes the distance between vertices x and y.
- We denote the group of integers modulo n under addition by $\mathbb{Z}_n$.
- We denote the sets of integers, real numbers, and complex numbers, respectively, by $\mathbb{Z}, \mathbb{R},$ and $\mathbb{C}$.
- Let x be a real number. We write $\lfloor x \rfloor$ for the greatest integer less than or equal to x. For example, $\lfloor 2.55 \rfloor = 2$.
- Let A and B are two sets. We write $A \setminus B$ for the set difference; that is $A \setminus B = \{x \in A | x \notin B\}$.

CONTENTS

1. INTRODUCTION

1.1 Background

Algebraic graph theory is a branch of Mathematics in which algebraic methods, particularly those employed in group theory and linear algebra, are use to solve graph-theoretic problems. An important subbranch of algebraic graph theory is spectral graph theory, which involves the study of the spectra of matrices associated with the graph such as its adjacency matrix, and its relation to the properties of the graph. Expander graph are graphs with the special property that any set of vertices S (unless very large) has a number of outgoing edges proportional to $|S|$. Expansion can be defined both with respect to the number of the edges or vertices on the boundary of S. We will stick with edge expansion, which is more directly related to eigenvalues The spectral gap of a graph is the difference in magnitude of the two largest eigenvalues of its adjacency matrix. Graphs which have large spectral gaps are of great interest from the viewpoint of communication networks, as they exhibit strong connectivity properties. As Ramanujan Graphs are known to maximize the spectral gap, they have been widely studied from an application perspective. However, Ramanujan graphs have also fascinated pure mathematicians alike, as they lie in the interface of Number Theory, Representation Theory, and Algebraic Geometry [6]. In 1988, a beautiful paper by A. Lubotsky, R. Phillips, and P. Sarnak [5], described the explicit construction of a Ramanujan Graph for every pair of distinct primes congruent to 1 modulo 4, thereby establishing the existence of an infinite family of such graphs. Expander graphs have found extensive applications in computer science, in designing algorithms, error correcting codes, extractors, pseudorandom generators, sorting networks

(Ajtai, Komls and Szemerdi (1983)) and robust computer networks. They have also been used in proofs of many important results in computational complexity theory, such as SL = L (Reingold (2008)) and the PCP theorem (Dinur (2007)). In cryptography, expander graphs are used to construct hash functions.

The primary goal of this thesis is to study some basic concepts in algebraic graph theory, with a view towards understanding expander graphs and their properties. We will mostly follow [3, 4], and we will use [2, 6] and [1] as additional references.

2. PRELIMINARIES

In this introductory chapter we provide the background to the material that we present more formally in later chapters. We will begin our discussions by describing some basic notations and definitions in graph theory.

2.1 Graphs

Graphs are mathematical structures used to model pairwise relations between objects. We will now formally define an undirected graph.

Definition 2.1. An **undirected graph** X is defined to be a pair $(V(X), E(X))$, where

(i) $V(X)$ is a set called the set of **vertices**, and

(ii) $E(X)$ is a set of unordered pairs of vertices (i.e $E(X) \subset \{\{x, y\} \mid x, y \in V(X)\}$ called the set of **edges**.

For a graph X, $|V(X)|$ is called the **order** of graph, which is denoted as $|X|$. We say a graph X is **finite** if $|X| < \infty$, otherwise, we say a graph is **infinite**.

Example 2.2 (**Undirected Graph**). Let $X = (V(X), E(X))$ be a graph with $V(X) = \{1, 2, 3, 4, 6\}$ and $E(X) = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{4, 6\}, \{4, 2\}, \{1, 4\}\}$. Figure 2.1 gives the pictorial representation of an undirected graph $X = (V(X), E(X))$.

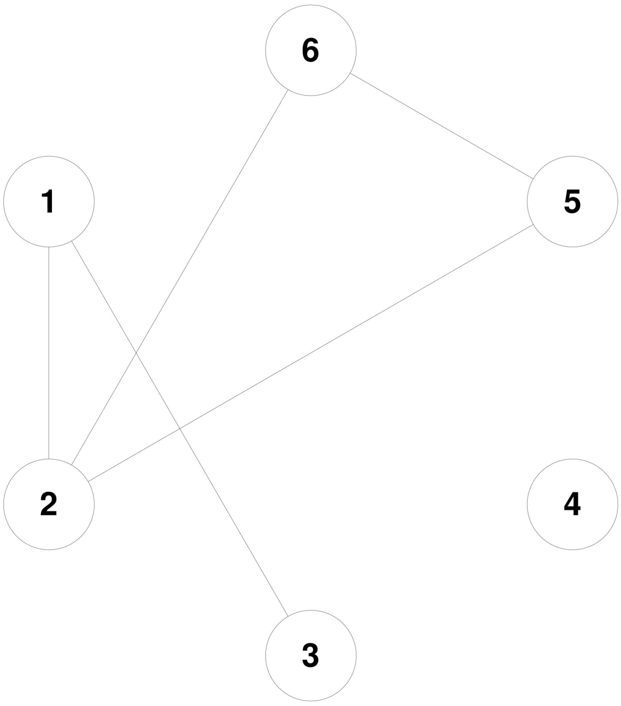

Fig. 2.1: An undirected graph

Since $\{2,1\}, \{2,6\}$ and $\{2,5\} \in E(X)$ we say that $1, 6$ and 5 are **adjacent** to 2. Also, note that since $4 \in V(X)$ has no adjacent vertices we will call it a isolated vertex. .

Remark 2.3. If the graph is **directed** then in $E(X)$ instead of taking unordered pairs of vertices we will take ordered pairs of vertices, that is,

$$E(X) = \{(x, y) | \ \ x, y \in V(X)\}.$$

Definition 2.4. Consider the graph $X = (V(X), E(X))$ .Then

(i) For $x, y \in V(X)$, we say x is **adjacent** to y if $\{x, y\} \in E(X)$.

(ii) An edge of the form $e = (x, x)$ or $e = \{x\}$ for $x \in V(X)$ is called a **loop**.

(iii) If $E(X)$ is a multiset then $X = (V(X), E(X))$ is called **multigraph**.

(iv) If X is not a multigraph and does not contain loops then it is called **simple**.

The following graph is an example of a multidirected graph(see Figure 2.2).

Example 2.5. Let $X = (V(X), E(X))$ be a graph with

$$\begin{aligned} V(X) = \; & \{1, 2, 3, 4, 5, 6, 7, 8\} \\ E(X) = \; & \{(1, 2), (1, 3), (2, 1), (4, 2), (5, 3), (5, 8), (8, 7), (3, 6), \\ & (7, 8), (7, 6), (6, 5), (1, 1), (4, 2), (6, 2), (1, 2)\}. \end{aligned}$$

Here $E(X)$ is a multiset with $(1, 2)$ repeating twice and $(1, 1)$ represents a loop. In Figure 2.2 we can visualize why $e_1 = (1, 2)$ and $e_2 = (2, 1)$ are not equal.

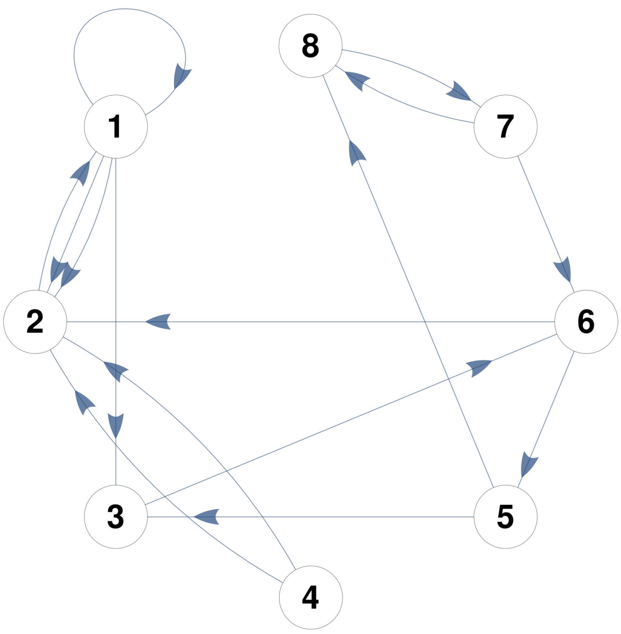

Fig. 2.2: A multigraph

Example 2.6. Let $X = (V(X), E(X))$ be a graph with

$V(X) = \{1, 2, 3, 4, 5, 6\}$ and $E(X) = \{\{1, 2\}, \{1, 6\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}$

This simple graph is called a 6-cycle as shown in Figure 2.3. We will discuss more about n-cycles in following chapter.

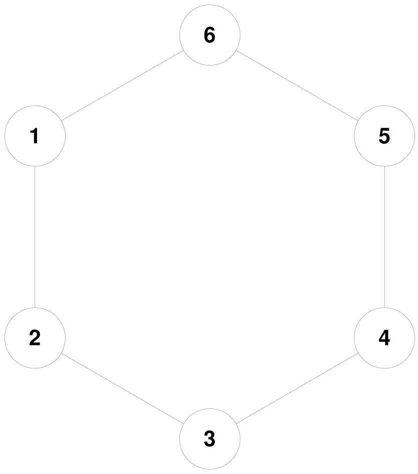

Fig. 2.3: A 6-cycle.

Remark 2.7. Throughout this thesis we will assume graphs to be finite, undirected, and simple, unless mentioned otherwise.

Definition 2.8. Let $X = (V(X), E(X))$ be a graph then a graph $Y = (V(Y), E(Y))$ is called a **subgraph** of X if $V(Y) \subseteq V(Y)$ and $E(X) \subseteq E(Y)$. If $V(Y) = V(X)$ then $Y = (V(Y), E(Y))$ is called a **spanning** subgraph of X. For a subgraph $Y = (V(Y), E(Y))$ of $X = (V(X), E(X))$ we will call $Y = (V(Y), E(Y))$ as a **induced subgraph**, if $\{x, y\} \in E(Y)$ iff $\{x, y\} \in E(X)$.

Example 2.9. The graph $Y = (V(Y), E(Y))$ shown in the Figure 2.4 is a subgraph of the 6-cycle(in Figure 2.3) with $V(X) = \{1, 2, 3, 4, 5\}$ and $E(X) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$.

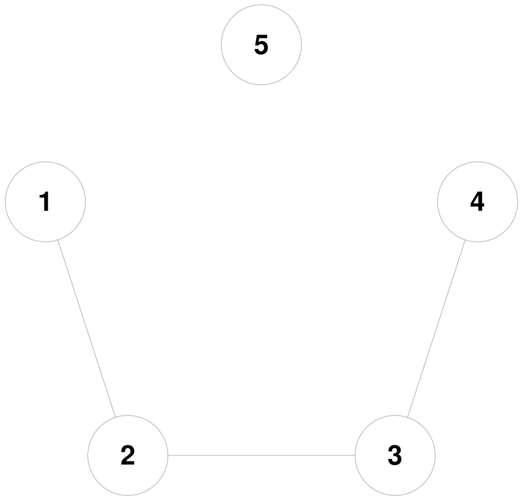

Fig. 2.4: A subgraph of the 6-cycle.

Note that Y is not a spanning subgraph of X, since vertex $6 \notin V(Y)$. Now we define new graph $Y_S = (V(Y_S), E(Y_S))$ (see Figure2.5) such that $V(Y_S) = V(Y) \cup \{6\}$ and $E(Y_S) = E(Y)$. Hence, Y_S is a spanning subgraph.

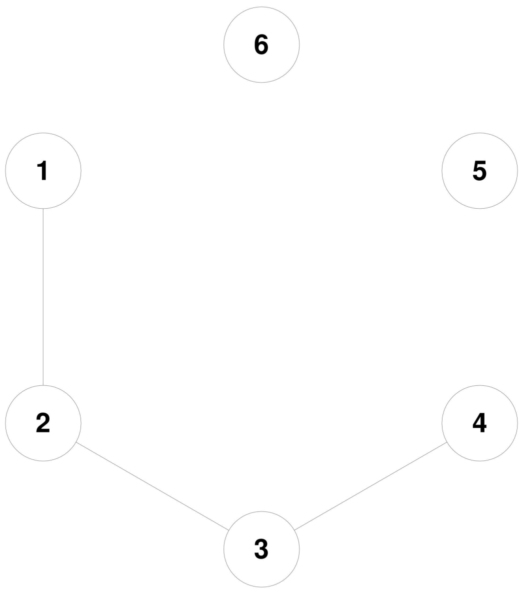

Fig. 2.5: A spanning graph of the 6-cycle.

Since $\{4, 5\} \notin E(Y_S)$ the graph Y is not an induced subgraph. The graph $Y_I = (V(Y_I), E(Y_I))$(see Figure 2.6) defined as $V(Y_I) = V(Y)$ and $E(Y_I) = E(Y) \cup \{\{4,5\}\}$ is an induced subgraph.

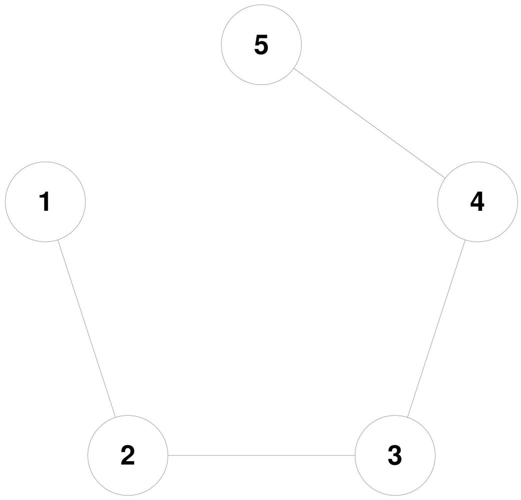

Fig. 2.6: An induced subgraph of the 6-cycle.

2.2 Degree and Regularity

Definition 2.10. Let X be a graph. Then for each $x \in V(X)$, consider the set

$$N_x = \{y \in V(X) | \{x, y\} \in E(X)\}.$$

The **degree** of a vertex $x \in V(X)$ is defined as

$$\deg(x) = |N_x|.$$

Degree of a vertex is also called its **valency**.

Definition 2.11. A graph X is called a **k-regular graph** if $\deg(x) = k,$ for all $x \in V(X)$.

Example 2.12. The following is an example of a k-regular graph.

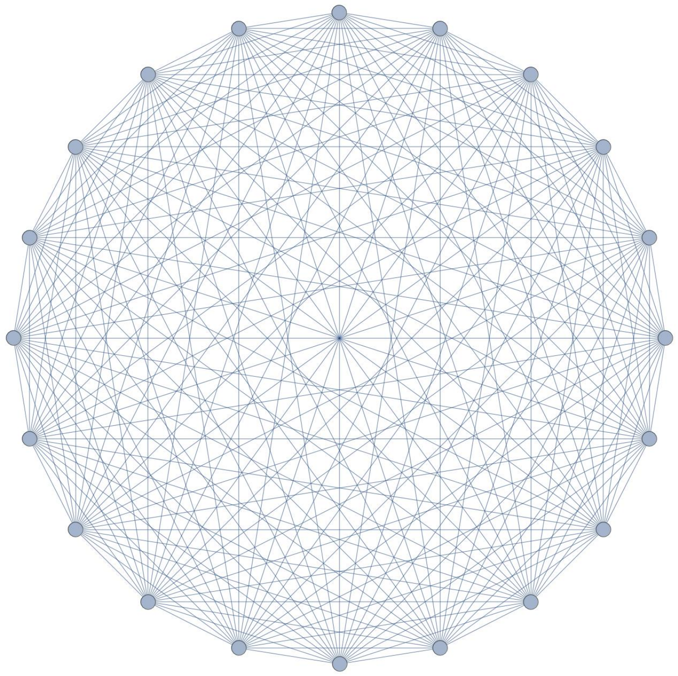

Fig. 2.7: 19-regular complete graph.

Complete graphs are ideal, as far as connectivity problems are concerned, as they yield good connectivity but they are not cost effective. We will see graphs with high connectivity in more detail while discussing expander graphs in Chapter 4.

2.3 Connected Graph and Path

Definition 2.13 (Path). Let X be a graph and $v, w \in V(X)$. Then a **path** from v to w is defined as a subgraph Y of X of the form

$$V(Y) = \{v, v_1, \dots, v_r, w\} \text{ and}$$

$$E(Y) = \{\{v, v_1\}, \{v_1, v_2\}, \dots, \{v_r, w\}\}.$$

The **length** of the path Y between vertices v and w is defined by

$$l(Y) = |E(Y)|.$$

If there exists a path between vertices v and w, then we write $v \sim w$.

Note that $\sim$ defines an equivalence relation on $V(X)$.

Example 2.14. Let $X = (V(X), E(X))$ be a graph with $V(X) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and
$E(X) = \{\{1, 2\}, \{1, 3\}, \{1, 8\}, \{2, 3\}, \{8, 3\}, \{8, 2\}, \{8, 4\}, \{8, 5\}, \{7, 3\}, \{7, 4\}, \{6, 4\}\},.$ as shown in Figure 2.8.

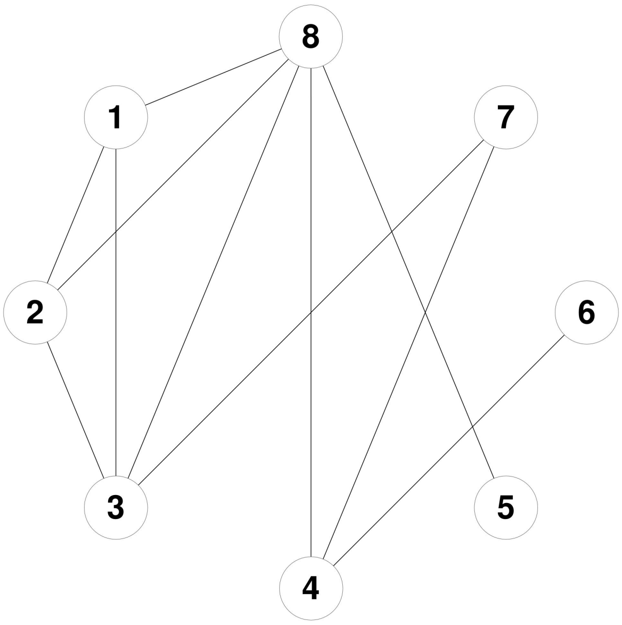

Fig. 2.8: $X = (V(X), E(X))$

The subgraph $Y = (V(X), E(Y))$ with $V(Y) = \{1, 8, 4, 6\}$ and
$E(Y) = \{\{1, 8\}, \{4, 8\}, \{4, 6\}\}$ is a path in X as indicated in Figure 2.9.

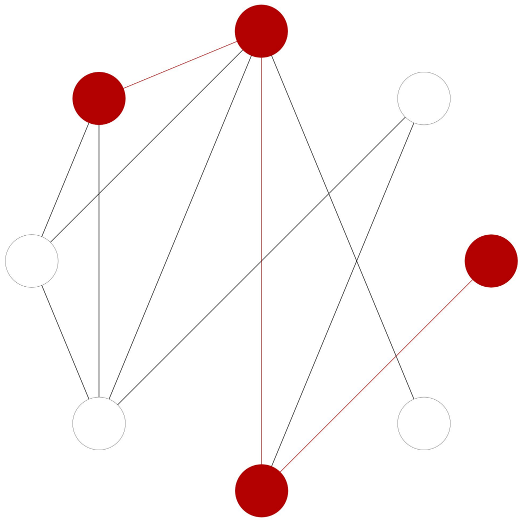

Fig. 2.9: A path in a graph.

Definition 2.15. A graph X is said to be **connected** if

$$x \sim y \text{ for all } x, y \in V(X).$$

If a graph is not connected, then we will call the graph **disconnected**.

Example 2.16. Consider graph Y (see Figure 2.10) with $V(Y) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $E(X) = \{\{1, 7\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 7\}, \{8, 5\}, \{5, 6\}\}$. As there is no path between vertices 8 and 1, it is disconnected. If we add the edge $e = \{1, 8\}$ to the edge set $E(Y)$, then Y is connected (see Figure 2.11).

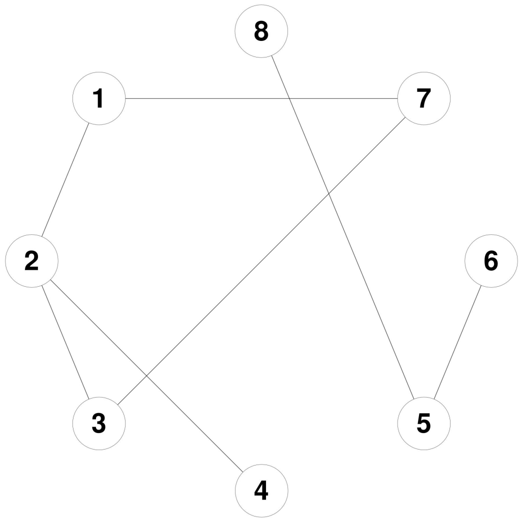

Fig. 2.10: A disconnected graph.

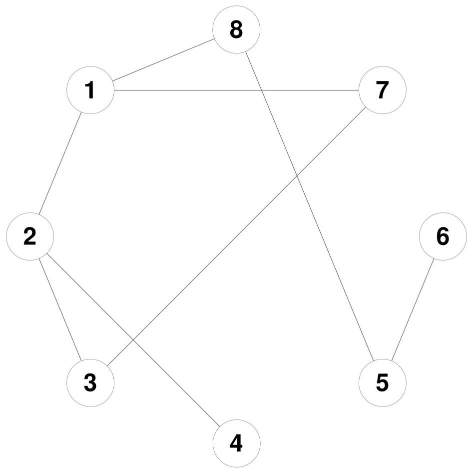

Fig. 2.11: Connected graph.

2.4 Adjacency Matrix

The adjacency matrix is the most important matrix associated with a graph from the perspective of spectral graph theory. It is formally defined in the following manner.

Definition 2.17. The **adjacency matrix** of a graph X of size n is defined as the matrix $A(X) = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \{i,j\} \in E(X), \\ 0, & \text{otherwise.} \end{cases}$$

Let us see some examples below.

Example 2.18. The adjacency matrix $A(X)$ of the graph X in example 2.14 is given by,

$$A(X) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Remark 2.19. Since for undirected graph $\{x,y\} \in E(X) \Leftrightarrow \{y,x\} \in E(X)$, we have that $a_{ij} = 1 \Leftrightarrow a_{ji} = 1$ in $A(X) = (a_{ij})$. So the adjacency matrix for the undirected graphs are symmetric.

Example 2.20. Let $X = (V(X), E(X))$ be a graph with $V(X) = \{1,2,3,4,5,6\}$ and $E(X) = \{(1,3),(1,4),(2,4),(2,3),(3,1),(3,6),(4,2),(4,5),(5,6),(5,1),(6,5),(6,2)\}$, as illustrated in Figure 2.12

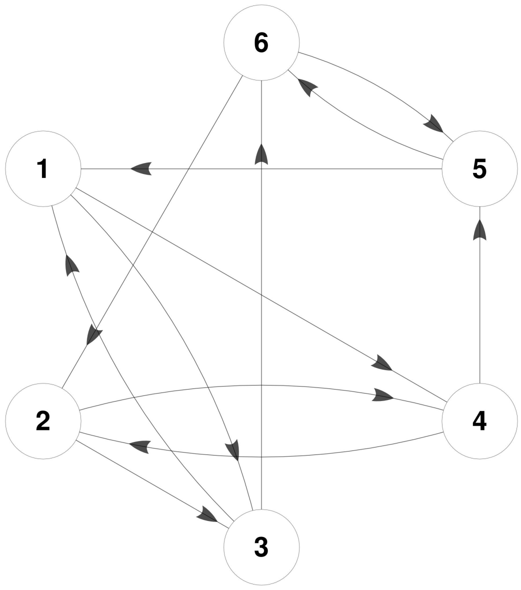

Fig. 2.12: A directed graph X.

Here,

$$A(X) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

From this, it is apparent that the adjacency matrix of a directed graph is not symmetric in general.

3. SPECTRUM AND CAYLEY GRAPH

3.1 Graph Isomorphism

Definition 3.1. Let X and Y be two graphs. Then a bijective map $f : V(X) \to V(Y)$ satisfying,

$$\{x, y\} \in E(X) \iff \{f(x), f(y)\} \in E(Y) \text{ for all } x, y \in V(X)$$

is called an **isomorphism** from X to Y. If such an f exists, then we say that graphs X and Y are **isomorphic**, which is denoted as $X \cong Y$.

Theorem 3.2 (Graph Isomorphism theorem)**.** *Let X and Y be two graphs. Then*

$$X \cong Y$$

iff there exists a permutation matrix P such that

$$P^T A(X) P = A(Y).$$

Proof. ($\implies$)Given $X \cong Y \Rightarrow \exists$ a bijective function $f : V(X) \to V(Y)$. Let $V(X) = \{1, 2, \ldots, n\}$ and $V(Y) = \{f(1), \ldots, f(n)\}$. Consider the adjacency matrices of X and Y,

$$A(X) = (a_{ij})_{n\times n} \text{ and } A(Y) = (a_{f(i)f(j)})_{n\times n}.$$

Define a permutation matrix P as follows,

$$P = \begin{pmatrix} e_{f(1)} & e_{f(2)} & \dots & e_{f(n)} \end{pmatrix}, \text{ where } e_{f(i)} = e_k \text{ for } k = f(i).$$

Then

$$P^T A(X) = \begin{pmatrix} e_{f(1)}^T \\ \vdots \\ e_{f(n)}^T \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{f(1)1} & \dots & a_{f(1)n} \\ \vdots & \ddots & \vdots \\ a_{f(n)1} & \dots & a_{f(n)n} \end{pmatrix}$$

So

$$\begin{aligned} P^T A(X) P &= \begin{pmatrix} a_{f(1)1} & \dots & a_{f(1)n} \\ \vdots & \ddots & \vdots \\ a_{f(n)1} & \dots & a_{f(n)n} \end{pmatrix} \begin{pmatrix} e_{f(1)} & \dots & e_{f(n)} \end{pmatrix} \\ &= \begin{pmatrix} a_{f(1)f(1)} & \dots & a_{f(1)f(n)} \\ \vdots & \ddots & \vdots \\ a_{f(n)f(1)} & \dots & a_{f(n)f(n)} \end{pmatrix} \\ &= A(Y). \end{aligned}$$

($\Longleftarrow$) Conversely, given a permutation matrix P such that $P^T A(X) P = A(Y)$, one can construct f from P by looking at where $f(i)$ goes for each $1 \leq i \leq n$. This can be achieved by just looking at the construction of P in the proof.

□

Example 3.3. Define $f : V(X) \to V(Y)$ such that $1 \mapsto a$, $2 \mapsto d$, $3 \mapsto b$,

$4 \mapsto e$, $5 \mapsto c$. Note that

$$\begin{aligned}
\{1,2\} \in E(X) &\Leftrightarrow \{f(1), f(2)\} \in E(Y) \\
\{2,3\} \in E(X) &\Leftrightarrow \{f(2), f(3)\} \in E(Y) \\
\{3,4\} \in E(X) &\Leftrightarrow \{f(3), f(4)\} \in E(Y) \\
\{4,5\} \in E(X) &\Leftrightarrow \{f(4), f(5)\} \in E(Y) \\
\{5,1\} \in E(X) &\Leftrightarrow \{f(5), f(1)\} \in E(Y).
\end{aligned}$$

Hence, f is an isomorphism.

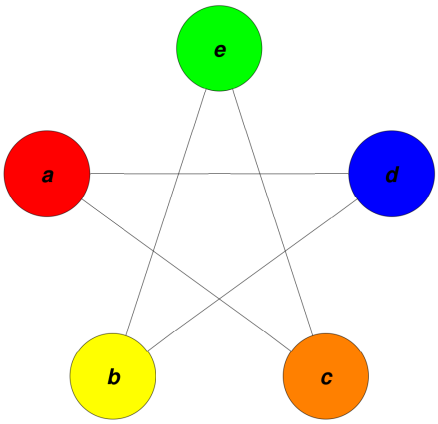

Fig. 3.1: Star pentagon.

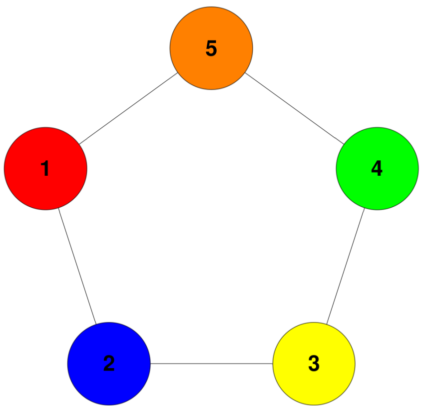

Fig. 3.2: 5-cycle.

3.2 Spectra of Graphs

Definition 3.4. Let X be a graph. Then the **spectrum** of X is defined as the multiset of eigenvalues of $A(X)$, written as

$$\operatorname{Spec}(X) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_k \\ m_1 & m_2 & \dots & m_k \end{pmatrix},$$

where for $1 \le i \le k$, λ_i are the distinct eigenvalues of $A(X)$ with multiplicity m_i, and $\sum_i m_i = n$.

Example 3.5. The graph X given in the Figure 3.3 is called the Chvatal graph. Using mathematica, one can compute the spectrum of X, and

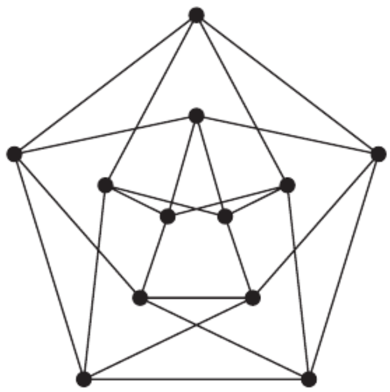

Fig. 3.3: Chvatal graph

,

$$\operatorname{Spec}(X) = \begin{pmatrix} -3 & \frac{-1-\sqrt{17}}{2} & -1 & 0 & 1 & \frac{-1+\sqrt{17}}{2} & 4 \\ 2 & 1 & 1 & 2 & 4 & 1 & 1 \end{pmatrix}.$$

Theorem 3.6. *Let X be a graph, and let*

$$\Delta(X) = \max\{\deg(x) \mid x \in V(X)\}$$

be its maximal degree. If λ is an eigenvalue of $A(X)$, then

$$|\lambda| \leq \Delta(X).$$

Proof. Let

$$U = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T$$

be an eigenvector corresponding to eigenvalue λ so that $AU = \lambda U$.
Assuming without loss of generality that

$$|x_1| = \max_{1 \leq i \leq n} |x_i|.$$

Then we have,

$$\begin{aligned} |\lambda||x_1| = |\sum_{j=1}^{n} a_{1j}x_j| &\leq |x_1||\sum_{j=1}^{n} a_{1j}| \\ \Rightarrow |\lambda||x_1| &\leq |x_1|deg(v_1) \leq |x_1|\Delta(X) \\ \Rightarrow |\lambda| &\leq \Delta(X). \end{aligned}$$

□

Theorem 3.7. *Let X be a k-regular graph. Then:*

(i) $\lambda = k$ is an eigenvalue of X,

(ii) the multiplicity of k is equal to the number of connected components of X, and

(iii) all the eigenvalues λ of $A(X)$ satisfy $|\lambda| \leq k$.

Proof. (i) Since X is a k-regular graph,

$$\sum_{j=1}^{n} a_{1j} = k.$$

For the eigenvalue $\lambda = k$, the vector $U = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^T$ is an eigenvector, since $A(X)U = kU$.

(ii) Let X have l connected components say $C_1, C_2, \ldots, C_l$ and $V(X) = \{v_1, v_2, \ldots, v_n\}$. Then for each i, $u_i = \begin{pmatrix} u_1^i & u_2^i & \ldots & u_n^i \end{pmatrix}^T$, where

$$u_j^i = \begin{cases} 1, & \text{if } v_j \in V(C_i) \\ 0, & \text{otherwise} \end{cases}$$

is an eigenvector for the eigenvalue k. Let $w = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^T$ be an eigenvector for the eigenvalue k. Assuming without loss of generality $x_1 = \max\limits_{v_j \in V(C_1)} |x_j|$ and $x_1 > 0$ and $v_1 \in C_1$, we have

$$kx_1 = \sum_{\{v_1, v_j\} \in E} x_j \leq k \max_{\{v_1, v_j\} \in E(X)} |x_j| \leq kx_1.$$

Hence $x_j = x_1$, if $v_j \in N_{v_1}$. Repeating the same argument for all $v_j \in C_1$, we can show that $x_j = x_1$. Applying a similar argument for all other components, we can show that w is a linear combination of u_i's.

(iii) follows immediately from previous theorem □

3.3 Bipartite Graph

Definition 3.8. A graph X is called **bipartite** if $\exists\ A, B \subseteq V(X)$ with $A \cap B = \emptyset$ such that,

$$V(X) = A \cup B \text{ and } E(X) = \{\{i, j\} | i \in A, j \in B\}.$$

Theorem 3.9. *Let X be a k-regular graph, then $-k$ is an eigenvalue of X iff X is a bipartite graph.*

Proof. ($\implies$)Let $u = \begin{pmatrix} u_1 \ldots u_n \end{pmatrix}^T$ be an eigenvector for the eigenvalue $-k$. Then

$$-ku_i = \sum_{\{i,j\} \in E(X)} u_j\ \forall\ i \in\ V(X).$$

Let $M = \max\limits_{i \in V(X)} |u_i|$, $A = \{i \mid u_i = M\}$, and $B = \{i \mid u_i = -M\}$. Then $V(X) = A \cup B$, and $A \cap B = \emptyset$.The converse is just a matter of reversing this argument.

□

Example 3.10. Consider the following graph X, with $A = \{1, 8, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ as shown in Figure 3.4

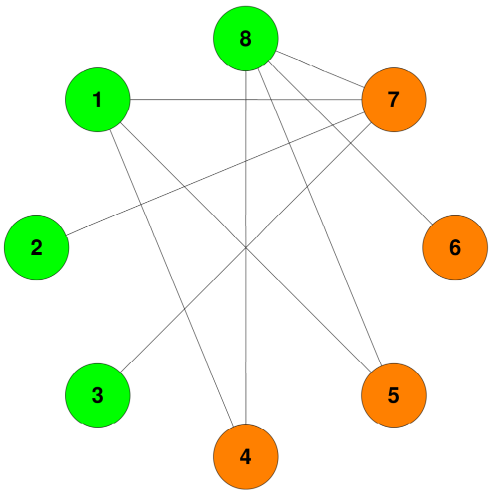

Fig. 3.4: A bipartite graph.

It is clear that any edge of X is of the form $\{x, y\}$ with $x \in A$ and $y \in B$, which shows that X is bipartite.

3.4 Graph Automorphism and Vertex Transitivity

Definition 3.11. Let X be a graph. Then an isomorphism from X onto itself is called an **automorphism** of X. The set of all such automorphisms of a graph X, denoted by $\mathrm{Aut}(X)$ forms a group under composition called the **automorphism group** of the graph X.

Definition 3.12. The **action of a group** G **on a graph** X is a homomorphism

$$\varphi : G \to \mathrm{Aut}(X).$$

Note that there is a natural action of $\mathrm{Aut}(X)$ on X.

Definition 3.13. Let X be a graph, then X is called **vertex-transitive** if for each pair of vertices $\{i, j\} \subset V(X)$, there exists $f \in \text{Aut}(X)$ such that $f(i) = j$.

For a vertex-transitive graph X, $\text{Aut}(X)$ acts transitively on X. All vertex-transitive graphs are regular, but the converse is not necessarily true.

Definition 3.14. Let $\Omega = \{1, 2, 3, 4, 5\}$. Then the graph X defined by

$$V(X) = \{A | A \subset \Omega \text{ and } |A| = 2\}, \text{ and}$$

$$E(X) = \{\{A, B\} | A, B \in V(X), A \cap B = \phi\}$$

is called the **Petersen graph**.(See Figure 3.5)

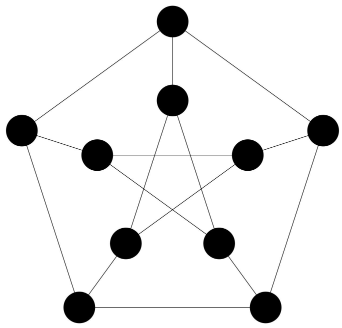

Fig. 3.5: Petersen graph.

Example 3.15.

Remark 3.16. Petersen graph is vertex-transitive

Since each vertex is labeled two-element subset of Ω, any permutation of Ω is going to induce a permutation of its vertices. Moreover, if $\{a, b\} \cap \{c, d\} = \phi$, and $\rho \in S_5$, then $\{\rho(a), \rho(b)\} \cap \{\rho(c), \rho(d)\} = \phi$. Hence, $S_5 \leq \text{Aut}(X)$ acts transitively on X.

3.5 Cayley graph

Definition 3.17. Let G be a group, and let $C \subseteq G$ that is closed under inverse and does not contain identity. Then the **Cayley graph of G with**

respect to C is defined as the graph

$$X(G,C) = (V(X), E(X)),$$

where

$$V(X) = G \text{ and } E(X) = \{\{h,g\} | h^{-1}g \in C\}.$$

Example 3.18. A Cayley graph of $G = S_4$ with respect to connection set $C = \{(12), (1234)\}$ is shown in the Figure 3.6.

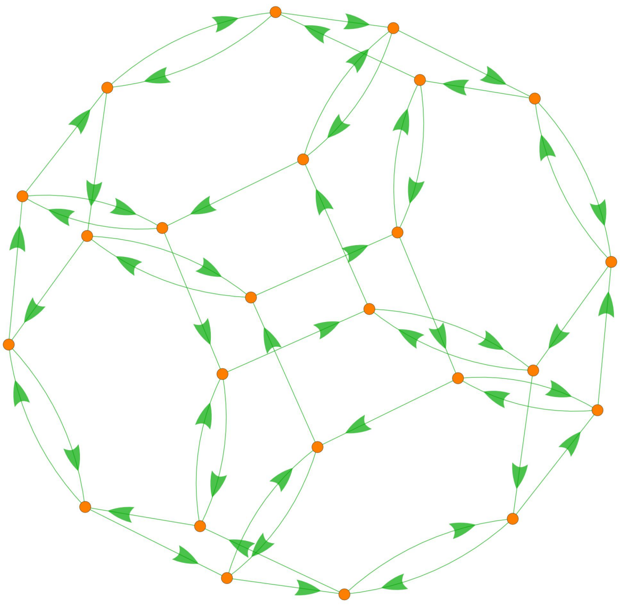

Fig. 3.6: Cay(S_4,$\{(12),(1234)\}$)

Definition 3.19. Let $G = \mathbb{Z}_n$ and $C \subseteq \mathbb{Z}_n \backslash \{0\}$ such that $C = C^{-1}$. Then the Cayley Graph $X(\mathbb{Z}_n, C)$ is called a **circulant graph**.

Definition 3.20. The Cayley graph $X(\mathbb{Z}_n, \{[1], [-1]\})$ is called the **undirected n-cycle**, and is denoted by C_n.

Example 3.21. In Figure 3.7, we have illustrated the circulant graph C_{10}

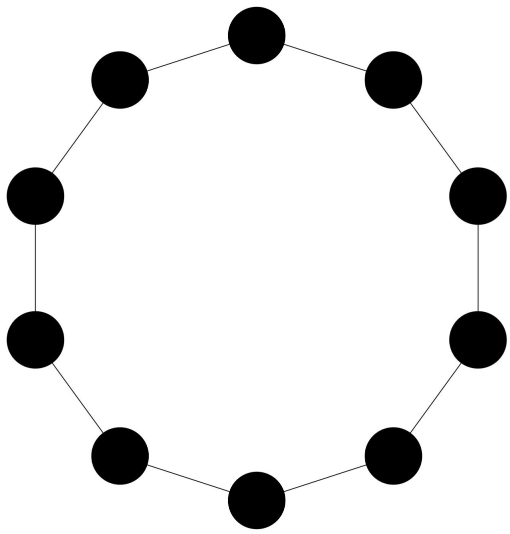

Fig. 3.7: The circulant graph C_n=Cay(Z_{10},$\{+1,-1\}$).

Theorem 3.22. *All Cayley graphs are vertex transitive.*

Proof. Let $X = X(G, C)$ be a Cayley graph, and let

$$\rho_g = gx, \; \forall x \in G.$$

Since $(gy)^{-1}(gx) = y^{-1}g^{-1}gx = y^{-1}x$, we have that

$$y^{-1}x \in C \iff (gy)^{-1}(gx) \in C.$$

So, $\{x, y\} \in E(X)$ iff $\{\rho_g(x), \rho_g(y)\} \in E(X)$. Hence, $\rho_g \in \mathrm{Aut}(X)$ and $H = \{\rho_g \mid g \in G\} \leq \mathrm{Aut}(X)$.
Furthermore, H acts transitively on X, as for any pair of vertices $\{g, h\} \subset V(X)$, we have $\rho_{hg^{-1}}(g) = h$. □

Theorem 3.23. *Let $X = X(G, C)$ be a Cayley graph. Then*

(i) X is $|C|$-regular, and

(ii) X is connected iff $G = \langle C \rangle$.

Proof. (i) Since $E(X) = \{\{g,h\} \mid g^{-1}h \in C\}$, we have

$$\{g,h\} \in E(X) \iff \exists\, c \in C \text{ such that } g^{-1}h = c.$$

In other words, $E(X) = \{\{g, gc\} \mid c \in C \text{ and } g \in G\}$. Hence, for each $g \in G$, we have

$$deg(g) = |C|.$$

(ii)

$$\begin{aligned} X \text{ is connected } &\iff \exists \text{ a path from each } g \in G \text{ to } 1 \\ &\iff \exists\, c_{g,1}, \ldots, c_{g,k(g)} \in C \text{ such that} \\ &\qquad g = \prod_{i=1}^{k(g)} c_{g,i} \\ &\iff G = \langle C \rangle. \end{aligned}$$

□

Definition 3.24. Let X be an undirected k-regular of order n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and let

$$\lambda(X) = \max\{|\lambda_i| \mid |\lambda_i| \neq k,\, 1 \leq i \leq n\}.$$

Then X is said to be a **Ramanujan graph** if,

$$\lambda(X) \leq 2\sqrt{k-1}.$$

Theorem 3.25. *The n-cycle C_n is Ramanujan.*

Proof. Since every $[i] \in V(C_n)$, the edges $\{[i],[i-1]\}$, $\{[i],[i+1]\} \in E(C_n)$, we can see that $A(C_n) = (a_{ij})_{n \times n}$, where

$$a_{ij} = \begin{cases} 1 & |i-j| \equiv 1 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $A(C_n) = W + W^T (= W^{-1})$, where W is defined as

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Note that W acts on a vector by shifting each entry up by one position, with the first entry becoming the last. Suppose $U = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}^T$ be an eigenvector with eigenvalue λ of W. Then we have

$$v_1 = \lambda v_n = \lambda^2 v_{n-1} = \cdots = \lambda^n v_1.$$

Hence, $\lambda^n = 1$, and so the eigenvalues of W are among the roots of unity. Furthermore, $\lambda_l = \omega^l$, for $1 \leq l \leq n$, where $\omega = \exp(2\pi i/n)$, is an eigenvalue of W with multiplicity 1. Let $v_1 = 1$, and

$$U_l = \begin{pmatrix} 1 & \omega^l & \omega^{2l} & \dots & \omega^{(n-1)l} \end{pmatrix}.$$

Since

$$\text{Spec}(W^k) = \begin{pmatrix} 1 & \omega^k & \omega^{2k} & \dots & \omega^{(n-1)k} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix},$$

we have

$$A(C_n)U_l = WU_l + W^{-1}U_l = (\omega^l + \omega^{-l})U_l.$$

Therefore,

$$\text{Spec}(C_n) = \begin{pmatrix} 2 & 2cos(2\pi/n) & \dots & 2cos((n-1)2\pi/n) \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

Since,

$$2|\cos(2\pi l/n)| \leq 2\sqrt{2-1},$$

for all l, C_n is Ramanujan.

□

Example 3.26. See Figure3.5.The spectrum of Petersen graph is given by,

$$\mathrm{Spec}(X) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 4 & 4 \end{pmatrix}.$$

Here $\lambda(X) = 2$ and $k = 3$. Since $2 \leq 2\sqrt{2}$, the Petersen graph is Ramanujan.

4. EXPANDER GRAPHS

The goal of this chapter is to understand some of the special properties of expander graphs, that will eventually lead to the proof of Alon-Boppana theorem. We will begin by describing some basic operators related to graphs.

4.1 Adjacency Operator

Definition 4.1. Let $X = (V(X), E(X))$ be a finite graph. We define the complex vector space $L^2(V(X))$ as

$$L^2(V(X)) = \left\{ f \middle| f : V(X) \to \mathbb{C} \right\}.$$

Remark 4.2. To see that $L^2(V(X))$ is a vector space take $f, g \in L^2(V(X))$ and $\alpha \in \mathbb{C}$. Then,

(a) the vector space sum is given by $(f + g)(x) = f(x) + g(x)$, and

(b) the scalar multiplication is given by $(\alpha f)(x) = \alpha f(x)$.

(c) The standard inner product and norm on $L^2(V(X))$ are given by,

$$\langle f, g\rangle_2 = \sum_{x \in V(X)} f(x)\overline{g(x)} \text{ and } \|f\|_2 = \sqrt{\langle f, g\rangle_2} = \sqrt{\sum_{x \in V(X)} |f(x)|^2}$$

This makes $L^2(V(X))$ a normed vector space.

Remark 4.3. Consider the graph $X = (V(X), E(X))$ with $V(X) = \{v_1, v_2, \ldots, v_n\}$.

Let $\beta = \{\partial_{v_1}, \partial_{v_2}, \ldots, \partial_{v_n}\}$ such that

$$\partial_{v_i}(v_j) = \begin{cases} 1, & \text{if } i = j, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

For $f \in L^2(V(X))$, we will write

$$f(x) = f(v_1)\partial_{v_1}(x) + f(v_2)\partial_{v_2}(x) + \cdots + f(v_n)\partial_{v_n}(x).$$

Then it is quite apparent that β forms a basis for $L^2(V(X))$.

Definition 4.4. For graph $X = (V(X), E(X))$ consider a linear transformation, $A : L^2(V(X)) \to L^2(V(X))$ defined as $(Af)(v) = \sum_{w \in V(X)} a_{vw} f(w)$, that is,

$$Af = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_n) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} f(v_j) \\ \vdots \\ \sum_{j=1}^{n} a_{nj} f(v_j) \end{pmatrix}, \tag{4.1}$$

is called **adjacency operator** of graph $X = (V(X), E(X))$.

Remark 4.5. It's not difficult to see that A is a linear operator, for

$$\begin{aligned} A(f+g)(v) &= \textstyle\sum_{w \in V(X)} a_{vw}(f+g)(w) \\ &= \textstyle\sum_{w \in V(X)} a_{vw} f(w) + \sum_{w \in V(X)} g(w) \end{aligned} .$$

Definition 4.6. Let $X = (V(X), E(X))$ be a graph. Then:

(a) The **gradient operator**

$$d : L^2(V) \to L^2(E)$$

defined by,

$$(df)(e) = f(e^+) - f(e^-), \text{ for all } f \in L^2(V).$$

(b) The **divergence operator**

$$d^*: L^2(E) \to L^2(V)$$

defined by,

$$(d * f)(v) = \sum_{e\in E, v=e^+} f(e) - \sum_{e\in E, v=e^-} f(e), \text{ for each } f \in L^2(E).$$

(c) The **laplacian operator**

$$\Delta : L^2(V) \to L^2(V)$$

defined by,

$$(\Delta f)(v) = d * d(f)(v), \text{ for all } f \in L^2(V).$$

The following lemma will gives a relationship between adjacency operator and laplacian operator.

Lemma 4.7. Let $X = (V(X), E(X))$ be a k-regular graph with $E(X)$ be a multi set. Then $\Delta = kI - A$.

Proof. Let $f \in L^2(V)$ and $x \in V(X)$. Then,

$$\begin{aligned}
(\Delta f)(x) &= (d * (df))(x) \\
&= \sum_{e\in E, x=e^+} (df)(e) - \sum_{e\in E, x=e^-} (df)(e) \\
&= (\sum_{e\in E, x=e^+} f(x) - \sum_{y=e^- \text{ and } x=e^+} f(y)) - \\
&\quad (\sum_{x=e^- \text{ and } y=e^+} f(y) - \sum_{e\in E, x=e^-} f(x)) \\
&= kf(x) - \sum_{y\in V} a_{xy} f(y) \\
&= kIf(x) - Af(x),
\end{aligned}$$

as required. □

Proposition 4.8. Let $X = (V(X), E(X))$ be a k-regular graph and let $|X| = n$. Then,

(i) The eigenvalues of Δ are given by

$$0 = k - \lambda_0(X) \leq k - \lambda_1(X) \leq \cdots \leq k - \lambda_{n-1}(X).$$

In particular, the eigenvalues of Δ lies in the interval $[0, 2k]$.

(ii) For $f \in L^2(V)$ and $g \in L^2(E)$, we have

$$\langle df, g\rangle_2 = \langle f, d * g\rangle_2 \text{ and } \langle \Delta f, f\rangle_2 = \sum_{e\in E} |f(e^+) - f(e^-)|.$$

Proof. (i) For $f \in L^2(V)$ then we know that,

$$Af = \lambda f \text{ iff } (kI - A)f = (k - \lambda)f.$$

Hence the eigenvalues of Δ are $k - \lambda_i$, for $0 \leq i \leq n - 1$.
We know that,

$$\begin{aligned}
\langle df, g\rangle_2 &= \sum_{e\in E} (df)(e)\overline{g(e)} \\
&= \sum_{e\in E} [f(e^+) - f(e^-)]\overline{g(e)} \quad \text{(by definition of gradient operator)} \\
&= \sum_{e\in E} f(e^+)\overline{g(e)} - \sum_{e\in E} f(e^+)\overline{g(e)} \\
&= \sum_{v\in V} f(v) \sum_{e\in E, v=e^+} \overline{g(e)} - \sum_{v\in V(X)} f(v) \sum_{e\in E, v=e^-} \overline{g(e)} \\
&= \sum_{v\in V} f(v)\overline{(d * g)(v)} \\
&= \langle f, d * g\rangle_2, \text{ and (i) follows.}
\end{aligned}$$

(ii) From (i), we have that,

$$\begin{aligned}\langle \Delta f, f\rangle_2 &= \langle d*df, f\rangle_2\\ &= \overline{\langle f, d*df\rangle_2}\\ &= \overline{\langle df, df\rangle_2}\\ &= \langle df, df\rangle_2\\ &= \|df\|_2^2.\end{aligned}$$

So,

$$\begin{aligned}\langle df, df\rangle_2 &= \sum_{e\in E}(f(e^+)-f(e^-))\overline{(f(e^+)-f(e^-))}\\ &= \sum_{e\in E}|f(e^+)-f(e^-)|., \text{ as required.}\end{aligned}$$

□

4.2 Isoperimetric Constant

Isoperimetric constant of a graph is intricately related to its connectivity. We will begin our discussion on the isoperimetric constant by defining the boundary of a graph.

Definition 4.9. Let $X = (V(X), E(X))$ be a graph. Consider $F \subseteq X$ be a sub-graph of $X = (V(X), E(X))$. Then the **boundary** of F denoted by ∂F, is defined by

$$\partial F = \{\{x, y\} \in E(X)|x \in V(F), y \in V(X)\setminus V(F)\}.$$

Definition 4.10. Let $X = (V(X), E(X))$ be a graph then isoperimetric constant $h(X)$ is defined as follows,

$$h(X) = \min\left\{\frac{|\partial F|}{|F|}\,\middle|\, F\subseteq X \text{ and } |F| \le \frac{|V|}{2}\right\}.$$

Equivalently,

$$h(X) = \min\left\{ \frac{|\partial F|}{\min\{|F|, |X \setminus F|\}} \middle| F \subseteq X \right\}.$$

Isoperimetric constant is also known as expansion constant or edge expansion constant or the conductance or the Cheeger's constant.

4.3 Expander families of graphs

In this section we will introduce some expander graphs and show that n-cycle's are not good expander graphs.

Definition 4.11. Let (X_n) be a sequence of $k-$regular graphs, such that $|X_n| \to \infty$ as $n \to \infty$. Then (X_n) are called **expander family** of graphs if the sequence $(h(X_n))$ is bounded away from 0.

Note that in general, regularity of graphs is not required in the definition of expander graphs. But in this report, we consider only regular families of graphs. It follows from the following theorem shows that $n-$cycles are not expanders.

Theorem 4.12. *There does not exist expander families of degree 2.*

Proof. let $X = (V(X), E(X))$ be a 2-regular graph connected graph with $|X| = n$. Then we know that, $X \cong C_n$. Take k such that, C_n, $1 \leq k \leq \frac{n}{2}$. Then,

$$\min\left\{ \frac{|\partial F|}{|F|} \middle| |F| = k, F \subset X \right\} = \frac{2}{k}.$$

Hence $h(C_n) = \min\left\{ \frac{2}{k} \right\}$, which is possible when the maximum value of k is $k = \frac{n}{2}$. So

$$h(C_n) = \begin{cases} \frac{4}{n}, & \text{if n is even, and} \\ \frac{4}{n-1}, & \text{if n is odd.} \end{cases}$$

The result now follows from the fact that $h(C_n) \to 0$, as $n \to \infty$. □

Definition 4.13. Let $X = (V(X), E(X))$ be a finite graph. Let

$$f_0 : V(X) \to \{1\}.$$

We define,

$$L^2(X, \mathbb{R}) = \Big\{ f : V(X) \to \mathbb{R} \Big\}$$

and

$$\begin{aligned} L_0^2(X, \mathbb{R}) &= \Big\{ f \in L^2(X, \mathbb{R}) \Big| \langle f, f_0 \rangle_2 = 0 \Big\} \\ &= \Big\{ f \in L^2(X, \mathbb{R}) \Big| \sum_{x \in V(X)} f(x) = 0 \Big\} \end{aligned}$$

Define the inner product for $f, g \in L_0^2(X, \mathbb{R})$ by,

$$\langle f, g \rangle_2 = \sum_{x \in V(X)} f(x) g(x) \text{ and } \|f(x)\|_2 = \sqrt{\sum_{x \in V(X)} f(x)^2}.$$

The following theorem which is one of the key ingredients for the Alon-Bopanna Theorem , will give us arelationship between second largest eigenvalue and the adjacency operator.

Theorem 4.14 (Rayleigh-Ritz)**.** *let* $X = (V(X), E(X))$ *be a d-regular graph, then*

$$\lambda_1(X) = \max_{f \in L_0^2(X, \mathbb{R})} \frac{\langle Af, f \rangle_2}{\|f\|_2^2} = \max_{f \in L_0^2(X, \mathbb{R}), \|f\|_2 = 1} \langle Af, f \rangle_2.$$

Equivalently we have,

$$d - \lambda_1(X) = \min_{f \in L_0^2(X, \mathbb{R})} \frac{\langle \Delta f, f \rangle_2}{\|f\|_2^2} = \max_{f \in L_0^2(X, \mathbb{R}), \|f\|_2 = 1} \langle \Delta f, f \rangle_2.$$

Proof. Let $\{f_0, f_1 \ldots, f_{n-1}\}$ be the orthonormal basis for $L^2(V, \mathbb{R})$ such that for each i,f_i is a real valued eigenfunction of A associated with the eigenvalue $\lambda_i(X)$. Consider $f \in L_0^2(V, \mathbb{R})$ with $\|f\|_2 = 1$. Suppose that,

$$f = c_0 f_0 + c_1 f_1 + \cdots + c_{n-1} f_{n-1}.$$

Then we have,

$$0 = \langle f, f_0 \rangle_2 = c_0 \langle f_0, f_0 \rangle_2 = c_1 \langle f_1, f_0 \rangle_2 = \cdots = c_{n-1} \langle f_{n-1}, f_0 \rangle_2 = c_0$$

$$\Rightarrow f = c_1 f_1 + c_1 f_1 + \cdots + c_{n-1} f_{n-1}.$$

This would imply that,

$$\begin{aligned}
\langle Af, f \rangle_2 &= \langle A \sum_{i=1}^{n-1} c_i f_i, \sum_{j=1}^{n-1} c_j f_j \rangle_2 \\
&= \langle \sum_{i=1}^{n-1} c_i \lambda_i f_i, \sum_{j=1}^{n-1} c_j f_j \rangle_2 \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_i c_j \lambda_i \langle f_i, f_j \rangle_2 \\
&= \sum_{i=1}^{n-1} c_i^2 \lambda_i \\
&\leq \lambda_1 \sum_{i=1}^{n-1} c_i^2 \\
&= \lambda_1 \|f\|_2^2 \\
&= \lambda_1.
\end{aligned}$$

Hence, we have

$$\lambda_1 \geq \max_{f \in L_0^2(X,\mathbb{R}), \|f\|_2 = 1} \langle Af, f \rangle_2$$

Note that $f_1 \in L_0^2(V, \mathbb{R})$, so

$$\|f_1\|_2 = 1 \text{ and } \langle Af_1, f_1 \rangle_2 = \langle \lambda_1 f_1, f_1 \rangle_2 = \lambda_1.$$

and we get

$$\lambda_1(X) = \max_{f \in L_0^2(X,\mathbb{R}), \|f\|_2 = 1} \langle Af, f \rangle_2.$$

The second part of the result follows directly from the fact that $\Delta = dI - A$. □

4.4 Spectral gap and the isoperimetric constant

In general, calculating $h(X)$ for large graphs is computationally intractable. So we will need to understand the relation between the Cheeger's constant and eigenvalues of graph. We derive this relation in this section.

Definition 4.15. Let $X = (V(X), E(X))$ be a graph. Define

$$f_0 : V(X) \to \{1\}$$

with

$$\begin{aligned} L_0^2(X) &= \left\{ f \in L^2(X) \middle| \langle f, f_0 \rangle_2 = 0 \right\} \\ &= \left\{ f \in L^2(X) \middle| \textstyle\sum_{x \in V(X)} f(x) = 0 \right\} \end{aligned}$$

Remark 4.16. Consider the graph $X = (V(X), E(X))$. Let A be the adjacency matrix of X, and let $f \in L_0^2(V(X))$. Then

$$\overline{\langle Af, f \rangle_2} = \overline{\langle f, Af \rangle_2} = \langle Af, f \rangle_2$$

Hence $\langle Af, f \rangle_2$ is real, and so we have

$$\lambda_1(X) = \max_{g \in L_0^2(V(X))} \frac{\langle Ag, g \rangle_2}{\|g\|_2^2}.$$

We will now define the spectral gap of a graph $X = (V(X), E(X))$

Definition 4.17. Let $X = (V(X), E(X))$ be a connected d-regular graph. Then $d - \lambda_1(X)$ is called the **spectral gap** of X.

The following proposition gives an explicit relation between $h(X)$ and the spectral gap of X.

Proposition 4.18. Let $X = (V(X), E(X))$ be a d-regular connected graph. Then,

$$\frac{d - \lambda_1(X)}{2} \leq h(X) \leq \sqrt{2d(d - \lambda_1(X))}.$$

Proof. First, we prove that,

$$\frac{d-\lambda_1(X)}{2} \leq h(X).$$

Consider $F \subset X$ such that $h(X) = \frac{|\partial F|}{|F|}$. Let

$$|F| = b \text{ and } |V \setminus F| = a.$$

Consider

$$g(x) = \begin{cases} a, & \text{if } x \in F, \text{ and} \\ -b, & \text{if } x \in V \setminus F, \end{cases}$$

and let $f = \frac{g}{\|g\|_2}$.
Since

$$\sum_{v\in V} g(v) = \sum_{v\in F} a - \sum_{v\in V\setminus F} b = ba - ab = 0,$$

we have,

$$\langle \Delta g, g\rangle_2 = \sum_{e\in E} \left| g(e^+) - g(e^-)\right|^2 = \sum_{e\in\partial F} (b+a)^2 = |\partial F|(b+a)^2.$$

Moreover, we have

$$\begin{aligned} \|g\|_2^2 &= \langle g, g\rangle_2 \\ &= \textstyle\sum_{x\in F} a^2 + \sum_{x\in V\setminus F} b^2 \\ &= a^2 b + b^2 a = ab(a+b) \end{aligned}$$

Furthermore,

$$b \leq \frac{(a+b)}{2} \implies 2b \leq a+b \implies b \leq a.$$

So, we have,

$$\langle \Delta f, f\rangle_2 = \frac{1}{\|g\|_2^2} \langle \Delta g, g\rangle_2 \text{ where } f \in L_0^2(V, \mathbb{R}) \text{ with } \|f\|_2 = 1.$$

Hence,

$$\langle \Delta f, f \rangle_2 = \frac{|\partial F|(b+a)}{ba}$$

$$= (1 + \frac{b}{a})h(X)$$

$$\leq 2h(X).$$

Therefore

$$d - \lambda_1(X) \ \Rightarrow \frac{d - \lambda_1(X)}{2} \leq h(X),$$

which concludes the first part.
For the second part we need to show that

$$h(X) \leq \sqrt{2d(d - \lambda_1(X))}.$$

Let $g \in L_0^2(V, \mathbb{R})$ be the real valued eigenfunction of adjacency operator of X associated with the eigenvalue $\lambda_1(X)$. Consider the set

$$V^+ = \{v \in V | g(x) \geq 0\}.$$

Note that as

$$\sum_{x \in V} g(x) = 0, \text{ we have}$$

$V^+ \neq V(X)$. We define function $f \in L_0^2(V, \mathbb{R})$ as follows

$$f(x) = \begin{cases} g(x), & \text{if } x \in V^+, \\ 0, & \text{otherwise} \end{cases}$$

$$\text{So that, } \frac{\langle \Delta f, f \rangle_2}{\langle f, f \rangle_2} \leq d - \lambda_1(X) \tag{4.2}$$

If $x \in V^+$, then we know that,

$$(\Delta f)(x) = df(x) - \sum_{y \in V^+} a_{xy} f(y)$$

$$= dg(x) - \sum_{y \in V^+} a_{xy} g(y)$$

$$\leq dg(x) - \sum_{y \in V} a_{xy} g(y) = (\Delta g)(x)$$

Thus,

$$\langle \Delta f, f \rangle_2 = \sum_{v \in V^+} (\Delta f)(v) f(v)$$

$$\leq \sum_{v \in V^+} (\Delta g)(v) g(v)$$

$$= \sum_{v \in V^+} (d - \lambda_1(X)) g(v)^2$$

$$= (d - \lambda_1(X)) \sum_{v \in V^+} f(x)^2$$

$$\leq (d - \lambda_1(X)) \langle f, f \rangle_2,$$

which actually proves the claim in (3.2). Now we will show that,

$$\frac{h(X)^2}{2d} \leq \frac{\langle \Delta f, f \rangle_2}{\langle f, f \rangle_2} \tag{4.3}$$

For proving this we will consider our graph to be oriented [1].

[1] "an oriented graph is a directed graph without multiple edges or loops"

Define,

$$B_f = \sum_{e\in E}(f(e^+)^2 - f(e^-)^2).$$

We will prove this claim in two steps,

(1) $B_f \le \sqrt{2d\langle \Delta f, f\rangle_2 \langle f, f\rangle_2}$.

(2) $h(x)\langle f, f\rangle_2 \le B_f$.

We now establish Step (1). We know that,

$$\begin{aligned}
B_f &= \sum_{e\in E}(f(e^+)^2 - f(e^-)^2)\\
&= \langle (f(e^+) + f(e^-)), (f(e^+) + f(e^-))\rangle_2\\
&= \sum_{e\in E}(f(e^+) + f(e^-))(f(e^+) - f(e^-))\\
&\le \sqrt{\sum_{e\in E}(f(e^+) + f(e^-))^2}\sqrt{\sum_{e\in E}(f(e^+) - f(e^-))^2} \quad \text{(By Cauchy- Schwarz inequality)}\\
&\le \sqrt{2\sum_{e\in E} f(e^+)^2 + f(e^-)^2}\sqrt{\langle \Delta f, f\rangle_2}\\
&= \sqrt{2d\sum_{v\in V} f(v)^2}\sqrt{\langle \Delta f, f\rangle_2} = \sqrt{2d\langle \Delta f, f\rangle_2\langle f, f\rangle_2},
\end{aligned}$$

and Step (1) follows.

To show Step (2), break vertices of X into level sets of f. Let

$$0 = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_r$$

be the values of f and $L_i = \left\{x \in V \middle| f(x) \geq \beta_i\right\}$. Note that,

$$L_r \subset L_{r-1} \subset \ldots L_2 \subset L_1 \subset L_0 = V \text{ and } L_i \subset V^+ \text{ for } i \geq 0.$$

Suppose that $e \in E(X)$ with $f(e^+) - f(e^-) \neq 0$ Then $f(e^-) = \beta_j$ and $f(e^+) = \beta_i$ with $j < i$, so we have so,

$$e \in \partial L_{j+1} \bigcap \partial L_{j+2} \bigcap \cdots \bigcap \partial L_i, \text{ and}$$

$$f(e^+)^2 - f(e^-)^2 = \beta_i^2 - \beta_j^2$$

$$= \beta_i^2 - \beta_{i-1}^2 + \beta_{i-1}^2 - \beta_{i-2}^2 + \cdots + \beta_{j+1}^2 - \beta_j^2.$$

Thus,

$$B_f = \sum_{\substack{e \in E \\ f(e^-)=\beta_j \\ f(e^+)=\beta_i \\ j<i}} \sum_{k=j+1}^{i} (\beta_k^2 - \beta_{k-1}^2) = \sum_{k=1}^{r} |\partial L_k| (\beta_k^2 - \beta_{k-1}^2).$$

Since $L_i \subset V^+$, we have for $i > 0$,

$$|V^+| \leq \frac{|V|}{2} \quad \partial L_i \geq h(X)|L_i|.$$

, Therefore,

$$B_f \geq h(X) \sum_{k=1}^{r} |L_k| (\beta_k^2 - \beta_{k-1}^2)$$

$$\geq h(X) \Big[|L_1|(\beta_1^2 - \beta_0^2) + |L_2|(\beta_2^2 - \beta_1^2) + \cdots + |L_{r-1}|(\beta_{r-1}^2 - \beta_{r-2}^2) + |L_r|(\beta_r^2 - \beta_{r-1}^2) \Big]$$

$$= h(X) \Big[|L_r| \beta_r^2 + \sum_{k=1}^{r-1} \beta_k^2 (|L_k| - |L_{k-1}|) \Big].$$

The fact that $x \in L_k \setminus L_{k+1}$ iff $f(x) = \beta_k$ implies that,

$$B_f \geq h(X) \sum_{i=0}^{r} \sum_{\substack{x \in V^+ \\ f(x)=\beta_i}} f(x)^2 = h(x)\langle f, f\rangle_2,$$

and Step (2) follows. Finally from (3.2) and (3.3), we have that,

$$\frac{h(X)^2}{2d} \leq \frac{\langle \Delta f, f\rangle_2}{\langle f, f\rangle_2} \leq d - \lambda_1(X)$$

$$h(X) \leq \sqrt{2d(d - \lambda_1(X))},$$

and the second part of the theorem follows. □

4.5 Properties of Expander graphs

Expander families of graphs have high connectivity, owing to the desirable property that they have controlled growth in diameter. In this section, we will show that this growth is in fact logarithmic.

Definition 4.19. Let $X = (V(X), E(X))$ be a graph. Then **diameter** of graph X is defined as follows,

$$\text{Diam}(X) = \max_{\{x,y\} \in E(X)} \{l(P_{x,y})| \text{ Where } P_{x,y} \text{ is shortest path between x and y }\}.$$

Definition 4.20. For a $X = (V(X), E(X))$ graph.Let $v \in V$ and r be a positive integer.Then a **closed ball** of radius r centered at v is defined as follows,

$$B_r[v] = \{w \in V(X)|dist(v, w) \leq r\}$$

and, **sphere** of radius r centered at v is defined as

$$S_r[v] = \{w \in V(X)|dist(v, w) = r\}.$$

The following lemma shows that for a sequence of d-regular graphs $\text{Diam}(X_n)$ grows atleast logarithmically.

Lemma 4.21. Let $X = (V(X), E(X))$ be a d-regular connected finite graph. Then for $d \geq 3$ and $\text{Diam}(X) \geq 3$ we have,

$$\text{Diam}(X) \geq \log_d |X|.$$

Proof. First we make few observations. Since X is d-regular we have $|S_0[v] = 1|$ and $|S_1[v]| = d$. For $j \geq 2$, consider vertex $w \in S_j[v]$. At least one edge incident to w is also incident to a vertex in $S_{j-1}[v]$. So for d edges incident to w, no more than $d-1$ of them are also incident to vertices in $S_{j+1}[v]$. Hence , we have

$$|S_{j+1}[v]| \leq (d-1)|S_j[v]|.$$

By inducting on j we get,

$$\Rightarrow |S_j[v]| \leq d(d-1)^{j-1}$$

But,

$$\begin{aligned} B_r[v] &= S_0[v] \cup S_1[v] \cup \ldots S_r[v] \cup S[v] \\ &\leq 1 + d(\textstyle\sum_{j=0}^{r-1}(d-1)^j). \end{aligned}$$

As the right hand side of this equation is a polynomial in d of degree r, we expect it to be controlled by d^r.
We claim that

$$|B_r[v]| \leq d^r, \text{ if } d \geq 3.$$

Observe that,

$$\begin{aligned} &\text{Since } (d-1)^3 \leq d^2(d-2), \text{ we have that} \\ &\text{so,} \\ (d-1)^r &= (d-1)^{r-3}(d-1)^3 \\ &\leq d^{r-3}d^2(d-2) \\ &\leq d_{r-1}(d-2). \end{aligned}$$

Hence,

$$d(d-1)^r - 2 \leq d(d-1)^r \leq d^r(d-2)$$
$$\Rightarrow 1 + d(\sum_{j=0}^{r-1}(d-1)^j) = 1 + d\Big[\frac{(d-1)^r - 1}{d-2}\Big] \leq d^r.$$

Let $k = \text{Diam}(X)$ then $|X| = |B_k[v]| \leq d^k$,
so we have,

$$\text{Diam}(X), \geq \log_d |X| \text{ as desired.}$$

□

The following lemma gives us an upper bound diameter of a connected[2] finite graph.

Lemma 4.22. Let $X = (V(X), E(X))$ be a connected finite graph. Suppose that for any vertex $v \in V$, we have $|B_r[v]| \geq a^r$ for $a > 1$ whenever $|B_{r-1}| \leq \frac{1}{2}|X|$. Then

$$\text{Diam}(X) \leq \frac{2}{\log a} \log |X|$$

Proof. Let $w_1, w_2 \in V(X)$. Let r_1 be the smallest non-negative integer such that $|B_{r_1}[w_1]| > \frac{1}{2}|X|$. Then

$$|B_{r_1}[w_1]| \geq a^{r_1}$$

But since $r_1 < r_1$, we have

$$|B_{r_1-1}[w_1]| \leq \frac{1}{2}|X|$$

Similarly,

$$|B_{r_2}[w_2]| > \frac{1}{2}|X|,$$

and by letting r_2 be the smallest non-negative integer we have that,$|B_{r_2}[w_2]| \geq a^{r_2}$
Then,

$$|B_{r_1}[w_1]| + |B_{r_2}[w_2] > |X|,$$

[2] We set diameter of disconnected graph to be ∞

we must have,

$$B_{r_1}[w_1] \cap B_{r_2}[w_2] \neq \phi, \text{ that is}$$

$$w_3 \in B_{r_1}[w_1] \cap B_{r_2}[w_2].$$

Hence $\text{dist}(w_1, w_3) \leq r_1$ and $\text{dist}(w_2, w_3) \leq r_2$ This implies,

$$\text{dist}(w_1, w_3) = r_1 + r_2$$

$$= \frac{\log |B_{r_1}[w_1]|}{\log(a)} + \frac{\log |B_{r_2}[w_2]|}{\log(a)}$$

$$\leq \frac{2}{\log(a)} \log |X|.$$

As this inequality holds for any w_1, w_2 hence we can conclude that

$$\text{Diam}(X) \leq \frac{2}{\log(a)} \log |X|.$$

□

The following proposition gives an upper bound on the diameter of a graph in terms of order of graph and isoperimetric constant.

Proposition 4.23. Let $X = (V(X), E(X))$ be a connected d-regular graph. Let $C = 1 + \frac{h(X)}{d}$. Then,

$$\text{Diam}(X) \leq \frac{2}{\log(C)} \log |X|.$$

Proof. Let $v \in V$, and suppose that $|B_{r-1}[v]| \leq \frac{1}{2}|X|$. We have,

$$|\partial B_{r-1}[v]| \geq h(X)|B_{r-1}[v]|$$

Any edge in $|\partial B_{r-1}[v]|$ must be incident to a vertex in $S_r[v]$, as X is d regular. So, we have

$$|S_r[v]| \geq \frac{|\partial B_{r-1}[v]|}{d} \geq \frac{h(X)}{d}|B_{r-1}[v]|.$$

Since

$$\begin{aligned} B_r[v] &= B_{r-1}[v] \cup S_r[v], \text{ we have} \\ |B_r[v]| &= |B_{r-1}[v]| + |S_r[v]| \\ &\geq |B_{r-1}[v]| + \frac{h(X)}{d}|B_{r-1}[v]| \\ &= C|B_{r-1}[v]| \end{aligned}$$

By induction on r, we get,

$$|B_r[v]| \geq C^r,$$

whenever

$$|B_{r-1}[v]| \leq \frac{1}{2}|X|.$$

From this it follows that,

$$\Rightarrow \text{Diam}(X) \leq \frac{2}{\log(a)} \log|X|.$$

□

Definition 4.24. Let (X_n) be a family of graphs. We say (X_n) has logarithmic diameter if

$$\text{Diam}(X) = O\Big(\log|X_n|\Big)$$

We will now prove the main result in this section which says that an expander family of graphs has logarithmic diameter.

Corollary 4.25. Let d be a non-negative integer. If (X_n) is a family of $d-$regular expanders, then (X_n) has logarithmic diameter.

Proof. Since (X_n) is family of expander graphs, there exists an $\epsilon > 0$ such that $h(X_n) \geq \epsilon \quad \forall n \in \mathbb{N}$.Define $C_n = 1 + \frac{h(X_n)}{d}$ and $C = 1 + \frac{\epsilon}{d}$. Since $\epsilon \leq h(X_n)$,
we have

$$\frac{2}{\log_{C_n}} \leq \frac{2}{\log_C}.$$

Since,

$$\log(C_n) \geq \log(C), \text{ we have}$$

$$\Big(\log(1+\frac{h(X_n)}{d})\geq\log(1+\frac{\epsilon}{d})\Big)$$

Hence by the Proposition 4.18, we have

$$\mathrm{Diam}(X_n)\leq\Big(\frac{2}{\log(C_n)}\Big)\log|X_n|\leq\Big(\frac{2}{\log(C)}\Big)\log|X_n|,$$

and the result follows. □

4.6 Diameter of Cayley graph

It is natural to look for expanders in Cayley graphs as they are symmetric and regular. In this section we derive bounds on the diameter of Cayley graphs and show that abelian Cayley graph do not yield expanders.

Definition 4.26. Let (G_n) be a sequence of finite groups. We say (G_n) has logarithmic diameter if for some positive integer d, there exist a sequence of (Γ_n) where for each n we have Γ_n is a symmetric subset of G_n with $|\Gamma_n| = d$ so that the sequence of Cayley graph $(\mathrm{Cay}(G_n, \Gamma_n))$ has logarithmic diameter.

Definition 4.27. Let Γ be a set, and $n \in \mathbb{Z}^+$.Then a **word of length** n in Γ is an element of the Cartesian product

$$\Gamma\times\Gamma\times\Gamma\times\ldots\Gamma=\Gamma^n.$$

If $\Gamma \subset G$ subset of some group G and $w = (w_1, w_2, \ldots, w_n)$ is a word in Γ, then we say that w evaluates to g if

$$g=w_1\cdot w_2\cdot w_3\cdot\cdots\cdot w_n,$$

and $l(w) = n$, if $w = (w_1, w_2, \ldots, w_n)$.

Definition 4.28. G be a group with $\Gamma \subseteq G$ and let $g \in G$ be expressible as a word in Γ. Then the **word norm** is defined by,

$$\|g\|_\Gamma=\min\{l(w)|\text{w evaluates to g}\}.$$

By convention, $l(w) = 0$ if w evaluates to 1, that is, $\|1\|_\Gamma = 0$.

Proposition 4.29. Let G be a finite group and let $\Gamma \subset G$ be a symmetric subset of G. Then for the Cayley graph $X = \mathrm{Cay}(G, \Gamma)$ we have:

(i) X is connected iff any element of G can be expressed as a word in Γ.

(ii) If $a, b \in G$ and there is walk in X from a to b, then distance from a to b is the word norm of $a^{-1}b$ in Γ.

(iii)

$$\mathrm{Diam}(X) = \max_{g \in G} \|g\|_\Gamma.$$

Proof. (i) We have already shown that if X is connected then $G = \langle \Gamma \rangle$.This implies that any element of G can be expressed as a word in Γ

(ii) For $a, b \in G$ consider a walk between them $(g_0, g_1, g_2, \ldots, g_n)$ of length n with $g_0 = a$ and $g_n = b$.Remember we said that $\{g_i, g_h\} \in E(X)$ iff $g_j^{-1} g_i \in \Gamma$. So call $g_1^{-1} g_0 = s_1$,$g_2^{-1} g_1 = s_2$ so on then $a^{-1}b = s_1 s_2 \ldots s_n$ and so the distance between a and $b = \|a^{-1}b\|_\Gamma$

(iii) This follows directly from the definition of $\mathrm{Diam}(X)$.

□

The following proposition gives us the reason why abelian groups do not yield expander graphs.

Proposition 4.30. No sequence of finite abelian groups has logarithmic diameter. Hence they do not give rise to expander family of Cayley graphs.

Proof. Let G be an abelian group. Let $\Gamma \subset G$ and $|\Gamma| = d$ with $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$. Let $X = \mathrm{Cay}(G, \Gamma)$ and $k = \mathrm{Diam}(X)$. We will rule out the condition that X is disconnected, since for a disconnected graph $k = \infty$. As $G = \langle \Gamma \rangle$ every element of G can be expressed as a word of length $\leq k$. Since G is abelian, we can write any $g \in G$ as

$$g = e^{a_0} \gamma_1^{a_1} \gamma_2^{a_2} \ldots \gamma_d^{a_d}.$$

Then

$$\sum_{i=0}^{d} a_i = k \text{ with } a_i \in \mathbb{Z}^+.$$

The number of distinct elements in G is bounded above by $\binom{k+d}{k}$, so that

$$|X| \leq \binom{k+d}{k} = \binom{k+d}{d} \leq (k+1)^d.$$

Hence,

$$\text{Diam}(X) \geq |X|^{\frac{1}{d}} - 1$$

So for any sequence (X_n) of d-regular Cayley graphs on abelian groups

$$\text{Diam}(X_n) \geq |X_n|^{\frac{1}{d}} - 1.$$

As root function grows faster than the logarithmic function,

$$\text{Diam}(X_n) \neq O(\log |X_n|).$$

So we can conclude that abelian groups do not yield expander families of graphs. □

4.7 Alon-Boppana Theorem

In this final section, we will discuss the Alon-Boppana theorem, and why Ramanujan graphs are natural expanders.

Definition 4.31. Let $X = (V(X), E(X))$ be a d-regular graph with $|X| = n$. Then if X is not bipartite then $\lambda_0(X) = d$ is called the *trivial eigenvalue* of X.If X is bipartite, then $\lambda_0(X) = d$ and $\lambda_{n-1}(X) = -d$ are called the *trivial eigenvalues* of X. By definition,

$$\lambda(X) = \begin{cases} \max\Big\{|\lambda_1(X)|, |\lambda_{n-1}(X)|\Big\}, & \text{if } X \text{ is bipartite} \\ \max\Big\{|\lambda_1(X)|, |\lambda_{n-2}(X)|\Big\}, & \text{otherwise.} \end{cases}$$

For proving the main theorem, we need the following proposition.

Proposition 4.32. Let (X_n) be a sequence of connected $d-$regular graphs with $|X_n| \to \infty$ as $n \to \infty$. Then

$$\liminf_{n\to\infty} \lambda_1(X_n) \geq 2\sqrt{d-1}.$$

Proof. Let X be a connected d-regular graph. If the $\text{Diam}(X) \geq 4$ then we claim that

$$\lambda_1(X) > 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2}\text{Diam}(X) - 1 \rfloor}$$

Before proving this claim, we will show that this implies our proposition. Since our graph is d-regular, the number of walks of length 1 starting at v is d. The number of walks of length 2 starting at v is d^2. Similarly number of walks starting at v of length a is d^a. Note that a walk of length a contains at most $a+1$ vertices. Hence,

$$|X| \leq (\text{Diam}(|X|) + 1)d^{\text{Diam}(X)}.$$

Now consider the following sequence of d-regular graphs (X_n),such that $|X_n| \to \infty$ as $n \to \infty$. By above inequality as $n \to \infty$ then $\text{Diam}(X_n) \to \infty$ Hence

$$\frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2}\text{Diam} - 1 \rfloor} \to 0, \text{ as } n \to \infty$$

. This show that our claim implies the proposition, we will now prove our claim. Let $b = \lfloor \frac{1}{2}\text{Diam}(X) - 1 \rfloor$ and $q = d - 1$..Consider $f \in L_0^2(V, (R))$. Let $v_1, v_2 \in V(X)$ such that $d(v_1, v_2) \geq 2b + 2$. Define,

$$A_i = \{v \in V(X) | \text{dist}(v, v_1) = i\} \text{ and } B_i = \{v \in V(X) | \text{dist}(v, v_2) = i\}.$$

For $i = 0, 1, 2, \ldots, b$. If $x \in A_i \cap B_i$ for some $0 \leq i, j \leq b$,then

$$\text{dist}(v_1, v_2) \leq \text{dist}(v_1, x) + \text{dist}(x, v_2) \leq 2b < 2b + 2.$$

As this cannot happen, $A_0, A_1, \ldots, A_b, B_0, B_1, \ldots, B_b$ are all disjoint sets. Let

$$A = \bigcup_{i=0}^{b} A_i \text{ and } B = \bigcup_{i=0}^{b} B_i$$

Suppose that $x \in A$ and $y \in B$ are adjacent (in other words $\{x, y\} \in E(X)$). Then,

$$\text{dist}(v_1, v_2) \leq \text{dist}(v_1, x) + \text{dist}(x, y) + \text{dist}(y, v_2) \leq 2b + 1 < 2b + 2$$

As this is impossible, $\nexists \{x, y\} \in E(X)$ such that $x \in A$ and $y \in B$
Define the function $f \in L_0^2(V, \mathbb{R})$ by,

$$f(x) = \begin{cases} \alpha, & \text{if } x \in A_0 \\ \alpha q^{-(i-1)/2}, & \text{if } x \in A_i, \text{ for } i \geq 1 \\ 1, & \text{if } x \in B_0 \\ q^{-(i-1)/2}, & \text{if } x \in B_i, \text{ for } i \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Here α is chosen in the next step. Let $f_0 : V \to \{1\}$.Then,

$$\begin{aligned} \langle f, f_0 \rangle_2 &= \alpha\Big(|A_0| + \textstyle\sum_{i=1}^{b} q^{-(i-1)/2}|A_i|\Big) + \Big(|B_0| + \textstyle\sum_{i=1}^{b} q^{-(i-1)/2}|B_i|\Big) \\ &= \alpha c_0 + c_1 \end{aligned}$$

For some real numbers $c_0, c_1 > 0$,let $\alpha = \frac{-c_1}{c_0}$, so that $\langle f, f_0 \rangle_2 = 0$. Now we will compute

$$\begin{aligned} \langle f, f \rangle_2 &= \textstyle\sum_{x \in V} f(x)\overline{f(x)} \\ &= \textstyle\sum_{1=0}^{b} \sum_{x \in A_i} |f(x)|^2 + \sum_{1=0}^{b} \sum_{x \in B_i} |f(x)|^2 \ , \\ &= S_A + S_B \end{aligned}$$

where,

$$S_A = \alpha^2 + \sum_{i=0}^{b} |A_i| \alpha^2 q^{-(i-1)/2}, \text{ and}$$

$$S_B = \alpha^2 + \sum_{i=0}^{b} |B_i|\alpha^2 q^{-(i-1)/2}.$$

Now we will find an upper bound for $\langle \Delta f, f\rangle_2$. For doing this we will orient the edges of the graph $X = (V(X), E(X))$ .By Proposition 4.18, we have

$$\langle \Delta f, f\rangle_2 = C_A + C_B,$$

where,

$$C_A = \sum_{e\in E, e^+ or e^- \in A} (f(e^+) - f(e^-))^2 \text{ and,}$$

$$C_B = \sum_{e\in E, e^+ or e^- \in B} (f(e^+) - f(e^-))^2.$$

We see that

$$C_A = \sum_{i=0}^{b-1} \sum_{x\in A_i} \sum_{y\in A_{i+1}} a_{xy}(f(x) - f(y))^2 + \sum_{x\in A_b} \sum_{y\notin A} a_{xy}(f(x) - 0)^2,$$

where $A(X) = (a_{xy})$ is the adjacency matrix.For each $x \in A_i$, there are at most q elements y in A_{i+1} that are adjacent to x. Thus,

$$C_A \leq \sum_{i=1}^{b-1} q|A_i|(q^{-(i-1)/2 - q^{-i/2}})^2\alpha^2 + q|A_b|q^{-(b-1)}\alpha^2.$$

Note that,

$$(q^{-(i-1)/2 - q^{-i/2}})^2 = (q^{1/2} - 1)^2 q^{-i} \text{ and } q = (q^{1/2} - 1)^2 + 2q^{1/2} - 1,$$

where the expression for q is obtained by putting $i = 0$. Thus, we have

$$\begin{aligned} C_A &\leq \alpha^2 \sum_{i=0}^{b-1} q|A_i|(q^{1/2} - 1)^2 q^{-i} + \alpha^2\Big((q^{1/2} - 1)^2 + 2q^{1/2} - 1\Big)|A_b|q^{-(b-1)} \\ &= \alpha^2(q^{1/2} - 1)^2\Big(\sum_{i=1}^{b} |A_i|q^{-(i-1)}\Big) + \alpha^2(2q^{1/2} - 1)|A_b|q^{-(b-1)}. \end{aligned}$$

As,

$$S_A - \alpha^2 = \alpha^2 \sum_{i=1}^{b} |A_i| q^{-(i-1)},$$

we have

$$C_A \leq (q^{1/2} - 1)^2 (S_A - \alpha^2) + \alpha^2 (\frac{2\sqrt{q} - 1}{b}) b |A_b| q^{-(b-1)}.$$

If $x \in A_i$ where $1 \leq i \leq b-1$, then by asimilar argument we had used previously the claim follows. (ie there is at least one vertex from A_{i-1} that is adjacent to x,and at most q vertices from A_{i+1} that are adjacent to x). Hence $|A_{i+1}| \leq q|A_i|$, for $1 \leq i \leq b-1$, and $|A_{i+1}| \leq q|A_i|$ with $1 \leq i \leq b-1$. So,

$$|A_1| \geq q^{-1}|A_2| \geq q^{-2}|A_3| \geq \cdots \geq q^{-(b-2)}|A_{b-1}| \geq q^{-(b-1)}|A_b|.$$

So that,

$$\alpha^2 b |A_b| q^{-(b-1)} = \alpha^2 \sum_{i=1}^{b} |A_b| q^{-(b-1)} \leq \alpha^2 \sum_{i=1}^{b} |A_i| q^{-(i-1)} = S_A - \alpha^2.$$

But X is connected and $\text{Diam}(X) \geq 4$ which implies that $d \geq 2$, so that $\frac{(2\sqrt{q}-1)}{b} > 0$ and $0 < (q^{1/2} - 1)^2 = q + 1 - 2\sqrt{q}$. Thus,

$$C_A \leq (q^{1/2} - 1)^2 (S_A - \alpha^2) + \frac{2\sqrt{q} - 1}{b} (S_A - \alpha^2)$$

$$= \left(q + 1 - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}\right)(S_A - \alpha^2)$$

$$< \left(q + 1 - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}\right) S_A.$$

Similarly,

$$C_B < \left(q + 1 - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}\right) S_B.$$

Hence ,

$$\langle \Delta f, f\rangle_2 = C_A + C_B < \Big(q + 1 - 2\sqrt{q} + \frac{2\sqrt{q}-1}{b}\Big)(S_A + S_B).$$

So by using Rayleigh-Ritz theorem, we have

$$\begin{aligned} d - \lambda_1(X) &= \min_{g\in L_0^2(X), \|g\|_2=1} \langle \Delta g, g\rangle_2 \\ &\le \frac{\langle \Delta f, f\rangle_2}{\langle f, f\rangle_2} \\ &= \frac{C_A + C_B}{S_A + S_B} \\ &< q + 1 - 2\sqrt{q} + \frac{2\sqrt{q}-1}{b} \\ &= d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{b} \\ \text{Hence, } \lambda_1(X) &> 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{b}, \end{aligned}$$

and the result follows. □

We are now in a position to present the proof of Alon Boppana theorem.

Theorem 4.33 (Alon-Boppana Theorem)**.** *Let* (X_n) *be a sequence of connected* d*-regular graphs with* $|X_n| \to \infty$ *as* $n \to \infty$*. Then*

$$\liminf_{n\to\infty} \lambda(X) \ge 2\sqrt{d-1}.$$

Proof. By definition,

$$\lambda(X) \ge \lambda_1(X), \text{ so that}$$

$$\liminf_{n\to\infty} \lambda(X) \ge \liminf_{n\to\infty} \lambda_1(X) \ge 2\sqrt{d-1},$$

and the result follows. □

The following corollary tells why Ramanujan graphs are natural for expanders.

Corollary 4.34. Any sequence of d-regular Ramanujan graphs for $d \ge 3$ is an expander family.

Proof. By Proposition 4.18, we know that,

$$\frac{d - \lambda_1(X)}{2} \leq h(X) \leq \sqrt{2d(d - \lambda_1(X))}.$$

Since (X_n) is Ramanujan, we have that

$$\lambda(X) \leq 2\sqrt{d-1}$$

As $\lambda_1(X) \leq \lambda(X)$, we have

$$h(X) \geq \frac{d - \lambda_1(X)}{2} \geq \frac{d - 2\sqrt{d-1}}{2} > 0.$$

Since $d \geq 3$, so for each n $h(X_n)$ is bounded away from 0 thereby showing that (X_n) is expander. □

Appendices

I Basic Definitions

Definition I.1. A **multiset** is a set A together with a map $m : A\mathbb{N}$ (here $\mathbb{N}$ does not include 0).For $a \in N$ the natural number $m(a)$ is called the *mutliplicity* of a.

Definition I.2. Consider (a_n) sequence of non-zero real numbers we say (a_n) is bounded away from 0 if $\exists$ a real number $\epsilon > 0$ such that $a_n \geq \epsilon, \forall n$.

Definition I.3 (Equivalence Relation)**.** A given binary relation $\sim$ on a set X is said to be an equivalence relation if and only if it is reflexive, symmetric and transitive. That is, for all a, b and $c \in X$

- $a \sim a$. (Reflexivity)
- $a \sim b$ if and only if $b \sim a$. (Symmetry)
- If $a \sim b$ and $b \sim c$ then $a \sim c$. (Transitivity)

II Additional Theorems

Theorem II.1 (Spectral theorem for symmetric real matrices)**.** *Suppose that A is an $n \times n$ symmetric matrix with real entries. Then,*

(i) All the eigenvalues of A are real.

(ii) There is an orthonormal basis of $\mathbb{C}^n$ consisting of n real eigenvectors of A.

(iii) If λ is an eigenvalue of A, the algebraic multiplicity of λ equals the geometric multiplicity of λ.

Proof. See[[4],pp.239] □

Theorem II.2. *The number of solutions to the equation $a_1+a_2+\cdots+a_n = k$, where the a_i are non-negative integers, is $\binom{n+k-1}{k}$*

Proof. Combinatorially we can give a proof to this theorem. Consider k dots and $n-1$ vertical bars separating them.Then number of ways of choosing k dots from $k+n-1$ is $\binom{n+k-1}{k}$. □

Theorem II.3. *If $a, b \in \mathbb{N}$ with $b \leq a$, then*

$$\binom{a}{b} \leq (a-b+1)^b$$

Proof. Observer that if $0 < q \leq p$, then $\frac{p+1}{q+1} \leq \frac{p}{q}$. Hence

$$\frac{a}{b} \leq \frac{a-1}{b-1} \leq \ldots \frac{a-b+2}{2} \leq \frac{a-b+1}{1}.$$

So

$$\binom{a}{b} = (\frac{a}{b})(\frac{a-1}{b-1}) \ldots (\frac{a-b+2}{2})(\frac{a-b+1}{1}) \leq (a-b+1)^b$$

□

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