Metacyclic Actions on Surfaces

A thesis submitted in partial fulfilment of the requirements

for the award of the degree of

DOCTOR OF PHILOSOPHY

by

APEKSHA SANGHI

1610402

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL

BHOPAL - 462066

April, 2022

Metacyclic Actions on Surfaces

A thesis submitted in partial fulfilment of the requirements

for the award of the degree of

DOCTOR OF PHILOSOPHY

by

APEKSHA SANGHI

1610402

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL

BHOPAL - 462066

April, 2022

CERTIFICATE

The undersigned have examined the Ph.D. thesis entitled:

Metacyclic Actions on Surfaces

presented by Miss Apeksha Sanghi, a candidate for the degree of Doctor of Philosophy in the department of Mathematics, and hereby certify that it is worthy of acceptance.

September, 2022
IISER, Bhopal

Prof. Parameswaran Sankaran
External Examiner

Dr. Kashyap Rajeevsarathy
Supervisor

Dr. Atreyee Bhattacharya
Member

Dr. Siddhartha Sarkar
Member

Dr. Varadharajan Srinivasan
Member

ACADEMIC INTEGRITY AND COPYRIGHT DISCLAIMER

I hereby declare that this thesis is my own work and, to the best of my knowledge, it contains no materials previously published or written by any other person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Bhopal or any other educational institution, except where due acknowledgement is made in the thesis.

I certify that all copyrighted material incorporated into this thesis is in compliance with the Indian Copyright (Amendment) Act, 2012 and that I have received written permission from the copyright owners for my use of their work, which is beyond the scope of the law. I agree to indemnify and save harmless IISER Bhopal from any and all claims that may be asserted or that may arise from any copyright violation.

April, 2022
IISER, Bhopal

Apeksha Sanghi

ACKNOWLEDGEMENTS

First and foremost, I want to express my gratitude to my supervisor, Dr. Kashyap Rajeevsarathy, for his consistent encouragement and belief in me. Without him, this thesis would not exist. Many discussions with him helped me learn about mapping class groups and several other topics. Just like any other Ph.D. student, I had numerous ups and downs. He aided me in getting through the most difficult part of my Ph.D.

I'd want to express my gratitude to my doctorate committee members for providing me with valuable advice on a variety of topics. It aided me in gaining new perspectives on the topic, improving my presentation abilities, and increasing my confidence. I'd also like to express my gratitude to the Department of Mathematics for providing me with a conducive atmosphere to do my research and for providing relevent courses to strengthen my basic mathematical knowledge.

I'd want to thank UGC and the institute for providing me with sufficient resources to pursue my research ambitions. In addition, the institute provided me with the necessary infrastructure and resources. I am also grateful for the institute's excellent sports facilities. Sports were an important part of my daily routine which helped me clear my mind and kept me going. Sidharth, Aniruddha, Bhavya, Bhati, Monika, Sannidhi, Swapnil, Pratap, Shivangi, and Kunal, as well as coach Hemant, all assisted me in improving my game.

I'd want to express my gratitude to all of my school and college teachers who encouraged me to pursue a career in mathematics research. They were always kind and encouraging of my desire to pursue more advanced mathematics.

I'd like to express my gratitude to my friends for being there for me throughout my life's ups and downs. I might not be able to name them all. First and foremost, I'd like to express my gratitude to Dr. Neeraj for his support during my Ph.D. journey. Many discussions with him led me to grasp the subject better and quicker. I'd want to express my gratitude to my friends Geetika, Azad, Surbhi, Garvita, Dr. Punit, Ashima, Rahul, Naina, Dr. Anshu, Dr. Ritu, Lokenath, Dr. Nupur, Sidharth, Aparna, Kamal, Kaustav, Dr. Abhijit, Pankaj, Renu, Naveen, Aditya, Riju, and Ujjal for their unwavering support.

I'd like to thank to some of my above friends for loving me the way I am. I'd also like to express my gratitude to Dr. Atul for constantly looking after me. I'd also like to thank my group members for having numerous discussions, talks, and meetings that proved beneficial to all of us. They were like family to me when I was on campus.

Finally, I'd like to thank my family for giving me time, love, and patience. My parents provide me with freedom and encouragement to follow my passion and dreams. They looked after me at all times and respected my choices. They are the constant pillars of support. I'd also like to thank to my dearest brother, Apoorav Sanghi. In my life, he is a constant source of motivation. I am at this position because of them.

DEDICATION

I want to dedicate this thesis to my parents and my brother.

ABSTRACT

Let $\mathrm{Mod}(S_g)$ be the mapping class group of the closed orientable surface S_g of genus $g \geq 2$. As the main result of this thesis, we derive necessary and sufficient conditions under which two torsion elements in $\mathrm{Mod}(S_g)$ will have conjugates that generate a finite metacyclic subgroup of $\mathrm{Mod}(S_g)$. This also yields a complete solution to the problem of liftability of periodic mapping classes under finite cyclic covers. As applications of the main result, we give a complete characterization of the finite dihedral and dicyclic subgroups of $\mathrm{Mod}(S_g)$, up to a certain equivalence that we will call weak conjugacy. Moreover, we provide a complete classification of the weak conjugacy classes of the non-abelian finite metacyclic subgroups of $\mathrm{Mod}(S_3)$ and $\mathrm{Mod}(S_5)$, and non-split metacyclic subgroups of $\mathrm{Mod}(S_{10})$ and $\mathrm{Mod}(S_{11})$. We also describe nontrivial geometric realizations of some of these actions.

Furthermore, we provide necessary and sufficient conditions under which a non-split metacyclic action on S_g lifts to a split metacyclic action under a regular cyclic cover. As another consequence of our main theorem, we show that any finite-order mapping class whose corresponding orbifold is a sphere has a conjugate that lifts under certain finite-sheeted regular cyclic covers of S_g. We further establish that $4g$ is a realizable upper bound on the order of a non-split metacyclic action on S_g and this bound is realized by the action of a dicyclic group. By applying this result, we show that every periodic mapping class in a non-split metacyclic subgroup of $\mathrm{Mod}(S_g)$ is reducible. Finally, for $g \geq 5$, we show the existence of an infinite dihedral subgroup of $\mathrm{Mod}(S_g)$ that is generated by an involution and a root of a bounding pair map of degree 3.

Keywords: surface; mapping class; finite order maps; metacyclic subgroups
Subjclass[2020]: Primary 57K20; Secondary 57M60

TABLE OF CONTENTS

List of Symbols

$S_{g,d}^b$	Closed orientable surface of genus g with b boundary components and d punctures
$\mathrm{Homeo}^+(S_{g,d}^b)$	Group of orientation-preserving self homeomorphisms on $S_{g,d}^b$ that restrict to identity on $\partial(S_{g,d}^b)$
$\mathrm{Mod}(S_{g,d}^b)$	Mapping class group of surface $S_{g,d}^b$
$\pi_0(S)$	Path components of surface S
$A \rtimes B$	Semidirect product of group B acting on group A
$\mathcal{M}(u,n,r,k)$	Metacyclic group (Extension of $\mathbb{Z}_{\frac{un}{r}}$ by $\mathbb{Z}_n$)
$\mathcal{O}_H$	Quotient orbifold obtained by H action on surface S_g
$\pi_1(S)$	Fundamental group of surface S
$\pi_1^{orb}(\mathcal{O}_H)$	Orbifold fundamental group of $\mathcal{O}_H$
$\Gamma(\mathcal{O}_H)$	Signature of orbifold fundamental group of $\mathcal{O}_H$
ϕ_H	Surface kernel epimorphism $\Gamma(\mathcal{O}_H) \to H$
$C_H(\mathcal{G})$	Centralizer of $\mathcal{G}$ in group H
$\mathbb{F}_{\mathcal{G}}(u'.m)$	Fixed points of $\mathcal{G}$ with induced rotation angle $\frac{2\pi u'}{m}$
$\mathrm{Teich}(S_g)$	Teichm̈uller space of surface S_g
$\mathrm{Fix}(H)$	Subspace of fixed points in the $\mathrm{Teich}(S_g)$ under the action of H
$K \lhd H$	K is a normal subgroup of H
$\mathbb{Z}_n^\times$	Group of units in $\mathbb{Z}_n$

$\mathrm{LMod}_p(S_{h,b})$	Liftable mapping class group of the cover $p : S_g \to S_{h,b}$
$\mathrm{SMod}_p(S_g)$	Symmetric mapping class group of the cover $p : S_g \to S_{h,b}$
Dic_n	Dicyclic group of order $4n$
D_{2n}	Dihedral group of order $2n$
Q_{2^n}	Generalized quaternion group of order 2^n
T_c	Left-handed Dehn twist about a simple closed curve c
$\mathbb{R}^n$	Real Euclidean space of dimension n
$Z(H)$	Center of group H
$\mathbb{Z}_m *_{\mathbb{Z}_k} \mathbb{Z}_n$	Amalgamated free product of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ over $\mathbb{Z}_k$
$[H, H]$	Commutator subgroup of the group H

List of Figures

List of Tables

CHAPTER 1

INTRODUCTION

Let $S := S^b_{g,d}$ be the compact orientable surface of genus $g \geq 0$ with $b \geq 0$ boundary components and $d \geq 0$ punctures (or marked points). Let $\text{Homeo}^+(S)$ be the group of orientation-preserving self homeomorphisms of S that restrict to identity on ∂S and preserve the set of punctures. The *mapping class group* $\text{Mod}(S)$ is defined to be the path components of $\text{Homeo}^+(S)$, that is,

$$\text{Mod}(S) = \pi_0(\text{Homeo}^+(S)).$$

As we will only be concerned of surfaces S with $\partial S = \emptyset$ (i.e. $b = 0$), we shall refrain from using the parameter b while referring to surfaces. Furthermore, when $d = 0$, we simply denote $S_{g,0}$ by S_g.

A simple closed curve in $S_{g,d}$ is called *essential* if it is not homotopic to a point or to a puncture. A *multicurve in* $S_{g,d}$ is the union of a finite collection of disjoint non-isotopic essential simple closed curves in $S_{g,d}$.

Definition 1.0.1. A mapping class $F \in \text{Mod}(S_{g,d})$ is said to be:

(a) *periodic*, if F is of finite order,

(b) *reducible*, if F is represented by an $\mathcal{F} \in \text{Homeo}^+(S_{g,d})$ such that $\mathcal{F}$ preserves a multicurve on $\text{Homeo}^+(S_{g,d})$, and

(c) *pseudo-Anosov*, if F is neither periodic nor reducible.

The Nielsen-Thurston classification theorem [47] asserts that these are the only possible kinds of mapping classes.

A mapping class that is not reducible is called an *irreducible* mapping class. From the Nielsen-Thurston classification, it is apparent that any irreducible mapping class is either periodic or pseudo-Anosov. We will be primarily dealing with periodic mapping classes in this thesis.

1.1 Background

The study of automorphisms on Riemann surfaces [29] has a rich history. In 1893, Hurwitz [24] showed that the order of the group of automorphisms of a compact Riemann surface S_g is finite and bounded above by $84(g-1)$. In 1895, Wiman [49] proved that the order of the cyclic group acting on S_g is bounded above by $4g+2$. Harvey gave a complete characterization of cyclic actions on S_g in 1966, and in the same work, he also recovered Wiman's result. Around the same time, Maclachlan [30] applied the combinatorial techniques developed by Hurwitz and Wiman to show that the order of a finite abelian group acting on S_g is bounded above by $4g+4$. Recently, a complete classification of finite group actions on S_g for $2 \leq g \leq 4$ has been given in [7, 9, 27]. Furthermore, the dihedral actions on S_g have been classified in [11]. A method for enumerating the conjugacy classes of abelian subgroups of $\text{Mod}(S_g)$ has been developed by Broughton-Wootton in [10]. Moreover, it has been shown in [44] that for $g \geq 3$, the bound on the order of an arbitrary metacyclic subgroup of $\text{Mod}(S_g)$ is $12(g-1)$. In fact, this bound is realized for infinitely many values of g by split metacyclic subgroups of $\text{Mod}(S_g)$. However, this bound does not apply for $\text{Mod}(S_2)$ as it contains a unique split metacyclic subgroup of order 16 admitting the presentation $\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^8 = \mathcal{G}^2 = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^3 \rangle$.

We will now briefly explore the developments in a parallel geometric viewpoint of group actions on surfaces. Given an arbitrary finite subgroup $H \leq \text{Mod}(S_g)$, let $\text{Fix}(H)$ denote the set of fixed points induced by the natural action of H on the Teichmüller space $\text{Teich}(S_g)$. In 1932, Nielsen [35] asked whether $\text{Fix}(H) \neq \emptyset$. This is popularly known as *Nielsen realization problem*. Nielsen [37] showed this assertion to be true for cyclic groups, which was later generalized to solvable groups by Fenchel [16, 17] in 1948. In 1959, Kravetz [28] claimed to have solved the Nielsen realization problem, but his proof was subsequently shown [33, 50] to be incorrect. In 1971, Harvey [21] showed that if $\text{Fix}(H) \neq \emptyset$, then $\text{Fix}(H) \approx \hat{\imath}(\text{Teich}(S_g/H))$, where $\text{Teich}(S_g/H)$ is defined in the sense of Bers [1], and $\hat{\imath}$ is the natural embedding induced by the branched cover $S_g \to S_g/H$.

Finally, in 1983, Kerckhoff [26] answered the Nielsen realization problem in the affirmative.

Theorem 1.1.1 (Nielsen-Kerckhoff theorem)**.** *For $g \geq 2$, suppose $H < \mathrm{Mod}(S_g)$ is a finite group. Then there exists a finite group $\tilde{H} < \mathrm{Homeo}^+(S_g)$ so that the natural projection $\mathrm{Homeo}^+(S_g) \to \mathrm{Mod}(S_g)$ restricts to an isomorphism $\tilde{H} \to H$. Further, $\tilde{H}$ can be chosen to be a subgroup of isometries of some hyperbolic metric on S_g.*

Hence, any finite subgroup $H < \mathrm{Mod}(S_g)$ can be viewed as an H-action on S_g. More recently, Parsad-Rajeevsarathy-Sanki [38] described a procedure to construct the hyperbolic structures on S_g that realize a given finite cyclic subgroup of $\mathrm{Mod}(S_g)$ as a group of isometries.

By applying ideas from the theory of group actions on surfaces [25, 29] and Thurston's orbifold theory [46, Chapter 13], Dhanwani-Rajeevsarathy [12] derived equivalent number-theoretic conditions under which a pair of periodic mapping classes will have conjugates that generate a (two-generator) abelian subgroup of $\mathrm{Mod}(S_g)$. Moreover, by applying the theory developed in [38] they also provided an algorithm to obtain the hyperbolic structures on S_g that realize two-generator abelian groups as groups of isometries. In this thesis, we generalize the above work to finite metacyclic subgroups of $\mathrm{Mod}(S_g)$.

1.2 Motivation

Given periodic elements $F, G \in \mathrm{Mod}(S_g)$ such that $\langle F, G \rangle$ is finite, an arbitrary pair of conjugates F', G' (of F, G resp.) may (or may not) generate a subgroup isomorphic to $\langle F, G \rangle$. Even if they generate isomorphic groups, they might not yield analogous actions on the surface. For example, consider the periodic mapping classes $F_1, G_1 \in \mathrm{Mod}(S_7)$ represented by the homeomorphisms $\mathcal{F}_1, \mathcal{G}_1$ as shown in first subfigure of Figure 1.1 below. For $i = 2, 3$, consider the conjugates F_i of F_1 and G_i of G_1 represented by the homeomorphisms $\mathcal{F}_i, \mathcal{G}_i \in \mathrm{Homeo}^+(S_7)$ illustrated in the second and third subfigures of Figure 1.1.

In the first subfigure, the surface S_7 is imbedded in $\mathbb{R}^3$ such that when we consider the aerial view the eight tubes connecting the spheres, they appear aligned along the vertices of a regular octagon as shown in the first subfigure of Figure 1.2 below. The homeomorphisms $\mathcal{F}_1, \mathcal{G}_1$ are the restriction of rotations on $\mathbb{R}^3$ to the (imbedded) surface S_7. Thus, it is apparent that $\langle F_1, G_1 \rangle \cong D_8$ (i.e. the dihedral group of order 8).

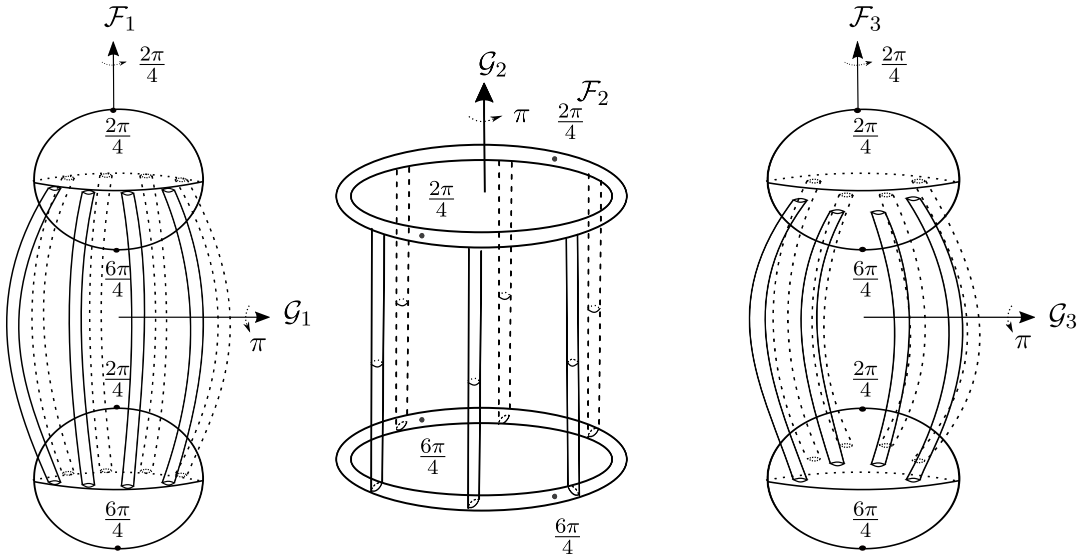

Figure 1.1: Split metacyclic actions on S_7 with conjugate generators.

In the second subfigure (of Figure 1.1), we have marked the fixed points of the conjugate $\mathcal{F}_2$ of $\mathcal{F}_1$ (with the same local rotation angles as that of $\mathcal{F}_1$). Here, note that $\mathcal{F}_2$ is not a restriction of a rotation on $\mathbb{R}^3$. Since the $\mathcal{F}_2$ and $\mathcal{G}_2$ commute, we have $\langle F_2, G_2 \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Therefore, distinct conjugates of two periodic maps can generate non-isomorphic subgroups.

In the third subfigure, S_7 is imbedded in $\mathbb{R}^3$ such that from a bird's-eye view of the eight tubes connecting the spheres, they appear aligned along the vertices of two concentric squares as shown in second subfigure of Figure 1.2. The homeomorphisms $\mathcal{F}_3, \mathcal{G}_3$ (shown in Figure 1.1) are the restrictions of rotations of $\mathbb{R}^3$ to the surface S_7. Hence, it is apparent that $\langle F_3, G_3 \rangle \cong D_8$. But the actions of $\langle \mathcal{F}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{G}_2 \rangle$ on S_7 are indeed different since $\mathcal{F}_1\mathcal{G}_1$ is not conjugate to $\mathcal{F}_3\mathcal{G}_3$. Thus, even when distinct conjugates of two periodic maps generate isomorphic subgroups, the actions of these subgroups can be different on the surface. We will revisit this example in more detail in Examples 4.2.1 and 4.2.9 of Chapter 4.

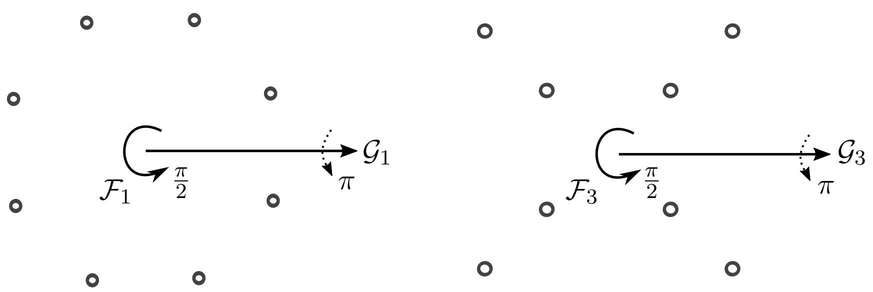

Figure 1.2: Tubes attached viewed from top view.

This example motivates the following natural question:

Question. Given a two generator finite group H, can one derive equivalent conditions under which two periodic elements $F, G \in \text{Mod}(S_g)$ will have conjugates F', G' (resp.) such that $\langle F', G' \rangle \cong H$?

The main theorem of the thesis answers this question in the affirmative when H is metacyclic by providing equivalent elementary number-theoretic conditions for the same. From Birman-Hilden theory [3, 4, 5, 6], it follows that every finite metacyclic subgroup of $\text{Mod}(S_g)$ is realized as the lift of a cyclic subgroup of $\text{Mod}(S_{h,b})$ under a finite cyclic cover $p : S_g \to S_{h,d}$. Thus, in order to prove our main result, we derive equivalent conditions under which a periodic element of $\text{Mod}(S_{h,b})$ would lift under p. Consequently, our theory also provides a complete solution to the problem of liftability of periodic mapping classes (up to conjugacy) under finite cyclic covers. This viewpoint also allows us to give non-trivial geometric realizations of some metacyclic actions on S_g.

1.3 Layout of Thesis

In Chapter 2, we introduce some basic concepts that lie at the foundation of the theory developed in this thesis. In particular, in Section 2.1, we provide key notions from the theory of group actions on surfaces [25, 29, 8] and in Section 2.2, we describe the theory of cyclic actions on surfaces [36, 34]. In Section 2.3, we briefly explain the theory developed in [38].

In Chapter 3, we provide an elementary introduction to metacyclic groups and some of their key properties. In Sections 3.1 and 3.5, we describe presentations for finite and infinite metacyclic groups, respectively. In Section 3.3, we provide some examples of metacyclic groups and in Sections 3.2 and 3.4, we establish some basic properties of metacyclic groups that are relevant to the thesis.

In Chapter 4, we prove the main result of this thesis. For this, we first defined metacyclic actions on surfaces. In Sections 4.1 and 4.2, we use the theory of group actions on surfaces [25, 29] and Thurston's orbifold theory [46, Chapter 13] to develop the necessary combinatorial machinery required to prove our main result. In Section 4.3, we establish our main theorem, and in Section 4.4, we characterize dihedral and dicyclic subgroups of $\text{Mod}(S_g)$. In Section 4.5, we classify the metacyclic subgroups of $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$, and non-split metacyclic subgroups of $\text{Mod}(S_{10})$ and $\text{Mod}(S_{11})$, up to a certain equivalence that we call *weak conjugacy* that arises naturally in our setting. In Section 4.6, we give geometric realizations of some metacyclic actions.

In Chapter 5, we provide liftability under regular cyclic covers. In Section 5.1, we provide an equivalent formulation of our main theorem in the context of liftability, which we had explained in the preceding section. In Section 5.2, we provide equivalent conditions under which a non-split metacyclic action on S_g would lift to a split metacyclic action under a regular cyclic cover. In Section 5.3, we give many applications concerning the liftability of periodic mapping classes under regular finite cyclic covers.

In Chapter 6, we provide some applications to our main theorem. As our first application, in Section 6.1, we derive the bound on the order of non-split metacyclic subgroups of $\mathrm{Mod}(S_g)$ is $4g$ and is realized by dicyclic groups. As an application of this result, we show that every periodic element in a non-split metacyclic subgroup of $\mathrm{Mod}(S_g)$ is reducible. In Section 6.2, we show that no root of a Dehn twist [31, 34, 40] can generate an infinite split metacyclic subgroup of $\mathrm{Mod}(S_g)$ that is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_{2m}$. We also show that a pseudo-periodic and a periodic mapping class can form an infinite split metacyclic subgroup of $\mathrm{Mod}(S_g)$ that is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_{2m}$. This thesis is essentially an amalgamation of the work carried out in [13] and [42].

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce some basic notions from Thurston's orbifold theory [46, Chapter 13], and the theory of group actions on surfaces [25, 29] which are crucial for the theory we develop in this thesis.

2.1 Fuchsian groups

Let $\mathrm{Homeo}^+(S_g)$ denote the group of orientation-preserving homeomorphisms of S_g, and let $H < \mathrm{Homeo}^+(S_g)$ be a finite group. A faithful and properly discontinuous H-action on S_g induces a branched covering

$$S_g \to \mathcal{O}_H := S_g/H$$

with ℓ cone points $x_1, x_2, \dots, x_\ell$ on the quotient orbifold $\mathcal{O}_H \approx S_{g_0}$ (which we will call the *corresponding orbifold*) of orders $n_1, n_2, \dots, n_\ell$, respectively. Then the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{O}_H)$ of $\mathcal{O}_H$ has a presentation given by

$$\left\langle \alpha_1, \beta_1, \dots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \dots, \xi_\ell \,|\, \xi_1^{n_1}, \dots, \xi_\ell^{n_\ell}, \prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] \right\rangle. \tag{2.1.1}$$

In classical parlance, $\pi_1^{\mathrm{orb}}(\mathcal{O}_H)$ is also known as a *Fuchsian group* [25, 29] with signature

$$\Gamma(\mathcal{O}_H) := (g_0; n_1, \dots, n_\ell),$$

the α_i, β_i are called the *hyperbolic generators* of $\pi_1^{\mathrm{orb}}(\mathcal{O}_H)$, the ξ_i are called the *elliptic generators* of $\pi_1^{\mathrm{orb}}(\mathcal{O}_H)$, and the relation $\prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i]$ appearing in its presentation is called the *long relation*. From Thurston's orbifold theory [46, Chapter 13], we obtain a short exact sequence

$$1 \to \pi_1(S_g) \to \pi_1^{\text{orb}}(\mathcal{O}_H) \xrightarrow{\phi_H} H \to 1, \tag{*}$$

where ϕ_H is known as the *surface kernel* epimorphism. In this context, we will require the following result due to Harvey [20].

Lemma 2.1.2. *A finite group H acts faithfully on S_g with $\Gamma(\mathcal{O}_H) = (g_0; n_1, \ldots, n_\ell)$ if and only if it satisfies the following two conditions:*

(i) $\dfrac{2g-2}{|H|} = 2g_0 - 2 + \displaystyle\sum_{i=1}^{\ell}\left(1 - \frac{1}{n_i}\right)$*, and*

(ii) there exists a surjective homomorphism $\phi_H : \pi_1^{\text{orb}}(\mathcal{O}_H) \to H$ that preserves the orders of all torsion elements of $\pi_1^{\text{orb}}(\mathcal{O}_H)$.

2.2 Cyclic actions on surfaces

For $g \geq 1$, let $F \in \text{Mod}(S_g)$ be of order n. The Nielsen-Kerckhoff theorem [26, 37] asserts that F is represented by a *standard representative* $\mathcal{F} \in \text{Homeo}^+(S_g)$ of the same order. We refer to both $\mathcal{F}$ and the group it generates, interchangeably, as a *$\mathbb{Z}_n$-action on S_g*. Each cone point $x_i \in \mathcal{O}_{\langle \mathcal{F} \rangle}$ lifts to an orbit of size n/n_i on S_g, and the local rotation induced by $\mathcal{F}$ around the points in each orbit is given by $2\pi c_i^{-1}/n_i$, where $\gcd(c_i, n_i) = 1$ and $c_i c_i^{-1} \equiv 1 \pmod{n_i}$. Further, it is known (see [20] and the references therein) that the exact sequence in (*) takes the following form

$$1 \to \pi_1(S_g) \to \pi_1^{\text{orb}}(\mathcal{O}_{\langle \mathcal{F} \rangle}) \xrightarrow{\phi_{\langle \mathcal{F} \rangle}} \langle \mathcal{F} \rangle \to 1,$$

where $\phi_{\langle \mathcal{F} \rangle}(\xi_i) = \mathcal{F}^{(n/n_i)c_i}$, for $1 \leq i \leq \ell$. We now introduce a tuple of integers that encodes the conjugacy class of a $\mathbb{Z}_n$-action on S_g.

Definition 2.2.1. A *data set of degree n* is a tuple

$$D = (n, g_0, d; (c_1, n_1), \ldots, (c_\ell, n_\ell)),$$

where $n \geq 2$, $g_0 \geq 0$, and $0 \leq d \leq n-1$ are integers, and each $c_i \in \mathbb{Z}_{n_i}^\times$ such that:

(i) $d > 0$ if and only if $\ell = 0$ and $\gcd(d, n) = 1$, whenever $d > 0$,

(ii) each $n_i \mid n$,

(iii) $\text{lcm}(n_1, \ldots, \widehat{n_i}, \ldots, n_\ell) = N$, for $1 \leq i \leq \ell$, where $N = n$ if $g_0 = 0$, and

(iv) $\displaystyle\sum_{j=1}^{\ell} \frac{n}{n_j} c_j \equiv 0 \pmod{n}$.

The number g determined by the Riemann-Hurwitz equation

$$\frac{2-2g}{n} = 2 - 2g_0 + \sum_{j=1}^{\ell} \left(\frac{1}{n_j} - 1 \right) \tag{2.2.2}$$

is called the *genus* of the data set, denoted by $g(D)$.

Note that quantity d (in Definition 2.2.1) will be non-zero if and only if D represents a free rotation of S_g by $2\pi d/n$, in which case, D will take the form $(n, g_0, d;)$. We will not include d in the notation of a data set, whenever $d = 0$.

By the Nielsen-Kerckhoff theorem, the canonical projection $\text{Homeo}^+(S_g) \to \text{Mod}(S_g)$ induces a bijective correspondence between the conjugacy classes of finite-order maps in $\text{Homeo}^+(S_g)$ and the conjugacy classes of finite-order mapping classes in $\text{Mod}(S_g)$. This leads us to the following lemma primarily due to Nielsen [36] (see also [43, Theorem 3.9]), which allows us to use data sets to describe the conjugacy classes of cyclic actions on S_g.

Lemma 2.2.3. *For $g \geq 1$ and $n \geq 2$, data sets of degree n and genus g correspond to conjugacy classes of $\mathbb{Z}_n$-actions on S_g.*

We will denote the data set corresponding to the conjugacy class of a periodic mapping class F by D_F or $D_{\langle F \rangle}$. For compactness of notation, we also write a data set D (as in Definition 2.2.1) as

$$D = (n, g_0, d; ((d_1, m_1), \alpha_1), \ldots, ((d_{\ell'}, m_{\ell'}), \alpha_{\ell'})),$$

where (d_i, m_i) are the distinct pairs in the multiset $S = \{(c_1, n_1), \ldots, (c_\ell, n_\ell)\}$, and the α_i denote the multiplicity of the pair (d_i, m_i) in the multiset S. Further, we note that every cone point $[x] \in \mathcal{O}_{\langle \mathcal{F} \rangle}$ corresponds to a unique pair in the multiset S appearing in D_F, which we denote by $\mathcal{P}_x := (c_x, n_x)$.

Given $u' \in \mathbb{Z}_m^\times$ and $\mathcal{G} \in H \leq \text{Homeo}^+(S_g)$ be of order m, let $\mathbb{F}_{\mathcal{G}}(u', m)$ denote the set of fixed points of $\mathcal{G}$ with induced rotation angle $2\pi u'/m$. Let $C_H(\mathcal{G})$ be the centralizer of $\mathcal{G} \in H$, and $\sim$ denote the conjugation relation between any two elements in H. We conclude this subsection by stating the following result from the theory of Riemann surfaces [8], which is crucial in proving our main theorem.

Lemma 2.2.4. *Let $H < \mathrm{Homeo}^+(S_g)$ of finite order with $\Gamma(\mathcal{O}_H) = (g_0; n_1, \dots, n_\ell)$, and let $\mathcal{G} \in H$ be of order m. Then for $u' \in \mathbb{Z}_m^\times$, we have*

$$|\mathbb{F}_{\mathcal{G}}(u', m)| = |C_H(\mathcal{G})| \cdot \sum_{\substack{1 \le i \le \ell \\ m | n_i \\ \mathcal{G} \sim \phi_H(\xi_i)^{n_i u'/m}}} \frac{1}{n_i}.$$

2.3 Hyperbolic structures realizing cyclic actions

Given a finite subgroup H of $\mathrm{Mod}(S_g)$, let $\mathrm{Fix}(H)$ denote the subspace of fixed points in the Teichm̈uller space $\mathrm{Teich}(S_g)$ under the action of H. When H is cyclic, a method for constructing the hyperbolic metrics representing the points in $\mathrm{Fix}(H)$ is described in [2] and [38], thereby yielding explicit solutions to the Nielsen realization problem [26, 37]. This method involves the construction of an arbitrary periodic element in $\mathrm{Mod}(S_g)$ (that is not realizable as a rotation of S_g) by the "compatibilities" of irreducible periodic components, which are uniquely realized as rotations of certain special hyperbolic polygons with side-pairings.

We recall that an $F \in \mathrm{Mod}(S_{g,d})$ is reducible if it is represented by an $\mathcal{F} \in \mathrm{Homeo}^+(S_{g,d})$ such that $\mathcal{F}$ preserves a multicurve on $S_{g,d}$, and a mapping class that is not reducible is called *irreducible*. Let $F \in \mathrm{Mod}(S_g)$ be of order n. Gilman [19] showed that F is irreducible if and only if $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle})$ has the form $(0; n_1, n_2, n_3)$ (i.e. the quotient orbifold $\mathcal{O}_{\langle \mathcal{F} \rangle}$ is a sphere with three cone points.) Following the nomenclature in [2, 38], F is *rotational* if $\mathcal{F}$ is either of order 2, or $\mathcal{F}$ has at most 2 fixed points. A non-rotational F is said to be of *Type 1* if $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}) = (g_0; n_1, n_2, n)$, otherwise, it is called a *Type 2* action. The following result describes a hyperbolic structure that realizes an irreducible Type 1 action.

Theorem 2.3.1 ([38, Theorem 2.7])**.** *For $g \geq 2$, consider an irreducible Type 1 action $F \in \mathrm{Mod}(S_g)$ with*

$$D_F = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n)).$$

Then F can be realized explicitly as the rotation $\theta_F = \dfrac{2\pi c_3^{-1}}{n}$ of a hyperbolic polygon $\mathcal{P}_F$

with a suitable side-pairing $W(\mathcal{P}_F)$*, where* $\mathcal{P}_F$ *is a hyperbolic* $k(F)$*-gon with*

$$k(F) := \begin{cases} 2n, & \text{if } n_1, n_2 \neq 2, \text{ and} \\ n, & \text{otherwise,} \end{cases}$$

and for $0 \leq m \leq n-1$*,*

$$W(\mathcal{P}_F) = \begin{cases} \prod_{i=1}^{n} a_{2i-1}a_{2i} \text{ with } a_{2m+1}^{-1} \sim a_{2z}, & \text{if } k(F) = 2n, \text{ and} \\ \prod_{i=1}^{n} a_i \text{ with } a_{m+1}^{-1} \sim a_z, & \text{otherwise,} \end{cases}$$

where $z \equiv m + qj \pmod{n}$ *with* $q = (n/n_2)c_3^{-1}$ *and* $j = n_2 - c_2$*.*

Moreover, it follows from the irreducibility of F (see also [2, Proposition 4.1]) that the structure in Theorem 2.3.1 is unique.

Further, it is shown [38] that the process of realizing an arbitrary non-rotational action F of order n using these unique hyperbolic structures realizing irreducible Type 1 components involves two broad types of processes.

(a) *k-compatibility.* In this process, for $i = 1, 2$, we take a pair of irreducible Type 1 mapping classes $F_i \in \text{Mod}(S_{g_i})$ such that the $\langle \mathcal{F}_i \rangle$-action on S_{g_i} induces a pair of *compatible orbits* of size k (where the induced local rotation angles add upto 0 modulo 2π). We remove (cyclically permuted) $\langle \mathcal{F}_i \rangle$-invariant disks around points in the compatible orbits and then identify the resulting boundary components realizing a periodic mapping class $F \in \text{Mod}(S_{g_1+g_2+k-1})$. An analogous construction can also be performed using a pair of orbits induced by a single $\langle \mathcal{F}' \rangle$-action on S_g to realize a periodic mapping class $F \in \text{Mod}(S_{g+k})$.

(b) *Permutation additions and deletions.* The *addition of a permutation component* involves the removal of (cyclically permuted) invariant disks around points in an orbit of size n induced by an $\langle \mathcal{F} \rangle$-action on S_g and then pasting n copies of $S_{g'}^1$ (i.e. $S_{g'}$ with one boundary component) to the resultant boundary components. This realizes an action on $S_{g+ng'}$ with the same fixed point and orbit data as F. The reversal of this process is called a *permutation deletion.*

Thus, in summary, we have the following:

Theorem 2.3.2. *[38, Theorem 2.24] For $g \geq 2$, a non-rotational periodic mapping class in $\mathrm{Mod}(S_g)$ can be realized through finitely many k-compatibilities, permutation additions, and permutation deletions on the unique structures of type $\mathcal{P}_F$ realizing irreducible Type 1 mapping classes.*

CHAPTER 3

METACYCLIC GROUPS

A *metacyclic group* H is a group extension of a cyclic group L by a cyclic group N. Equivalently, H is a metacyclic group if there is a short exact sequence

$$1 \to N \xrightarrow{i} H \xrightarrow{q} L \to 1, \tag{3.0.1}$$

where N and L are cyclic groups. Thus, H is metacyclic if and only if there is an $N \lhd H$ such that both N and $H/N(\cong L)$ are cyclic. If a metacyclic group H fits into a short exact sequence as in (3.0.1) that also splits, then we say that H is a *split metacyclic group* and in this case, $H \cong N \rtimes L$. Note that abelian groups are metacyclic if and only if they are generated by one or two elements. As these groups are not of interest to us, throughout this thesis we will use the term 'metacyclic group' to denote a non-abelian finite metacyclic group. When we have occasion to discuss non-finite metacyclic groups we will specifically identify them as 'infinite metacyclic groups'.

3.1 Metacyclic groups

Given integers $u, n \in \mathbb{N}$, $r \in \mathbb{N}$ such that $r \mid n$ and $k \in \mathbb{Z}_n^\times$ such that $k^u \equiv 1 \pmod{n}$ and $r(k-1) \equiv 0 \pmod{n}$, we define a group $\mathcal{M}(u, n, r, k)$ of order $u \cdot n$ that admits the following presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = 1, \mathcal{F}^r = \mathcal{G}^u, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle. \tag{3.1.1}$$

Here, note that the relation $\mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k$ implies that $\langle \mathcal{F} \rangle \lhd \mathcal{M}(u, n, r, k)$ and $\mathcal{M}(u, n, r, k)/\langle \mathcal{F} \rangle$ is a cyclic group. Hence, the group $\mathcal{M}(u, n, r, k)$ is a metacyclic group. We will call the multiplicative class k the *twist factor* and r the *amalgam* of the group $\mathcal{M}(u, n, r, k)$. Note that $m := |\mathcal{G}| = \frac{un}{r}$ in $\mathcal{M}(u, n, r, k)$. The following theorem provides a characterization of metacyclic groups.

Theorem 3.1.2. *Every metacyclic group H is isomorphic to $\mathcal{M}(u, n, r, k)$ for some integers u, n, r, k as described above. However, this representation is not unique.*

Proof. As H be a metacyclic group, there exists a short exact sequence

$$1 \to \langle y \rangle \xrightarrow{i} H \xrightarrow{q} H/\langle y \rangle \to 1,$$

such that $H/\langle y \rangle$ is cyclic. Since i is injective and $\ker(q) = \mathrm{im}(i)$, we have $\langle y \rangle \lhd H$. Now as $H/\langle y \rangle$ is cyclic, let $x \in H$ such that $\langle x\langle y \rangle \rangle = H/\langle y \rangle$. This implies that H is the union of the cosets $x^a\langle y \rangle$, for $0 \le a < |H/\langle y \rangle|$. Thus, every element of H is of the form $x^a y^b$ for some $a, b \in \mathbb{N}$, and so we have $H = \langle x, y \rangle$. Moreover, the relation $x^{-1}yx = y^k$ holds because $\langle y \rangle \lhd H$. Setting $|y| = n, |x| = m$, and $|H/\langle y \rangle| = u$, it can be seen that $H \cong \mathcal{M}(u, n, r, k)$, where $r := mu/n$.

The non-uniqueness of the representation for H can be seen from the following example. For an odd positive integer n, consider the metacyclic groups

$$H_1 = \langle \mathcal{F}_1, \mathcal{G}_1 \mid \mathcal{F}_1^n = \mathcal{G}_1^4 = 1, \mathcal{G}_1^{-1}\mathcal{F}_1\mathcal{G}_1 = \mathcal{F}_1^{-1} \rangle = \mathcal{M}(4, n, n, -1) \text{ and}$$

$$H_2 = \langle \mathcal{F}_2, \mathcal{G}_2 \mid \mathcal{F}_2^{2n} = 1, \mathcal{F}_2^n = \mathcal{G}_2^2, \mathcal{G}_2^{-1}\mathcal{F}_2\mathcal{G}_2 = \mathcal{F}_2^{-1} \rangle = \mathcal{M}(2, 2n, n, -1).$$

We claim that the map $\psi : H_1 \to H_2$ defined by $\psi(\mathcal{F}_1) = \mathcal{F}_2^2$, $\psi(\mathcal{G}_1) = \mathcal{G}_2$, and $\psi(1) = 1$ is an isomorphism. From the relations

$$\psi(\mathcal{F}_1^n) = 1 = \mathcal{F}_2^{2n} = (\psi(\mathcal{F}_1))^n,$$

$$\psi(\mathcal{G}_1^4) = 1 = \mathcal{G}_2^4 = (\psi(\mathcal{G}_1))^4, \text{ and}$$

$$\psi(\mathcal{G}_1^{-1}\mathcal{F}_1\mathcal{G}_1\mathcal{F}_1) = 1 = \mathcal{G}_2^{-1}\mathcal{F}_2^2\mathcal{G}_2\mathcal{F}_2^2 = \psi(\mathcal{G}_1^{-1})\psi(\mathcal{F}_1)\psi(\mathcal{G}_1)\psi(\mathcal{F}_1),$$

it follows that ψ is a homomorphism. To see that ψ is injective, for $\mathcal{G}_1^{i_1}\mathcal{F}_1^{j_1}, \mathcal{G}_1^{i_2}\mathcal{F}_1^{j_2} \in H_1$, assume that $\psi(\mathcal{G}_1^{i_1}\mathcal{F}_1^{j_1}) = \psi(\mathcal{G}_1^{i_2}\mathcal{F}_1^{j_2})$. Then it follows that $\mathcal{G}_2^{i_1-i_2}\mathcal{F}_2^{2(j_1-j_2)} = 1$, and so we have $i_1 - i_2 \equiv 2c \pmod 4$ and $2(j_1 - j_2) \equiv -cn \pmod{2n}$, for some $c \in \mathbb{N}$. Hence, we have $i_1 \equiv i_2 \pmod 4$ and $j_1 \equiv j_2 \pmod n$. Thus, $\mathcal{G}_1^{i_1}\mathcal{F}_1^{j_1} = \mathcal{G}_1^{i_2}\mathcal{F}_1^{j_2}$, from which the injectivity of ψ follows. The surjectivity of ψ is a consequence of the fact that $|H_1| = |H_2|$. Therefore, ψ is an isomorphism. □

Metacyclic groups have been completely classified by Hempel in [22, 23]. However, as this classification is highly technical, we will give a more elementary and accessible introduction to metacyclic groups in next sections, which will be sufficient for the theory we will develop in this thesis.

3.2 Basic Properties of metacyclic groups

In this section, we derive several basic properties of metacyclic groups that will be extensively used in this thesis. We begin by deriving some properties that follow from the presentations described in (3.1.1).

Lemma 3.2.1. *For a metacyclic group $H = \langle \mathcal{F}, \mathcal{G} \rangle$, where $\mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k$, we have $\mathcal{F}^b\mathcal{G}^a = \mathcal{G}^a\mathcal{F}^{bk^a}$ for $a, b \in \mathbb{N}$.*

Proof. As $\mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k$, it will imply that

$$\begin{aligned} \mathcal{G}^{-a}\mathcal{F}^b\mathcal{G}^a &= (\mathcal{G}^{-a}\mathcal{F}\mathcal{G}^a)^b \\ &= (\mathcal{G}^{-a+1}\mathcal{G}^{-1}\mathcal{F}\mathcal{G}\mathcal{G}^{a-1})^b \\ &= (\mathcal{G}^{-a+1}\mathcal{F}^k\mathcal{G}^{a-1})^b \\ &= (\mathcal{G}^{-a+1}\mathcal{F}\mathcal{G}^{a-1})^{bk} \\ &= \mathcal{F}^{bk^a} \qquad \text{(inductively)}. \end{aligned}$$

Hence, $\mathcal{F}^b\mathcal{G}^a = \mathcal{G}^a\mathcal{F}^{bk^a}$. □

Lemma 3.2.2. *For a metacyclic group $H = \langle \mathcal{F}, \mathcal{G} \rangle$, where $\mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k$, we have*

$$\mathcal{G}^{a_1}\mathcal{F}^{b_1}\mathcal{G}^{a_2}\mathcal{F}^{b_2}\cdots\mathcal{G}^{a_l}\mathcal{F}^{b_l} = \mathcal{G}^A\mathcal{F}^B,$$

where $A = \sum_{i=1}^{l} a_i$, $B = \sum_{i=1}^{l} b_i \prod_{j=i+1}^{l} k^{a_j}$, and $a_i, b_i \in \mathbb{N}$ for $1 \le i, j \le l$. In particular,

$$(\mathcal{G}^a\mathcal{F}^b)^l = \mathcal{G}^{al}\mathcal{F}^{\left(b\sum_{i=1}^{l} k^{a(l-i)}\right)}$$

for $a, b \in \mathbb{N}$.

Proof. By Lemma 3.2.1, we have that $\mathcal{F}^b\mathcal{G}^a = \mathcal{G}^a\mathcal{F}^{bk^a}$ for $a, b \in \mathbb{N}$. Hence, by repeated application of the $\mathcal{F}^b\mathcal{G}^a = \mathcal{G}^a\mathcal{F}^{bk^a}$, we see that:

$$\begin{aligned}\mathcal{G}^{a_1}\mathcal{F}^{b_1}\mathcal{G}^{a_2}\mathcal{F}^{b_2}\cdots\mathcal{G}^{a_l}\mathcal{F}^{b_l} &= \mathcal{G}^{a_1+a_2}\mathcal{F}^{b_1k^{a_2}+b_2}\mathcal{G}^{a_3}\mathcal{F}^{b_3}\cdots\mathcal{G}^{a_l}\mathcal{F}^{b_l}\\ &= \mathcal{G}^{a_1+a_2+a_3}\mathcal{F}^{b_1k^{a_2}k^{a_3}+b_2k^{a_3}+b_3}\mathcal{G}^{a_4}\mathcal{F}^{b_4}\cdots\mathcal{G}^{a_l}\mathcal{F}^{b_l}\\ &= \mathcal{G}^A\mathcal{F}^B,\end{aligned}$$

and our assertion follows. □

Lemma 3.2.3. *Given a metacyclic group* $H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = 1, \mathcal{F}^r = \mathcal{G}^u, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k\rangle$, *we have* $k^u \equiv 1 \pmod n$ *and* $r(k-1) \equiv 0 \pmod n$.

Proof. By substituting $a = u,\ b = 1$ in Lemma 3.2.1, we get

$$\mathcal{F}^{k^u} = \mathcal{G}^{-u}\mathcal{F}\mathcal{G}^u = \mathcal{F}\ (\text{as } \mathcal{F}^r = \mathcal{G}^u).$$

Therefore, we have $k^u \equiv 1 \pmod n$. Again, by plugging in $a = 1,\ b = r$ in Lemma 3.2.1, we see that

$$\mathcal{F}^{kr} = \mathcal{G}^{-1}\mathcal{F}^r\mathcal{G} = \mathcal{F}^r\ (\text{as } \mathcal{F}^r = \mathcal{G}^u),$$

from which it follows that $r(k-1) \equiv 0 \pmod n$. □

As an immediate consequence of the Lemma 3.2.2, we have the following lemma that describes the order of an arbitrary element in a metacyclic group H.

Lemma 3.2.4. *Given a metacyclic group* $H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = 1, \mathcal{F}^r = \mathcal{G}^u, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k\rangle$, *an element* $\mathcal{G}^a\mathcal{F}^b \in H$ *is of order* s *if and only if* s *is the least positive integer such that the following conditions hold for some* $t \in \mathbb{N}$*:*

(i) $as \equiv tu \pmod{\frac{un}{r}}$ *and*

(ii) $b\sum_{i=1}^{s} k^{a(s-i)} \equiv -tr \pmod n$.

3.3 Examples of metacyclic groups

Given integers $u, n \in \mathbb{N}$, a metacyclic group $\mathcal{M}(u, n, n, k)$ of order $u \cdot n$ admits the following presentation

$$H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = \mathcal{G}^u = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k\rangle,$$

where $k \in \mathbb{Z}_n^\times$ such that $k^u \equiv 1 \pmod{n}$. Note that $\mathcal{M}(u,n,n,k)$ is a split metacyclic group, and we will write $\mathcal{M}(u,n,n,k) \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_u$. We will now give some basic examples of split metacyclic groups.

Example 3.3.1. The abelian group $\mathbb{Z}_n \times \mathbb{Z}_m$ is isomorphic to the split metacyclic group $\mathcal{M}(m,n,n,1)$ that admits the presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = \mathcal{G}^m = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F} \rangle.$$

Example 3.3.2. The *dihedral group* D_{2n} of order $2n$ is isomorphic to the split metacyclic group $\mathcal{M}(2,n,n,-1)$ that admits the presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = \mathcal{G}^2 = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{-1} \rangle.$$

Example 3.3.3. The *dicyclic group* Dic_n of order $4n$ is isomorphic to the metacyclic group $\mathcal{M}(2,2n,n,-1)$ that admits a presentation given by

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^{2n} = 1, \mathcal{G}^2 = \mathcal{F}^n, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{-1} \rangle.$$

When n is odd, Dic_n is isomorphic to the split metacyclic group $\mathcal{M}(4,n,n,-1) \cong \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_4$ (as seen in Theorem 3.1.2).

Remark 3.3.4. From the presentation in (3.1.1), it is apparent that a finite metacyclic group is abelian if and only if $k = 1$. Note that any two-generator finite abelian group is a split metacyclic group.

A finite metacyclic group that is not split is called a *non-split metacyclic group.* Given a non-split metacyclic group $\mathcal{M}(u,n,r,k)$ with a presentation as in (3.1.1), it follows that $u, n \geq 2$, $r \geq 2$, $r \neq n$, and $k \neq 1$. However, the converse does not hold true in general as we saw in Example 3.3.3. We will now give two fundamental examples of finite non-split metacyclic groups.

Example 3.3.5. The *quaternion group* Q_8 of order 8 is isomorphic to the non-split metacyclic group $\mathcal{M}(2,4,2,-1)$ that admits the presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^4 = 1, \mathcal{G}^2 = \mathcal{F}^2, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{-1} \rangle.$$

The group Q_8 generalizes to a group of order 2^{n+1} for $n \geq 2$ known as a *generalized quaternion group*, which is isomorphic to the non-split metacyclic group $\mathcal{M}(2, 2^n, 2^{n-1}, -1)$ with the presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^{2^n} = 1, \mathcal{G}^2 = \mathcal{F}^{2^{n-1}}, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{-1} \rangle.$$

The generalized quaternions belong to the much broader family $\{\mathrm{Dic}_n : n \text{ is even}\}$ of non-split dicyclic groups. When n is even, $\mathrm{Dic}_n \cong \mathcal{M}(2, 2n, n, -1)$ with the presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^{2n} = 1, \mathcal{G}^2 = \mathcal{F}^{n}, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{-1} \rangle.$$

For example, the group Dic_6 is a non-quaternionic and non-split metacyclic group of order 24.

Example 3.3.6. The group $\mathbb{Z}_3 \times Q_8$ of order 24 is isomorphic to the non-split metacyclic group $\mathcal{M}(2, 12, 6, 7)$ that admits the presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^{12} = 1, \mathcal{G}^2 = \mathcal{F}^{6}, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{7} \rangle.$$

The group $\mathbb{Z}_3 \times Q_8$ belongs to a broader family $\{\mathbb{Z}_p \times \mathrm{Dic}_n : n \text{ is even and } \gcd(p, n) = 1\}$ of non-split metacyclic groups. Note that $\mathbb{Z}_p \times \mathrm{Dic}_n \cong \mathcal{M}(2, 2np, np, k)$ with the presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^{2np} = 1, \mathcal{G}^2 = \mathcal{F}^{np}, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{k} \rangle,$$

where $k \in \mathbb{Z}_{2n}^{\times}$ such that $k - 1 \equiv 0 \pmod{p}$ and $k + 1 \equiv 0 \pmod{2n}$.

3.4 Subgroups and quotients of metacyclic groups

In this section, we derive some properties pertaining to certain subgroups and quotients of metacyclic groups.

Lemma 3.4.1. *The subgroups and quotients of metacyclic groups are metacyclic.*

Proof. Let H be a metacyclic group with $N \lhd H$ such that both N and H/N are cyclic. For a subgroup $K < H$, the Second Isomorphism Theorem implies that

$$KN/N \cong K/K \cap N.$$

Moreover, we have $K \cap N$ is cyclic as $K \cap N < N$, which is cyclic. Also, $K/K \cap N$ is cyclic as $KN/N < H/N$, which is cyclic. Thus, it follows that K is a metacyclic group.

Now we consider the quotient group H/K, where $K \lhd H$. We claim that $KN/K \lhd H/K$ such that both KN/K and $(H/K)/(KN/K)$ are cyclic. The group KN/K is cyclic as it is the image of the cyclic group N under the canonical projection $H \to H/K$. By the Third Isomorphism Theorem, we have that $KN/K \lhd H/K$ and

$$(H/K)/(KN/K) \cong H/KN \cong (H/N)/(NK/N).$$

Furthermore, $(H/K)/(KN/K)$ is cyclic as it is isomorphic to the quotient of the cyclic group H/N. Therefore, it follows that H/K is a metacyclic group. $\square$

Lemma 3.4.2. *The center $Z(H)$ of a finite metacyclic group $H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = 1, \mathcal{F}^r = \mathcal{G}^u, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle$ is given by*

$$Z(H) = \langle \mathcal{G}^\alpha, \mathcal{F}^\beta \rangle,$$

where $\alpha \in \mathbb{N}$ such that $|k| = \alpha$ and β is least positive integer satisfying $\beta(k-1) \equiv 0 \pmod{n}$.

Proof. First, we will show that $\mathcal{G}^\alpha, \mathcal{F}^\beta \in Z(H)$. It suffices to show that $\mathcal{G}^\alpha \mathcal{F} = \mathcal{F}\mathcal{G}^\alpha$ and $\mathcal{F}^\beta \mathcal{G} = \mathcal{G}\mathcal{F}^\beta$. Since $|k| = \alpha$, it follows from Lemma 3.2.1 that $\mathcal{F}\mathcal{G}^\alpha = \mathcal{G}^\alpha \mathcal{F}^{k^\alpha} = \mathcal{G}^\alpha \mathcal{F}$. By a similar argument, we get $\mathcal{F}^\beta \mathcal{G} = \mathcal{G}\mathcal{F}^\beta$. Hence, it follows that $\langle \mathcal{G}^\alpha, \mathcal{F}^\beta \rangle \subseteq Z(H)$.

Conversely, let $\mathcal{G}^a \mathcal{F}^b \in Z(H)$. By Lemma 3.2.1, we have that

$$\mathcal{G}^{a+1}\mathcal{F}^{bk} = \mathcal{G}^a \mathcal{F}^b \mathcal{G} = \mathcal{G}\mathcal{G}^a \mathcal{F}^b = \mathcal{G}^{a+1}\mathcal{F}^b \text{ and}$$
$$\mathcal{G}^a \mathcal{F}^{b+1} = \mathcal{G}^a \mathcal{F}^b \mathcal{F} = \mathcal{F}\mathcal{G}^a \mathcal{F}^b = \mathcal{G}^a \mathcal{F}^{k^a + b}.$$

This implies that $k^a \equiv 1 \pmod{n}$ and $b(k-1) \equiv 0 \pmod{n}$. Therefore, we have $\mathcal{G}^a \mathcal{F}^b \in \langle \mathcal{G}^\alpha, \mathcal{F}^\beta \rangle$, which completes our argument. $\square$

Lemma 3.4.3. *Every non-split metacyclic group is isomorphic to a quotient of a split metacyclic group.*

Proof. Let H be a non-split metacyclic group admitting the presentation

$$H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = 1, \mathcal{F}^r = \mathcal{G}^u, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle.$$

Then, we see that $H \cong \tilde{H}/N$, where

$$\tilde{H} = \langle \tilde{\mathcal{F}}, \tilde{\mathcal{G}} \mid \tilde{\mathcal{F}}^n = \tilde{\mathcal{G}}^{un/r} = 1, \tilde{\mathcal{G}}^{-1}\tilde{\mathcal{F}}\tilde{\mathcal{G}} = \tilde{\mathcal{F}}^k \rangle$$

and $N = \langle \tilde{\mathcal{G}}^u \tilde{\mathcal{F}}^{-r} \rangle$. The fact that $N \lhd \tilde{H}$ can be seen from Lemma 3.4.2 and Lemma 3.2.3. □

The following result describes the commutator subgroup of an arbitrary metacyclic group.

Lemma 3.4.4. *The commutator subgroup of an arbitrary metacyclic group $H = \langle \mathcal{F}, \mathcal{G} \rangle$ with $\langle \mathcal{F} \rangle \lhd H$ is cyclic and is generated by $\langle \mathcal{F}^{k-1} \rangle$, where k satisfies $\mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k$.*

Proof. To show that the commutator subgroup $[H, H] = \langle \mathcal{F}^{k-1} \rangle$, it suffices to show that $[x, y] \in \langle \mathcal{F}^{k-1} \rangle$ for any $x, y \in H$ and that there exists $x', y' \in H$ such that $[x', y'] = \mathcal{F}^{k-1}$. Let $x = \mathcal{G}^{i_1}\mathcal{F}^{j_1}, y = \mathcal{G}^{i_2}\mathcal{F}^{j_2} \in H$. By Lemma 3.2.2, we have

$$\begin{aligned} x^{-1}y^{-1}xy &= \mathcal{F}^{-j_1}\mathcal{G}^{-i_1}\mathcal{F}^{-j_2}\mathcal{G}^{-i_2}\mathcal{G}^{i_1}\mathcal{F}^{j_1}\mathcal{G}^{i_2}\mathcal{F}^{j_2} \\ &= \mathcal{F}^{(-j_1 - j_2 k^{i_1} + j_1 k^{i_2} + j_2)} \\ &= \mathcal{F}^{(-j_2(k^{i_1}-1) + j_1(k^{i_2}-1))} \in \langle \mathcal{F}^{k-1} \rangle. \end{aligned}$$

Since $[\mathcal{G}, \mathcal{F}^{-1}] = \mathcal{G}^{-1}\mathcal{F}\mathcal{G}\mathcal{F}^{-1} = \mathcal{F}^{k-1}$, we follows that $[H, H] = \langle \mathcal{F}^{k-1} \rangle$. □

We conclude this section with the following technical proposition which will be helpful in Chapter 6.

Proposition 3.4.5. *Let $H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{G}^u = \mathcal{F}^r, \mathcal{F}^n = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle$. Suppose that there exists $x, y \in H$ such that $H = \langle x, y \rangle$ and at least one of x or y is of prime order. Then H is a split metacyclic group.*

Proof. Let us assume without loss of generality that x is of prime order p. Since H is a metacyclic group, there is a short exact sequence

$$1 \to \langle \mathcal{F} \rangle \xrightarrow{i} H \xrightarrow{q} H/\langle \mathcal{F} \rangle \to 1,$$

such that $|H/\langle \mathcal{F} \rangle| = u$. As x is of prime order, $q(x)$ is either trivial or an element of order p. Therefore, either $x = \mathcal{F}^\alpha$, for some α, or $x = (\mathcal{G}')^\beta$, for some β, where $\mathcal{G}'$ is a premiage of $\bar{\mathcal{G}}$ under q with $\langle \bar{\mathcal{G}} \rangle = H/\langle \mathcal{F} \rangle$ and $H = \langle \mathcal{F}, \mathcal{G}' \rangle$.

When $x = \mathcal{F}^\alpha$, we claim that H must be a split metacyclic group. To see this, first note that $\mathcal{F}^\alpha$ is of prime order and $\langle \mathcal{F}^\alpha \rangle \lhd H$. Suppose that $(\mathcal{F}^\alpha)^a = y^b$ for some $a, b \in \mathbb{N}$. Then $p \mid a$ implies that H is a split metacyclic group. However, if $\gcd(a, p) = 1$, then $\langle \mathcal{F}^\alpha \rangle \subseteq \langle y \rangle$, and hence $H = \langle y \rangle$. This establishes our claim.

When $x = (\mathcal{G}')^\beta$, we first claim that $\gcd(p, \frac{|\mathcal{G}'|}{u}) = 1$. To show this, suppose we assume on the contrary that $\gcd(p, \frac{|\mathcal{G}'|}{u}) \neq 1$. Then $x = \mathcal{G}'^\beta = \mathcal{G}'^{\frac{|\mathcal{G}'|}{up} uc} = \mathcal{G}'^{duc}$, where $\gcd(c, p) = 1$, $\beta = |\mathcal{G}'|c/p$, and $d = \frac{|\mathcal{G}'|}{up} \in \mathbb{N}$. This would imply that $q(x)$ is trivial (as $q((\mathcal{G}')^u) = 1$), which is a contradiction. Thus, our claim follows.

It follows from the preceding claim that $H = \langle x, y \rangle = \langle \mathcal{G}'^\beta, y \rangle = \langle \mathcal{F}, (\mathcal{G}')^{p^\gamma}, (\mathcal{G}')^{\frac{|\mathcal{G}'|}{p^\gamma}} \rangle$, where $\gamma \in \mathbb{N}$ such that $p^\gamma \mid |\mathcal{G}'|$ but $p^{\gamma+1} \nmid |\mathcal{G}'|$. Thus, either $\langle y \rangle = \langle \mathcal{F}, (\mathcal{G}')^{p^\gamma}, (\mathcal{G}')^{\frac{|\mathcal{G}'|}{p^\gamma}} \rangle = H$ (when $\gamma \neq 1$) or $\langle \mathcal{F}, (\mathcal{G}')^{p^\gamma} \rangle \subseteq \langle y \rangle$ (when $\gamma = 1$). When $\gamma \neq 1$, H is cyclic, and hence, is a split metacyclic group. When $\gamma = 1$, since $\langle \mathcal{F}, (\mathcal{G}')^{p^\gamma} \rangle \subseteq \langle y \rangle$, $\langle \mathcal{F}, (\mathcal{G}')^{p^\gamma} \rangle$ is a split metacyclic group. By arguments similar to the ones used for the case when $x = \mathcal{F}^\alpha$, we see that $\langle \mathcal{F}, (\mathcal{G}')^{\frac{|\mathcal{G}'|}{p^\gamma}} \rangle$ is a split metacyclic group. Therefore, as both $\langle \mathcal{F}, (\mathcal{G}')^{p^\gamma} \rangle$ and $\langle \mathcal{F}, (\mathcal{G}')^{\frac{|\mathcal{G}'|}{p^\gamma}} \rangle$ are split metacyclic groups and $\gcd(|(\mathcal{G}')^{p^\gamma}|, |(\mathcal{G}')^{\frac{|\mathcal{G}'|}{p^\gamma}}|) = 1$, H is also a split metacyclic group. □

3.5 Infinite metacyclic group

Even though this thesis mostly concerns finite metacyclic groups, we have included this section for completion. However, the results in this section will be pertinent in Section 6.2, where we describe a family of infinite metacyclic subgroups of $\text{Mod}(S_g)$. An *infinite metacyclic group* is a metacyclic group of infinite order. It is known [22, Chapter 7] that such a group admits exactly one of the following presentations:

$$\begin{aligned} \langle \mathcal{F}, \mathcal{G} \mid \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle &\cong \mathbb{Z} \rtimes_k \mathbb{Z}, \text{ for } k = \pm 1, \\ \langle \mathcal{F}, \mathcal{G} \mid \mathcal{G}^{2m} = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle &\cong \mathbb{Z} \rtimes_k \mathbb{Z}_{2m}, \text{ for } k = -1, m \in \mathbb{N}, \text{ and} \\ \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle &\cong \mathbb{Z}_n \rtimes_k \mathbb{Z}, \text{ for } k \in \mathbb{Z}_n^\times, n \in \mathbb{N}. \end{aligned} \tag{3.5.1}$$

Thus, any infinite metacyclic group is a split metacyclic group. Note that Lemma 3.2.1 and 3.2.2 are also true for infinite metacyclic group admitting the presentations described in (3.5.1).

We conclude this chapter by giving two elementary examples of infinite metacyclic groups.

Example 3.5.2. The two generator infinite abelian groups $\mathbb{Z}_n \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are infinite metacylic groups admitting the presentations

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F} \rangle \text{ and } \langle \mathcal{F}, \mathcal{G} \mid \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F} \rangle,$$

respectively.

Example 3.5.3. The *infinite dihedral group* $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_2$ admits the presentation

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{G}^2 = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{-1} \rangle.$$

CHAPTER 4

METACYCLIC ACTIONS ON SURFACES

In this chapter, we will establish the main result of this thesis. Given integers $u, n \in \mathbb{N}$, $r \mid n$, $k \in \mathbb{Z}_n^\times$ such that $k^u \equiv 1 \pmod{n}$, and $r(k-1) \equiv \pmod{n}$, a *finite metacyclic action of order un* (written as $u \cdot n$) on S_g is a tuple $(H, (\mathcal{G}, \mathcal{F}))$, where $H < \text{Homeo}^+(S_g)$, and

$$H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^r = \mathcal{G}^u, \mathcal{F}^n = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle \cong \mathcal{M}(u, n, r, k).$$

The main focus of this thesis is to analyze the finite metacyclic actions on S_g. Since the two generator finite abelian group actions on surfaces have been analyzed in [12], we will focus our attention on the finite non-abelian metacyclic subgroups of $\text{Mod}(S_g)$. By Lemma 3.2.3, these are groups of the form $\mathcal{M}(u, n, r, k)$, where $u, n, r \geq 2$, $r \mid n$, and $k \in \mathbb{Z}_n^\times$ such that $k \neq 1$, $k^u \equiv 1 \pmod{n}$, and $r(k-1) \equiv 0 \pmod{n}$.

It is well known that $\text{Mod}(S_0)$ is trivial and $\text{Mod}(S_1) \cong \text{SL}(2, \mathbb{Z})$, which is isomorphic to the amalgamated product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ (see [45]). Therefore, $\text{Mod}(S_1)$ does not have any metacyclic subgroups as any finite subgroup would inject into either $\mathbb{Z}_4$ or $\mathbb{Z}_6$. Thus, we will assume from here on that $g \geq 2$.

4.1 Induced automorphisms

Since $\langle \mathcal{F} \rangle \lhd H$, $\mathcal{G}$ would induce a $\bar{\mathcal{G}} \in \text{Homeo}^+(\mathcal{O}_{\langle \mathcal{F} \rangle})$ (see [48]) that restricts to an order-preserving bijection on the set of cone points in $\mathcal{O}_{\langle \mathcal{F} \rangle}$. We will call $\bar{\mathcal{G}}$, *the induced automorphism on* $\mathcal{O}_{\langle \mathcal{F} \rangle}$ *by* $\mathcal{G}$, and we formalize this notion in the following definition.

Definition 4.1.1. Let $H < \text{Homeo}^+(S_g)$ be a finite cyclic group with $|H| = n$ with data set D_H. We say a $\bar{\mathcal{G}} \in \text{Homeo}^+(\mathcal{O}_H)$ is a *data set automorphism of* $\mathcal{O}_H$ *of multiplicity* k if there exists an integer $k \in \mathbb{Z}_n^\times$ such that for any $[x], [y] \in \mathcal{O}_H$ satisfying $\bar{\mathcal{G}}([x]) = [y]$, we have:

(i) $n_x = n_y$, and

(ii) $c_x = kc_y$.

We denote the group of data set automorphisms of $\mathcal{O}_H$ of multiplicity k by $\mathrm{Homeo}_k(D_H)$.

We note that the concept of a data set automorphism in Definition 4.1.1 is more general than the one that was used in the abelian case ([12]), which required a more rigid condition that $c_x = c_y$. The following lemma, which provides some basic properties of the induced map $\bar{\mathcal{G}}$, is a metacyclic analog of [12, Lemma 3.1].

Lemma 4.1.2. *Let $\mathcal{G}, \mathcal{F} \in \mathrm{Homeo}^+(S_g)$ be maps of orders m, n, respectively, such that $\mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k$, and let $H = \langle \mathcal{F} \rangle$. Then:*

(i) $\mathcal{G}$ induces a $\bar{\mathcal{G}} \in \mathrm{Homeo}_k(D_H)$ such that

$$\mathcal{O}_H/\langle \bar{\mathcal{G}} \rangle = S_g/\langle \mathcal{F}, \mathcal{G} \rangle,$$

(ii) $|\bar{\mathcal{G}}|$ divides $|\mathcal{G}|$, and

(iii) $|\bar{\mathcal{G}}| < m$ if and only if $\mathcal{F}^r = \mathcal{G}^u$, for some $0 < r < n$ and $0 < u < m$.

Proof. Given $[x] \in S_g/\langle \mathcal{F} \rangle$, we define $\bar{\mathcal{G}}([x]) := [\mathcal{G}(x)]$. Then the assertion (i) follows immediately. As $|\mathcal{G}| = m$, (ii) follows from the fact that $\bar{\mathcal{G}}^m([x]) = [\mathcal{G}^m(x)] = [x]$, for $[x] \in Sg/\langle \mathcal{F} \rangle$. To prove (iii), we first assume that $t := |\bar{\mathcal{G}}| < m$. Then

$$[\mathcal{G}^t(x)] = \bar{\mathcal{G}}^t([x]) = [x],$$

for all $[x] \in \mathcal{O}_H$. Thus, for each $[x] \in \mathcal{O}_H$, there exists $1 \leq l_x \leq n$ such that $\mathcal{G}^t\mathcal{F}^{l_x}(y) = y$, for all $y \in S_g$ in the preimage of $[x]$ under the branched cover $S_g \to \mathcal{O}_H$. Suppose we assume on the contrary that $\mathcal{F}^r \neq \mathcal{G}^u$, for any $1 \leq r < n$ and $1 \leq u < m$. Then, since $u < m$, for each l_x, $\mathcal{G}^t\mathcal{F}^{l_x}$ is a non-trivial homeomorphism. This would imply that every point of S_g is fixed by some element of the metacyclic group $\langle \mathcal{F}, \mathcal{G} \rangle$ of order mn, which is impossible (as the action of $\langle \mathcal{F}, \mathcal{G} \rangle$ on S_g is properly discontinuous). The converse follows directly from the definition of $\bar{\mathcal{G}}$. □

4.2 Encoding metacyclic actions

We now recall the example described in Chapter 1, where we described different metacyclic actions on S_7.

Example 4.2.1. In Figure 4.1 below, note that $D_{F_1} = D_{F_2} = D_{F_3} = (4, 1; ((1, 4), 2), ((3, 4), 2))$ and $D_{G_1} = D_{G_2} = D_{G_3} = (2, 3, 1;)$.

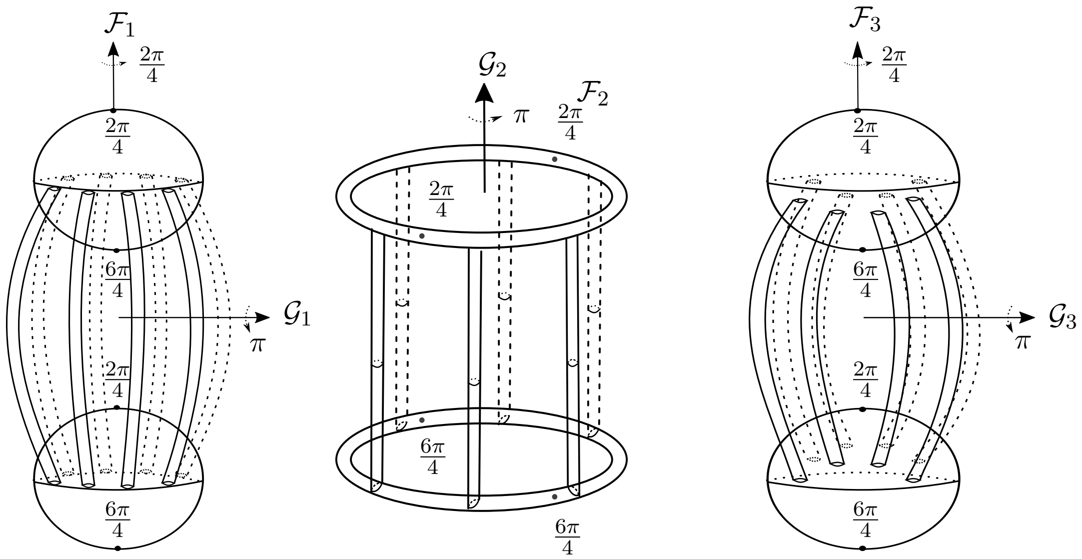

Figure 4.1: Split metacyclic actions on S_7 with conjugate generators.

Now, from Section 4.1, we know that $\mathcal{G}$ induces a data set automorphism $\bar{\mathcal{G}} \in \text{Homeo}_k(D_F)$ that maps the cone point corresponding to $(i, 4)$ to the cone point corresponding to $(k \cdot i, 4)$ for $k, i \in \{1, 3\}$, and the automorphism $\bar{\mathcal{G}}$ on $S_{1,4}$ is either the hyperelliptic involution, or a free rotation. Moreover, both of these automorphisms will induce two orbits of size two among the four cone points of $S_{1,4}$. Thus, the orbifold $S_{1,4}/\langle\bar{\mathcal{G}}\rangle$ is either homeomorphic to $S_{0,6}$ with signature $(0; 2, 2, 2, 2, 4, 4)$, or homeomorphic to $S_{1,2}$ with signature $(1; 4, 4)$. In our example, the group $\langle\mathcal{F}_1, \mathcal{G}_1\rangle \cong D_8$ acts on S_7 with orbifold signature $(1; 4, 4)$, while the groups $\langle\mathcal{F}_2, \mathcal{G}_2\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, and $\langle\mathcal{F}_3, \mathcal{G}_3\rangle \cong D_8$ act on S_7 with the same orbifold signature $(0; 2, 2, 2, 2, 4, 4)$. Following the nomenclature as in Section 2.1, the surface kernel epimorphisms corresponding to these actions are as follows:

$$\phi_{\langle\mathcal{F}_1,\mathcal{G}_1\rangle}(\alpha_1) = \mathcal{G}_1, \quad \phi_{\langle\mathcal{F}_1,\mathcal{G}_1\rangle}(\beta_1) = \mathcal{G}_1, \quad \phi_{\langle\mathcal{F}_1,\mathcal{G}_1\rangle}(\xi_1) = \mathcal{F}_1, \quad \phi_{\langle\mathcal{F}_1,\mathcal{G}_1\rangle}(\xi_2) = \mathcal{F}_1^{-1},$$

for $1 \leq i \leq 4$, we have

$$\phi_{\langle \mathcal{F}_2, \mathcal{G}_2 \rangle}(\xi_i) = \mathcal{G}_2 \mathcal{F}_2^2, \qquad \phi_{\langle \mathcal{F}_2, \mathcal{G}_2 \rangle}(\xi_5) = \mathcal{F}_2, \qquad \phi_{\langle \mathcal{F}_2, \mathcal{G}_2 \rangle}(\xi_6) = \mathcal{F}_2^{-1}, \text{ and}$$
$$\phi_{\langle \mathcal{F}_3, \mathcal{G}_3 \rangle}(\xi_i) = \mathcal{G}_3 \mathcal{F}_3, \qquad \phi_{\langle \mathcal{F}_3, \mathcal{G}_3 \rangle}(\xi_5) = \mathcal{F}_3, \qquad \phi_{\langle \mathcal{F}_3, \mathcal{G}_3 \rangle}(\xi_6) = \mathcal{F}_3^{-1}.$$

This example illustrates that distinct conjugates of two periodic maps, in addition to generating non-isomorphic subgroups, may also generate isomorphic subgroups that can act differently on the surface. Thus, we need to derive conditions under which distinct conjugates of two periodic maps will generate isomorphic subgroups that will have analogous actions on the surface. This motivates the following definition.

Definition 4.2.2. Two finite metacyclic actions $(H_1, (\mathcal{G}_1, \mathcal{F}_1))$ and $(H_2, (\mathcal{G}_2, \mathcal{F}_2))$ of order $u \cdot n$, amalgam r and twist factor k are said to be *weakly conjugate* if there exists an isomorphism $\psi : \pi_1^{\text{orb}}(\mathcal{O}_{H_1}) \cong \pi_1^{\text{orb}}(\mathcal{O}_{H_2})$, and an isomorphism $\chi : H_1 \to H_2$ such that the following conditions hold.

(i) $\chi((\mathcal{G}_1, \mathcal{F}_1)) = (\mathcal{G}_2, \mathcal{F}_2)$.

(ii) For $i = 1, 2$, let $\phi_{H_i} : \pi_1^{\text{orb}}(\mathcal{O}_{H_i}) \to H_i$ be the surface kernel epimorphisms. Then $(\chi \circ \phi_{H_1})(g)$ is conjugate to $(\phi_{H_2} \circ \psi)(g)$ in H_2, whenever $g \in \pi_1^{\text{orb}}(\mathcal{O}_{H_1})$ is of finite order. In other words, the following diagram commutes up to conjugacy:

$$\begin{array}{ccc} \pi_1^{orb}(\mathcal{O}_{H_1}) & \xrightarrow[\psi]{\cong} & \pi_1^{orb}(\mathcal{O}_{H_2}) \\ \downarrow{\scriptstyle \phi_{H_1}} & & \downarrow{\scriptstyle \phi_{H_2}} \\ H_1 & \xrightarrow[\chi]{} & H_2 \end{array}$$

The notion of weak conjugacy defines an equivalence relation on metacyclic actions on S_g and the equivalence classes thus obtained will be called *weak conjugacy classes*.

We note that, from the Definition 4.2.2, the pair $(\mathcal{G}_1, \mathcal{F}_1)$ will be conjugate (component-wise) to the pair $(\mathcal{G}_2, \mathcal{F}_2)$ (may not necessarily be conjugate with the same conjugating homeomorphism in $\text{Homeo}^+(S_g)$).

Remark 4.2.3. We will see later in Proposition 4.3.1 that conditions (i)-(ii) of Definition 4.2.2 are indeed required to ensure that the actions of the $\langle \mathcal{F}_i, \mathcal{G}_i \rangle$ on S_g are equivalent as topological actions in sense that the induced data set automorphisms $\bar{\mathcal{G}}_i$ on the homeomorphic orbifolds $\mathcal{O}_{\langle \mathcal{F}_i \rangle}$ are conjugate.

Remark 4.2.4. By virtue of the Nielsen-Kerckhoff theorem, the notion of weak conjugacy in Definition 4.2.2 naturally extends to an analogous notion in $\text{Mod}(S_g)$ via the natural association

$$(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F})) \leftrightarrow (\langle F, G \rangle, (G, F)).$$

For simplicity, we will now introduce the following notation.

Definition 4.2.5. Let $F, G \in \text{Mod}(S_g)$ be of finite order with $|F| = n$. Then (F, G) is *metacyclically realized* if there exists conjugates F', G' (of F, G resp.) such that $\langle F', G' \rangle \cong \mathcal{M}(u, n, r, k)$.

We will see later (in Corollary 6.1.5) that when $F \in \text{Mod}(S_g)$ is irreducible such that $S_g / \langle \mathcal{F} \rangle \approx S_{0,3}$ has three cone points of distinct orders, then no conjugate of F can form a metacyclic group with any nontrivial $G \in \text{Mod}(S_g)$. Hence (F, G) is not metacyclically realized for any nontrivial $G \in \text{Mod}(S_g)$.

Remark 4.2.6. Let $H < \text{Mod}(S_g)$ be a finite metacyclic subgroup, and let $I(H)$ denote the isomorphism class of H (in $\text{Mod}(S_g)$). By Remark 4.2.4, we have

$$I(H) = \{H' : H' \cong H \text{ and } (H', (G', F')) \text{ represents a weak conjugacy class}$$
$$\text{for some } F', G' \in H' \text{ such that } H' = \langle F', G' \rangle\}.$$

Consequently, periodic mapping classes $F, G \in \text{Mod}(S_g)$ such that (F, G) is metacyclically realized if and only if there exists conjugates F', G' (of F, G resp.) such that the triple $(\langle F', G' \rangle, (G', F'))$ represents a weak conjugacy class associated with a finite metacyclic subgroup (of order $u \cdot n$, twist factor k and amalgam r) of $\text{Mod}(S_g)$.

Let $H \cong \mathcal{M}(u, n, r, k)$ be a finite metacyclic group. By Lemma 2.1.2, if H acts faithfully on S_g with $\Gamma(\mathcal{O}_H) = (g_0; n_1, \ldots, n_\ell)$, then there exists a surjective homomorphism $\phi_H : \pi_1^{\text{orb}}(\mathcal{O}_H) \to H$ that preserves the orders of all torsion elements of $\pi_1^{\text{orb}}(\mathcal{O}_H)$. Fixing the presentation in (2.1.1) for $\pi_1^{\text{orb}}(\mathcal{O}_H)$ and presentation in (3.1.1) for H, $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \to H = \langle \mathcal{F}, \mathcal{G} \rangle$ would map

$$\xi_i \overset{\phi_H}{\longmapsto} \mathcal{G}^{c_{i1} \frac{m}{n_{i1}}} \mathcal{F}^{c_{i2} \frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell,$$

where $n_{i1} \mid m := |\mathcal{G}|$, $n_{i2} \mid n$, and for $j = 1, 2$, $\gcd(c_{ij}, n_{ij}) = 1$. Thus, in order to combinatorially encode the action of H on S_g, we need to encode the map ϕ_H, which is determined by the integer parameters associated with the metacyclic group H, the orbifold signature $\Gamma(\mathcal{O}_H)$, the n_{ij}, and the c_{ij}. Thus, we will now introduce an abstract tuple of integers satisfying certain number-theoretic conditions which will help to encode the weak conjugacy class of a metacyclic action $(H, (\mathcal{G}, \mathcal{F}))$.

Definition 4.2.7. A *metacyclic data set of degree $u \cdot n$, twist factor k, amalgam r and genus $g \geq 2$* is a tuple

$$\mathcal{D} = ((u \cdot n, r, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

where $u, n \geq 2$, the n_{ij} are positive integers for $1 \leq i \leq \ell,\ 1 \leq j \leq 2$, the $c_{ij} \in \mathbb{Z}_{n_{ij}}$, $r \mid n$ and $k \in \mathbb{Z}_n^\times$ such that $k^u \equiv 1 \pmod{n}$ and there exists a $w \in \mathbb{Z}$, satisfying the following conditions.

(i) $\dfrac{2g-2}{un} = 2g_0 - 2 + \displaystyle\sum_{i=1}^{\ell} \left(1 - \frac{1}{n_i}\right)$.

(ii) (a) For each i, j, $n_{i1} \mid \dfrac{un}{r} := m$, $n_{i2} \mid n$, either $\gcd(c_{ij}, n_{ij}) = 1$ or $c_{ij} = 0$, and $c_{ij} = 0$ if and only if $n_{ij} = 1$.

(b) For each i, $n_i = s_i$, where s_i is the least positive integer satisfying the following conditions for some $t_i \in \mathbb{N}$:

i. $c_{i1} \frac{m}{n_{i1}} s_i \equiv t_i u \pmod{m}$.

ii. $c_{i2} \frac{n}{n_{i2}} (k^{c_{i1} \frac{m}{n_{i1}} (s_i - 1)} + \cdots + k^{c_{i1} \frac{m}{n_{i1}}} + 1) \equiv -t_i r \pmod{n}$.

(iii) $\displaystyle\sum_{i=1}^{\ell} c_{i1} \frac{m}{n_{i1}} \equiv wu \pmod{m}$.

(iv) Defining $A := \displaystyle\sum_{i=1}^{\ell} c_{i2} \frac{n}{n_{i2}} \prod_{s=i+1}^{\ell} k^{c_{s1} \frac{m}{n_{s1}}}$ and $d := \gcd(n, k-1)$, we have

$$A \equiv \begin{cases} -wr \pmod{n}, & \text{if } g_0 = 0, \text{ and} \\ d\theta - wr \pmod{n}, \text{ for } \theta \in \mathbb{Z}_n, & \text{if } g_0 \geq 1. \end{cases}$$

(v) If $g_0 = 0$, there exists $(p_1, p_2, \ldots, p_{\ell v}), (q_1, q_2 \ldots, q_{\ell v}) \in (\mathbb{N} \cup \{0\})^{\ell v}$, $v \in \mathbb{N}$, and $a, b \in \mathbb{Z}$ such that the following conditions hold.

(a) $\displaystyle\sum_{i'=1}^{\ell v} p_{i'} c_{i1} \frac{m}{n_{i1}} \equiv 1 + au \pmod{m}$ and

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left(\sum_{s=1}^{p_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (p_{i'} - s)} \right) \left(\prod_{t'=i'+1}^{\ell v} k^{p_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv -ar \pmod{n}.$$

(b) $\displaystyle\sum_{i'=1}^{\ell v} q_{i'} c_{i1} \frac{m}{n_{i1}} \equiv bu \pmod{m}$ and

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left(\sum_{s=1}^{q_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (q_{i'} - s)} \right) \left(\prod_{t'=i'+1}^{\ell v} k^{q_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv 1 - br \pmod{n}, \text{ where}$$

$$i \equiv \begin{cases} i' \pmod{\ell}, & \text{if } i' \not\equiv 0 \pmod{\ell}, \\ \ell & \text{otherwise}, \end{cases} \qquad t \equiv \begin{cases} t' \pmod{\ell}, & \text{if } t' \not\equiv 0 \pmod{\ell}, \text{ and} \\ \ell, & \text{otherwise}. \end{cases}$$

(vi) If $g_0 = 1$, there exists $(p_1, p_2, \ldots, p_{\ell v}), (q_1, q_2, \ldots, q_{\ell v}) \in (\mathbb{N} \cup \{0\})^{\ell v}$, $m', n', a, b \in \mathbb{Z}$, and $v \in \mathbb{N}$ such that $m' \mid m$ and $n' \mid n$, satisfying the following conditions.

(a) $\displaystyle\sum_{i'=1}^{\ell v} p_{i'} c_{i1} \frac{m}{n_{i1}} \equiv m' + au \pmod{m}$ and

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left(\sum_{s=1}^{p_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (p_{i'} - s)} \right) \left(\prod_{t'=i'+1}^{\ell v} k^{p_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv -ar \pmod{n}.$$

(b) $\displaystyle\sum_{i'=1}^{\ell v} q_{i'} c_{i1} \frac{m}{n_{i1}} \equiv bu \pmod{m}$ and

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left(\sum_{s=1}^{q_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (q_{i'} - s)} \right) \left(\prod_{t'=i'+1}^{\ell v} k^{q_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv n' - br \pmod{n}, \text{ where}$$

$$i \equiv \begin{cases} i' \pmod{\ell}, & \text{if } i' \not\equiv 0 \pmod{\ell}, \\ \ell, & \text{otherwise}, \end{cases} \qquad t \equiv \begin{cases} t' \pmod{\ell} & \text{if } t' \not\equiv 0 \pmod{\ell}, \text{ and} \\ \ell & \text{otherwise}. \end{cases}$$

(c) $A \equiv -\beta k^{\alpha} + \beta - wr \pmod{n}$ for some non-negative integers α, β, where

$$\operatorname{lcm}\left(\frac{m}{m'}, \frac{m}{\gcd(m, \alpha)} \right) = m \text{ and } \operatorname{lcm}\left(\frac{n}{n'}, \frac{n}{\gcd(n, \beta)} \right) = n.$$

Furthermore, we set $\alpha = 1$, when $m' = 0$, and $\beta = 1$, when $n' = 0$.

A metacyclic data set $\mathcal{D}$ is said to be *split* when $r = n$.

As a notational convention, we will refrain from including the parameter r in a split metacyclic data set $\mathcal{D}$.

Remark 4.2.8. Note that a metacyclic data set $\mathcal{D}$ as in Definition 4.2.7 determines an orbifold $\mathcal{O}_\mathcal{D}$ with $\Gamma(\mathcal{O}_\mathcal{D}) = (g_0; n_1, n_2, \dots n_\ell)$.

We will now revisit the actions in Example 4.2.1 and describe their data sets.

Example 4.2.9. In Example 4.2.1, the data sets corresponding to the actions $(\langle \mathcal{F}_1, \mathcal{G}_1 \rangle, \mathcal{G}_1, \mathcal{F}_1)$, $(\langle \mathcal{F}_2, \mathcal{G}_2 \rangle, \mathcal{G}_2, \mathcal{F}_2)$, and $(\langle \mathcal{F}_3, \mathcal{G}_3 \rangle, \mathcal{G}_3, \mathcal{F}_3)$ are given by

$$\mathcal{D} = ((2 \cdot 4, -1), 1; [(0,1),(1,4),4],[(0,1),(3,4),4]),$$
$$\mathcal{D}' = ((2 \cdot 4, 1), 0; [(1,2),(1,2),2],[(1,2),(1,2),2],[(1,2),(1,2),2],[(1,2),(1,2),2],$$
$$[(0,1),(1,4),4],[(0,1),(3,4),4]), \text{ and}$$
$$\tilde{\mathcal{D}} = ((2 \cdot 4, -1), 0; [(1,2),(1,4),2],[(1,2),(1,4),2],[(1,2),(1,4),2],[(1,2),(1,4),2],$$
$$[(0,1),(1,4),4],[(0,1),(3,4),4]),$$

respectively. For the data sets above, it is straightforward to verify conditions (i) - (iv) of Definition 4.2.7. For $\mathcal{D}$, condition (vi) can be verified by taking $(p_1,p_2),(q_1,q_2) = (0,0),(1,0)$, $m' = 0$, $n' = 1$, $a = b = 0$, $v = 1$. For $\mathcal{D}'$, condition (v) follows by taking $(p_1,p_2,p_3,p_4,p_5,p_6),(q_1,q_2,q_3,q_4,q_5,q_6) = (1,0,0,0,2,0),(0,0,0,0,1,0)$, $a = b = 0, v = 1$. For $\tilde{\mathcal{D}}$, condition (v) is satisfied by taking $(p_1,p_2,p_3,p_4,p_5,p_6),(q_1,q_2,q_3,q_4,q_5,q_6) = (1,0,0,0,0,1),(0,0,0,0,1,0)$, $a = b = 0, v = 1$.

The following definition provides number-theoretic conditions for an equivalence of two metacyclic data sets. We will show later in Proposition 4.3.1 that the equivalence classes of metacyclic data sets correspond to the weak conjugacy classes of metacyclic actions.

Definition 4.2.10. Two metacyclic data sets

$$\mathcal{D} = ((u \cdot n, r, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell])$$
and
$$\mathcal{D}' = ((u \cdot n, r, k), g_0; [(c'_{11}, n'_{11}), (c'_{12}, n'_{12}), n'_1], \dots, [(c'_{\ell 1}, n'_{\ell 1}), (c'_{\ell 2}, n'_{\ell 2}), n'_\ell])$$

are said to be *equivalent* ($\mathcal{D} \sim \mathcal{D}'$) if for each tuple $[(c'_{i1}, n'_{i1}), (c'_{i2}, n'_{i2}), n'_i]$, there exists a tuple $[(c_{j_i 1}, n_{j_i 1}), (c_{j_i 2}, n_{j_i 2}), n_{j_i}]$ such that $j_{i_1} \neq j_{i_2}$ whenever $i_1 \neq i_2$ satisfying the following conditions:

(i) $n'_i = n_{j_i}$,

(ii) $c'_{i1}\frac{m}{n'_{i1}} \equiv c_{j_i1}\frac{m}{n_{j_i1}} + au \pmod{m}$, where $m = u\frac{n}{r}$, and

(iii) $c'_{i2}\frac{n}{n'_{i2}} \equiv c_{j_i2}\frac{n}{n_{j_i2}}k^{a_i} + b_i(k^{c_{j_i1}\frac{m}{n_{j_i1}}} - 1) - ar \pmod{n}$ for some $a_i, b_i, a \in \mathbb{Z}$.

4.3 Main theorem

Our main result provides equivalent conditions under which torsion elements $F, G \in \mathrm{Mod}(S_g)$ be such that (F, G) is metacyclically realized. A key ingredient in the proof of the main result is the following proposition.

Proposition 4.3.1. *For integers $n, u, g, r \geq 2$ such that $r \mid n$ and $k \in \mathbb{Z}_n^{\times}$, the equivalence classes of metacyclic data sets of degree $u \cdot n$ with twist factor k, amalgam r, and genus g correspond to the weak conjugacy classes of $\mathcal{M}(u, n, r, k)$-actions on S_g.*

Proof. Let $\mathcal{D}$ be a representative of an equivalence class of metacyclic data set of degree $u \cdot n$ with twist factor k, amalgam r and genus g (as in Definition 4.2.7). We need to show that $\mathcal{D}$ corresponds to the weak conjugacy class of a $\mathcal{M}(u, n, r, k)$-action on S_g represented by $(H, (\mathcal{G}, \mathcal{F}))$, where $H = \langle \mathcal{F}, \mathcal{G} \rangle$. To see this, we first show the existence of an epimorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_{\mathcal{D}}) \to H$ that preserves the orders of the torsion elements. Let H and $\pi_1^{orb}(\mathcal{O}_{\mathcal{D}})$ have presentations given by

$$\langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = 1, \mathcal{G}^u = \mathcal{F}^r, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle \cong \mathcal{M}(u, n, r, k) \text{ and}$$
$$\langle \alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \ldots, \xi_\ell \mid \xi_1^{n_1} = \ldots = \xi_\ell^{n_\ell} = \prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] = 1 \rangle,$$

respectively.

We consider the map

$$\xi_i \overset{\phi_H}{\longmapsto} \mathcal{G}^{c_{i1}\frac{m}{n_{i1}}} \mathcal{F}^{c_{i2}\frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell,$$

where $m := \dfrac{un}{r}$. Then condition (ii) of Definition 4.2.7 and Lemma 3.2.4 would imply that ϕ_H is a map which is order-preserving on torsion elements. For clarity, we break the argument for the surjectivity of ϕ_H into the following three cases.

Case 1: $g_0 = 0$. Then it follows from conditions (iii)-(iv) and Lemma 3.2.2 that ϕ_H satisfies the long relation $\prod_{i=1}^{\ell} \xi_i = 1$. Moreover, the surjectivity of ϕ_H follows from condition (v) and Lemma 3.2.2.

Case 2: $g_0 \geq 2$. In this case, $\pi_1^{orb}(\mathcal{O}_\mathcal{D})$ has additional hyperbolic generators (viewing them as isometries of the hyperbolic plane), namely the α_i and the β_i. Extending ϕ_H by mapping $\alpha_1 \xmapsto{\phi_H} \mathcal{G}, \beta_1 \xmapsto{\phi_H} \mathcal{F}$ yields an epimorphism. Moreover, by carefully choosing the images of the α_i and the β_i, for $i \geq 2$, conditions (iii)-(iv), Lemma 3.2.2 and Lemma 3.4.4 would ensure that the long relation $\prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] = 1$ is satisfied.

Case 3: $g_0 = 1$. In this case, $\pi_1^{orb}(\mathcal{O}_\mathcal{D})$ would have two additional hyperbolic generators, namely the α_1 and the β_1. We extend ϕ_H by defining $\alpha_1 \xmapsto{\phi_H} \mathcal{G}^\alpha$ and $\beta_1 \xmapsto{\phi_H} \mathcal{F}^{-\beta}$. We then apply conditions (iii), (iv), (vi), Lemma 3.2.2 and Lemma 3.4.4 to obtain the desired epimorphism.

Now we show that $\mathcal{D}$ determines $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$, up to conjugacy. By carefully applying Lemma 2.2.4, we see that

$$D_F = (n, g_1; ((v_{ij}^{-1}, t_i), \frac{t_i |\mho_{\mathcal{F}^{\frac{n}{t_i}}}(v_{ij}, t_i)|}{n}) : v_{ij} \in \mathbb{Z}_{t_i}^\times, \, t_i \mid n),$$

where

$$|\mho_{\mathcal{F}^{\frac{n}{t_i}}}(v_{ij}, t_i)| = |\mathbb{F}_{\mathcal{F}^{\frac{n}{t_i}}}(v_{ij}, t_i)| - \sum_{\substack{t_{i'} \in \mathbb{N} \\ t_{i'} \neq t_i \\ t_i | t_{i'} | n}} \sum_{\substack{(v_{i'j'}, t_{i'}) = 1 \\ v_{ij} \equiv v_{i'j'} (\text{mod}\, t_i)}} |\mho_{\mathcal{F}^{\frac{n}{t_{i'}}}}(v_{i'j'}, t_{i'})|$$

and g_1 is determined by Riemann-Hurwitz equation, and

$$D_G = (m, g_2; ((u_{ij}^{-1}, m_i), \frac{m_i |\mho_{\mathcal{G}^{\frac{m}{m_i}}}(u_{ij}, m_i)|}{m}) : u_{ij} \in \mathbb{Z}_{m_i}^\times, \, m_i \mid m),$$

where

$$|\mho_{\mathcal{G}^{\frac{m}{m_i}}}(u_{ij}, m_i)| = |\mathbb{F}_{\mathcal{G}^{\frac{m}{m_i}}}(u_{ij}, m_i)| - \sum_{\substack{m_{i'} \in \mathbb{N} \\ m_{i'} \neq m_i \\ m_i | m_{i'} | m}} \sum_{\substack{(u_{i'j'}, m_{i'}) = 1 \\ u_{ij} \equiv u_{i'j'} (\text{mod}\, m_i)}} |\mho_{\mathcal{G}^{\frac{m}{m_{i'}}}}(u_{i'j'}, m_{i'})|$$

and g_2 is determined by Riemann-Hurwitz equation.

It remains to be shown that if $\mathcal{D} \sim \mathcal{D}'$, then they corresponds to same weak conjugacy class. To see this, let $\mathcal{D}$ (resp. $\mathcal{D}'$) corresponds to the weak conjugacy class of a $\mathcal{M}(u, n, r, k)$-action on S_g represented by $(H_1, (\mathcal{G}_1, \mathcal{F}_1))$ (resp. $(H_2, (\mathcal{G}_2, \mathcal{F}_2))$), where $H_i = \langle \mathcal{F}_i, \mathcal{G}_i \rangle$ for $i = 1, 2$. We need to show that $(H_1, (\mathcal{G}_1, \mathcal{F}_1))$ is weakly conjugate to

$(H_2,(\mathcal{G}_2,\mathcal{F}_2))$. Clearly $H_1 \cong H_2$. Also, as $\mathcal{D} \sim \mathcal{D}'$, there exist $\psi : \pi_1^{orb}(\mathcal{O}_{\mathcal{D}}) \to \pi_1^{orb}(\mathcal{O}_{\mathcal{D}'})$ such that $\psi(\xi_i) \sim \xi'_{j_i}$ and hence $(\chi \circ \phi_{H_1})(\xi_i) \sim_{H_2} (\phi_{H_2} \circ \psi)(\xi_i)$, where ξ_i (resp. ξ'_i) are elliptic generators of $\pi_1^{orb}(\mathcal{O}_{\mathcal{D}})$ (resp. $\pi_1^{orb}(\mathcal{O}_{\mathcal{D}'})$). Thus, from Lemma 2.2.4, it can be seen that $D_{F_1} = D_{F_2}$ and $D_{G_1} = D_{G_2}$.

Conversely, consider the weak conjugacy class of $\mathcal{M}(u,n,r,k)$-actions on S_g represented by $(H,(\mathcal{G},\mathcal{F}))$, where $H = \langle \mathcal{F},\mathcal{G}\rangle$. By Theorem 2.1.2, there exists a surjective homomorphism

$$\phi_H : \pi_1^{orb}(\mathcal{O}_H) \to H : \xi_i \overset{\phi_H}{\longmapsto} \mathcal{G}^{c_{i1}\frac{m}{n_{i1}}}\mathcal{F}^{c_{i2}\frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell,$$

which is order-preserving on the torsion elements. This corresponds to a metacyclic data set $\mathcal{D}$ of degree $u \cdot n$ with twist factor k, amalgam r and genus g as in Definition 4.2.7. By Theorem 2.1.2, $\mathcal{D}$ satisfies condition (i) of Definition 4.2.7. Moreover, condition (ii) follows from the fact that ϕ_H is order-preserving on torsion elements and using Lemma 3.2.2. Furthermore, conditions (iii)-(iv) follow from the long relation satisfied by $\pi_1^{orb}(\mathcal{O}_H)$ and using Lemma 3.2.2, and condition (v)-(vi) are implied by the surjectivity of ϕ_H and by using Lemma 3.2.2 and Lemma 3.4.4. Thus, we obtain the metacyclic data set $\mathcal{D}$ of degree $u \cdot n$ with twist factor k, amalgam r and genus g.

Now it remains to be seen that if $(H_1,(\mathcal{G}_1,\mathcal{F}_1))$ is weakly conjugate to $(H_2,(\mathcal{G}_2,\mathcal{F}_2))$ and they corresponds to metacyclic data sets $\mathcal{D}$ and $\mathcal{D}'$ of degree $u \cdot n$ with twist factor k, amalgam r and genus g, then $\mathcal{D} \sim \mathcal{D}'$. As $(H_1,(\mathcal{G}_1,\mathcal{F}_1))$ is weakly conjugate to $(H_2,(\mathcal{G}_2,\mathcal{F}_2))$, there exists $\psi : \pi_1^{orb}(\mathcal{O}_{H_1}) \to \pi_1^{orb}(\mathcal{O}_{H_2})$ such that $\psi(\xi_i) \sim \xi'_{j_i}$ and $(\chi \circ \phi_{H_1})(\xi_i) \sim_{H_2} (\phi_{H_2} \circ \psi)(\xi_i)$, where ξ_i (resp. ξ'_i) are elliptic generators of $\pi_1^{orb}(\mathcal{O}_{H_1})$ (resp. $\pi_1^{orb}(\mathcal{O}_{H_2})$). This implies that $\mathcal{D} \sim \mathcal{D}'$, and our assertion follows. □

From here on, we will use metacyclic data sets (up to equivalence) to encode the weak conjugacy classes of metacyclic actions.

Notation 4.3.2. We denote the data sets D_F and D_G (representing the cyclic factors of H) derived from the metacyclic data set $\mathcal{D}$ appearing in the proof of Proposition 4.3.1 by $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

We will now state our main theorem, which follows from Proposition 4.3.1.

Theorem 4.3.3 (Main theorem)**.** *Let $F, G \in \text{Mod}(S_g)$ be of orders n, m, respectively. Then (F, G) is metacyclically realized if and only if there exists a metacyclic data set $\mathcal{D}$ of degree $u \cdot n$, twist factor k, amalgam r, and genus g such that $\mathcal{D}_1 = D_F$ and $\mathcal{D}_2 = D_G$.*

We conclude this section with an example of a split metacyclic action of order 16 on S_5.

Example 4.3.4. The split metacyclic data set $\mathcal{D} = ((4 \cdot 4, -1), 1; [(0,1), (1,2), 2])$ encodes the weak conjugacy class of a $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$-action on S_5 represented by $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$, where

$$D_F = (4, 1; (1,2), (1,2), (1,2), (1,2)) \text{ and } D_G = (4, 2, 1;).$$

The geometric realization of this action is illustrated in Figure 4.2 below.

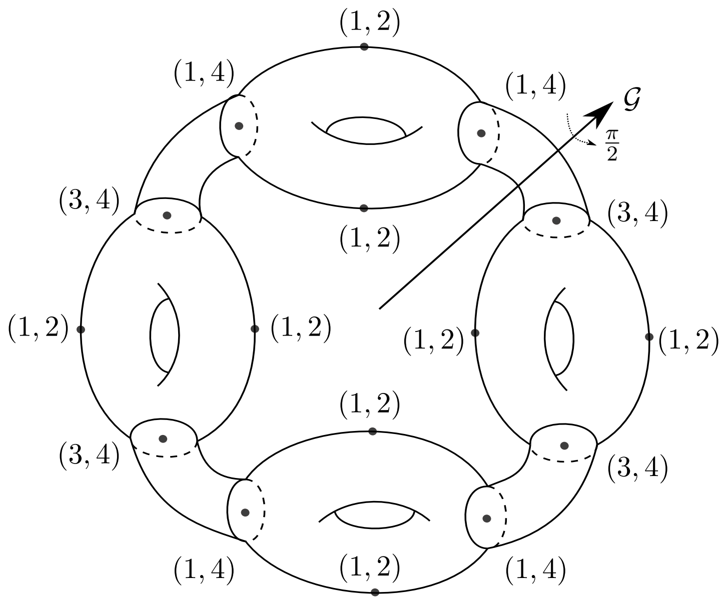

Figure 4.2: Realization of a $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$-action on S_5.

Note that the pairs of integers appearing in Figure 4.2 represent the compatible orbits involved in the realization of $\mathcal{F}$. Here, the action $\mathcal{F}$ is realized via two 1-compatibilities between the action $\mathcal{F}'$ on two copies of S_2 with $D_{F'} = (4, 0; ((1,2), 2), (1,4), (3,4))$. Furthermore, the action $\mathcal{F}'$ is realized by a 1-compatibility between the actions $\mathcal{F}''$ and $\mathcal{F}'''$ on two copies of S_1 with $D_{F''} = (4, 0; (1,2), (1,4), (1,4))$ and $D_{F'''} = (4, 0; (1,2), (3,4), (3,4))$.

4.4 Dihedral and Dicyclic subgroups of $\text{Mod}(S_g)$

In this section, as applications of our main result, we characterize the dihedral and dicyclic subgroups of $\text{Mod}(S_g)$.

4.4.1 Dihedral subgroups of $\mathrm{Mod}(S_g)$

Let $D_{2n} = \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_2$ be the dihedral group of order $2n$. We will call a split metacyclic data set of degree $2 \cdot n$ and twist factor -1, a *dihedral data set.* A simple computation reveals that a dihedral data set

$$((2 \cdot n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \cdots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell])$$

have the property that $(c_{j1}, n_{j1}) \in \{(0,1), (1,2)\}$, for $1 \leq j \leq \ell$. The following is an immediate consequence of Proposition 4.3.1.

Corollary 4.4.1. *For $g \geq 2$ and $n \geq 3$, dihedral data sets of degree $2 \cdot n$ and genus g correspond to the weak conjugacy classes of D_{2n}-actions on S_g.*

The following proposition provides an alternative characterization of a D_{2n}-action in terms of the generator of its factor subgroup of order n.

Proposition 4.4.2. *Let $F \in \mathrm{Mod}(S_g)$ be of order n. Then there exists an involution $G \in \mathrm{Mod}(S_g)$ such that $\langle F, G \rangle \cong D_{2n}$ if and only if D_F has the form*

$$(n, g_0, r; ((c_1, n_1), (-c_1, n_1), \ldots, (c_s, n_s), (-c_s, n_s)). \tag{**}$$

Proof. Suppose that D_F has the form $(**)$. Then $\mathcal{O}_{\langle \mathcal{F} \rangle}$ is an orbifold of genus g_0 with $2s$ cone points $[x_1], [y_1], \ldots, [x_s], [y_s]$, where $\mathcal{P}_{x_i} = (c_i, n_i)$ and $\mathcal{P}_{y_i} = (-c_i, n_i)$, for $1 \leq i \leq s$. Up to conjugacy, let $\bar{\mathcal{G}} \in \mathrm{Homeo}_k(D_{\langle \mathcal{F} \rangle})$ be the hyperelliptic involution so that $\bar{\mathcal{G}}([x_i]) = [y_i]$, for $1 \leq i \leq s$. To prove our assertion, it would suffice to show the existence of an involution $\mathcal{G} \in \mathrm{Homeo}^+(S_g)$ that induces $\bar{\mathcal{G}}$. This amounts to showing that there exists a split metacyclic data set $\mathcal{D}$ of degree $2 \cdot n$ with twist factor -1 encoding the weak conjugacy class $(H, (\mathcal{G}, \mathcal{F}))$ so that D_G has degree 2. Consider the tuple

$$\mathcal{D} = ((2 \cdot n, -1), 0; \underbrace{[(1,2), (0,1), 2], \ldots, [(1,2), (0,1), 2]}_{t-2 \text{ times}}, [(1,2), (c_{(t-1)2}, n_{(t-1)2}), 2]$$

$$[(1,2), (c_{t2}, n_{t2}), 2], [(0,1), (c_1, n_1), n_1], \ldots, [(0,1), (c_s, n_s), n_s]),$$

where $t = 2g_0 + 2$,

$$(c_{(t-1)2}n/n_{(t-1)2}, c_{t2}n/n_{t2}) = \begin{cases} (0, -\sum_{i=1}^{s} c_i \frac{n}{n_i} (\bmod n)), & \text{if } g_0 = 0, \text{ and} \\ (1, 1 - \sum_{i=1}^{s} c_i \frac{n}{n_i} (\bmod n)), & \text{if } g_0 > 0. \end{cases}$$

It follows immediately that $\mathcal{D}$ satisfies conditions (i)-(iv) of Definition 4.2.7. As $t \geq 2$, by taking $v = 1$, we may choose $(p_1, p_2, \ldots, p_{t+s}) = (1, 0, \ldots, 0)$ to conclude that $\mathcal{D}$ also satisfies condition (v)(a). Since $t = 2 \iff g_0 = 0$, and when $g_0 = 0$, we have that $\text{lcm}(n_1, \ldots, n_s) = n$, from which condition (v)(b) follows. Finally, for the case when $g_0 \neq 0$, (v)(b) follows by choosing $(q_1, \ldots, q_{t-2}, q_{t-1}, \ldots, q_{t+s}) = (0, \ldots, 1, 1, \ldots, 0)$. Thus, it follows that $\mathcal{D}$ is a split metacyclic data set. Further, a direct application of Theorem 4.3.3 would show that $\mathcal{D}$ indeed encodes the weak conjugacy represented by $(H, (\mathcal{G}, \mathcal{F}))$, as desired.

The converse follows immediately from Remark 4.2.4 and Proposition 4.3.1. □

We now provide a couple of examples of dihedral actions along with their realizations.

Example 4.4.3. Consider the $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_2$-action $\langle \mathcal{F}, \mathcal{G} \rangle$ on S_3 illustrated in Figure 4.3 below, where

$$D_F = (3, 1; (1, 3), (2, 3)) \text{ and } D_G = (2, 1; (1, 2), (1, 2), (1, 2), (1, 2)).$$

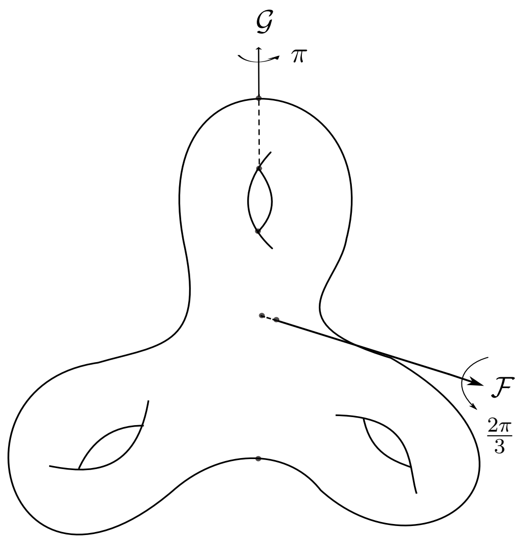

Figure 4.3: Realization of a D_6-action on S_3.

The weak conjugacy class of the action $(\langle \mathcal{F}, \mathcal{G}\rangle, (\mathcal{G}, \mathcal{F}))$ is encoded by

$$\mathcal{D} = ((2\cdot 3, -1), 0; [(1,2),(0,1),2], [(1,2),(0,1),2], [(1,2),(0,1),2], [(1,2),(1,3),2], [(0,1),(2,3),3]).$$

Example 4.4.4. Consider the $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_2$-actions $\langle \mathcal{F}, \mathcal{G}\rangle$ and $\langle \mathcal{F}, \mathcal{G}'\rangle$ on S_3 illustrated in Figure 4.4 below, where $D_F = (4, 0; (1,4),(3,4),(1,4),(3,4))$, $D_G = (2, 1; (1,2),(1,2),(1,2),(1,2))$, and $D_{G'} = (2, 2, 1;)$.

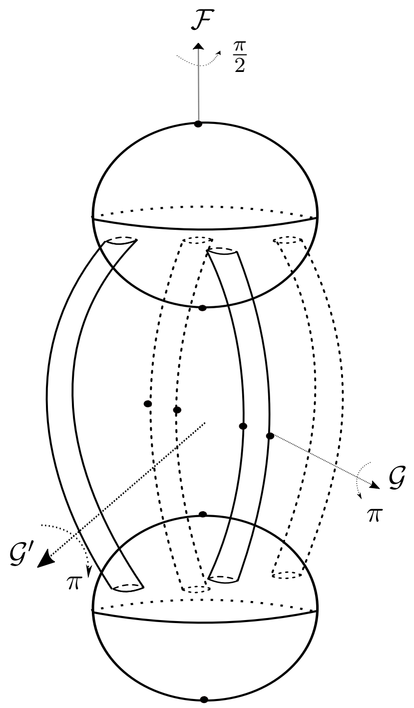

Figure 4.4: Realization of a D_8-action on S_3.

The weak conjugacy classes $(\langle \mathcal{F}, \mathcal{G}\rangle, (\mathcal{G}, \mathcal{F}))$ and $(\langle \mathcal{F}, \mathcal{G}'\rangle, (\mathcal{G}', \mathcal{F}))$ are encoded by

$$((2\cdot 4, -1), 0; [(1,2),(0,1),2], [(1,2),(0,1),2], [(0,1),(1,4),4], [(0,1),(3,4),4])$$

and

$$((2\cdot 4, -1), 0; [(1,2),(1,4),2], [(1,2),(1,4),2], [(0,1),(1,4),4], [(0,1),(3,4),4]),$$

respectively.

4.4.2 Dicyclic subgroups of $\mathrm{Mod}(S_g)$

In this subsection, we will characterize non-split dicyclic actions on S_g. We recall that a dicyclic group of order $4n$ is given by $\mathrm{Dic}_n := \mathcal{M}(2, 2n, n, -1)$. We will call a metacyclic data set of degree $2 \cdot 2n$, amalgam n and twist factor -1, a *dicyclic data set*. Note that a dicyclic group is a non-split metacyclic group if and only if n is even. Thus, throughout this subsection, n is assumed to be even. The following is an immediate consequence of Proposition 4.3.1.

Corollary 4.4.5. *For $g \geq 2$ and $n \geq 3$, dicyclic data sets of degree $2 \cdot 2n$ and genus g correspond to the weak conjugacy classes of Dic_n-actions on S_g.*

Remark 4.4.6. Let $H = \mathrm{Dic}_n = \langle F, G \rangle < \mathrm{Mod}(S_g)$. Then $\bar{\mathcal{G}}$ cannot fix a regular point in the orbifold $S_g/\langle \mathcal{F} \rangle$. In fact, $\bar{\mathcal{G}}$ fixes only an order 2 cone point in $S_g/\langle \mathcal{F} \rangle$. To see this, suppose we assume on the contrary that $\bar{G}([x]) = [x]$, where $[x]$ is a regular point (or any other cone point of order $p \neq 2$) in $\mathcal{O}_{\langle \mathcal{F} \rangle}$. Then, $\mathrm{Stab}_{\langle G,F \rangle}(x) = \langle G^2 \rangle = \langle F^n \rangle$, which implies that $[x]$ is an order 2 cone point in $\mathcal{O}_{\langle \mathcal{F} \rangle}$, thereby yielding a contradiction.

The following proposition provides a number-theoretic characterization of a Dic_n-action on S_g .

Proposition 4.4.7. *Let $F \in \mathrm{Mod}(S_g)$ be of order $2n$. Then there exists a $G \in \mathrm{Mod}(S_g)$ of order 4 such that $\langle F, G \rangle \cong \mathrm{Dic}_n$ if and only if D_F has the form*

$$(2n, g_0, d; ((c_1, n_1), (-c_1, n_1), \ldots, (c_s, n_s), (-c_s, n_s))) \qquad (*)$$

satisfying the following conditions.

(i) When g_0 is even, there exists an i such that $(c_i, n_i) = (-c_i, n_i) = (1, 2)$.

(ii) When g_0 is odd, at least one of the following statements hold true.

(a) There exists i, j with $i \neq j$ such that $(\pm c_i, n_i) = (\pm c_j, n_j) = (1, 2)$.

(b) $g_0 \geq 3$ and $\sum_{i=1}^{s} c_i \frac{2n}{n_i} \equiv 2a \pmod{2n}$ for some $a \in \mathbb{Z}$.

(c) $g_0 = 1$ and $\sum_{i=1}^{s} c_i \frac{2n}{n_i} \equiv 2a \pmod{2n}$ for some a such that $\gcd(a, n) = 1$.

(d) $g_0 = 1$, $\sum_{i=1}^{s} c_i \frac{2n}{n_i} \equiv 2a \pmod{2n}$ for some $a \in \mathbb{Z}$, and $\mathrm{lcm}(n_1, \ldots, n_s) = 2n$.

Proof. Suppose that D_F has the form (*). Then $\mathcal{O}_{\langle \mathcal{F} \rangle}$ is an orbifold of genus g_0 with $2s$ cone points $[x_1], [y_1], \ldots, [x_s], [y_s]$, where $\mathcal{P}_{x_i} = (c_i, n_i)$ and $\mathcal{P}_{y_i} = (-c_i, n_i)$, for $1 \leq i \leq s$. If D_F satisfies condition (i), then we may assume without loss of generality that $(c_1, n_1) = (-c_1, n_1) = (1, 2)$. Then up to conjugacy, let $\bar{\mathcal{G}} \in \mathrm{Homeo}_k(D_{\langle \mathcal{F} \rangle})$ be an involution such that $\bar{\mathcal{G}}([x_i]) = [y_i]$, for $2 \leq i \leq s$, $\bar{\mathcal{G}}([x_1]) = [x_1]$, and $\bar{\mathcal{G}}([y_1]) = [y_1]$. To prove our assertion, it would suffice to show the existence of an involution $\mathcal{G} \in \mathrm{Homeo}^+(S_g)$ that induces $\bar{\mathcal{G}}$. This amounts to showing that there exists a metacyclic data set $\mathcal{D}$ of degree $2 \cdot 2n$ with amalgam n and twist factor -1 encoding the weak conjugacy class $(H, (\mathcal{G}, \mathcal{F}))$ so that D_G has degree 4.

Consider the tuple

$$\mathcal{D} = ((2 \cdot 2n, n, -1), g_0/2; [(1,4), (0,1), 4], [(3,4), (c', n'), 4], \\ [(0,1), (c_2, n_2), n_2], \ldots, [(0,1), (c_s, n_s), n_s]),$$

where $c' \frac{2n}{n'} \equiv -\sum_{i=2}^{s} c_i \frac{2n}{n_i} \pmod{2n}$. It follows immediately that $\mathcal{D}$ satisfies conditions (i)-(iv) of Definition 4.2.7. By taking $v = 1$, we may choose $(p_1, \ldots, p_{s+1}) = (1, 0, \ldots, 0)$ to conclude that $\mathcal{D}$ also satisfies either condition (v)(a) or (vi)(a), based on the choice of g_0. If $g_0 = 0$, we have that $\mathrm{lcm}(n_1, n_2, \ldots, n_s) = 2n$, from which condition (v)(b) follows. If $g_0 \neq 0$, then by carefully defining ϕ_H (as in Proposition 4.3.1) on the hyperbolic elements of $\pi_1^{orb}(\mathcal{O}_H)$, our claim is true. Thus, it follows that $\mathcal{D}$ is a metacyclic data set.

If D_F satisfies condition (ii)(a), then by a similar argument as above, we obtain the metacyclic data set

$$\mathcal{D} = ((2 \cdot 2n, n, -1), (g_0+1)/2; [(1,4), (0,1), 4], [(1,4), (0,1), 4], [(1,4), (1,2n), 4], \\ [(1,4), (c'', n''), 4], [(0,1), (c_3, n_3), n_3], \ldots, [(0,1), (c_s, n_s), n_s]),$$

where $c'' \frac{2n}{n''} \equiv 1 - \sum_{i=3}^{s} c_i \frac{2n}{n_i} \pmod{2n}$. Suppose that D_F satisfies conditions (ii)(b)-(d). Then again by an analogous argument as above, we obtain the metacyclic data set

$$\mathcal{D} = ((2 \cdot 2n, n, -1), (g_0+1)/2; [(0,1), (c_1, n_1), n_1], \ldots, [(0,1), (c_s, n_s), n_s]).$$

Further, a direct application of Theorem 4.3.3 would show that $\mathcal{D}$ indeed encodes the weak conjugacy represented by $(H, (\mathcal{G}, \mathcal{F}))$, as desired.

The converse follows immediately from Remark 4.2.4, Remark 4.4.6 and Proposition 4.3.1. □

4.5 Classification of the weak conjugacy classes in $\mathrm{Mod}(S_g)$ for $g = 3, 5, 10, 11$

A complete classification of finite group actions on S_g for $2 \leq g \leq 4$ up to conjugacy is given in [7, 9, 27]. The metacyclic subgroups of $\mathrm{Mod}(S_2)$ ([9, Table 4]) are:

(a) the dihedral groups D_3, D_4, and D_6,

(b) the split metacyclic groups $\mathcal{M}(2,8,8,3)$ and $\mathcal{M}(4,3,3,2)$, and

(c) the quaternion group $\mathcal{M}(2,4,2,3)$.

The surface S_3 admits some interesting split metacyclic actions whose geometric realizations are relatively easy to illustrate. Also, the fact that the finite group actions on S_g for $2 \leq g \leq 4$ have been classified [7, 9, 27], it makes sense to verify our theory for surfaces with genera greater than four. Furthermore, our computations indicate that most non-split metacyclic actions on S_g for $g < 10$ are quaternionic. Thus, we will use Theorem 4.3.3 to classify the weak conjugacy classes of the metacyclic subgroups of $\mathrm{Mod}(S_3)$ and $\mathrm{Mod}(S_5)$ and weak conjugacy classes of the non-split metacyclic subgroups of $\mathrm{Mod}(S_{10})$ and $\mathrm{Mod}(S_{11})$.

For achieving our classification for $g = 3, 5, 10, 11$, we will need to first analyze the metacyclic groups that can act on S_g. In this regard, we use the signatures $\Gamma(\mathcal{O}_H)$ for all finite group actions on S_g for $2 \leq g \leq 48$ available at [39] and the list of metacyclic groups of order up to 500 available at [14]. These lists provide the GAP ids [18] of the groups acting on the surfaces along with the signatures $\Gamma(\mathcal{O}_H)$. Using this data and the basic properties of metacyclic groups (detailed in Section 3.2), we either construct all possible surface kernel maps $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \to H$ or compute all possible metacyclic data sets of genus g with the chosen signature, depending upon the relative ease of computation.

For example, in $g = 3$, when $H = \mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_2$, we have three possible signatures: $(1; 2), (0; 2, 2, 4, 4), (0; 2, 2, 2, 2, 2)$. For signature $(1; 2)$, condition (iii) of Definition 2.2.1 ensures $(c_{11}, n_{11}) = (0, 1)$ and condition (ii) (b) ensures $(c_{12}, n_{12}) = (1, 2)$. Hence, the metacyclic data set corresponding to signature $(1; 2)$ is

$$((2 \cdot 4, -1), 1; [(0, 1), (1, 2), 2]).$$

However, for signature $(0; 2, 2, 4, 4)$, it is easier to compute the possible maps $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \to H$, which are:

$$\begin{array}{llll} \xi_1 \mapsto G, & \xi_2 \mapsto G, & \xi_3 \mapsto F, & \xi_4 \mapsto F^3, \\ \xi_1 \mapsto G, & \xi_2 \mapsto G, & \xi_3 \mapsto F^3, & \xi_4 \mapsto F, \\ \xi_1 \mapsto GF, & \xi_2 \mapsto GF, & \xi_3 \mapsto F, & \xi_4 \mapsto F^3, \text{ and} \\ \xi_1 \mapsto GF, & \xi_2 \mapsto GF, & \xi_3 \mapsto F^3, & \xi_4 \mapsto F. \end{array}$$

By Definition 4.2.10, permutations of images of finite order generators lead to equivalent data sets, and so we have the following 2 equivalence classes of data sets:

$$((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4]) \text{ and}$$
$$((2 \cdot 4, -1), 0; [(1, 2), (1, 4), 2]_2, [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4]),$$

where the suffix refers to the multiplicity of the tuple in the metacyclic data set. For signature $(0; 2, 2, 2, 2, 2)$, by a similar argument, we can deduce that the only possible data set upto equivalence is

$$((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(1, 2), (1, 4), 2], [(1, 2), (3, 4), 2], [(0, 1), (1, 2), 2]).$$

Using similar computations, we will now provide a classification of the weak conjugacy classes of finite metacyclic subgroups of $\mathrm{Mod}(S_3)$ and $\mathrm{Mod}(S_5)$ (up to this equivalence) in Tables 4.1 and 4.2, respectively and classification of the weak conjugacy classes of finite non-split metacyclic subgroups of $\mathrm{Mod}(S_{10})$ and $\mathrm{Mod}(S_{11})$ (up to this equivalence) in Tables 4.3 and 4.4, respectively.

Group	**Weak conjugacy classes in** $\mathrm{Mod}(S_3)$	**Cyclic factors** $[D_G; D_F]$
$\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 3,-1),1;[(0,1),(1,3),3])$	$[(2,2,1;);(3,1;(1,3),(2,3))]$
	$((2\cdot 3,-1),0;[(1,2),(0,1),2]_3,[(1,2),(1,3),2],[(0,1),(2,3),3])^*$	$[(2,1;((1,2),4));(3,1;(1,3),(2,3))]$
$\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 4,-1),1;[(0,1),(1,2),2])$	$[(2,2,1;);(4,1;((1,2),2))]$
	$((2\cdot 4,-1),0;[(1,2),(0,1),2]_2,[(0,1),(1,4),4],[(0,1),(3,4),4])$	$[(2,1;((1,2),4));(4,0;((1,4),2),((3,4),2))]$
	$((2\cdot 4,-1),0;[(1,2),(1,4),2]_2,[(0,1),(1,4),4],[(0,1),(3,4),4])$	$[(2,2,1;);(4,0;((1,4),2),((3,4),2))]$
	$((2\cdot 4,-1),0;[(1,2),(0,1),2]_2,[(1,2),(1,4),2],[(1,2),(3,4),2],[(0,1),(1,2),2])$	$[(2,1;((1,2),4));(4,1;((1,2),2))]$
$\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4$	$((4\cdot 3,-1),0;[(1,4),(0,1),4],[(1,4),(1,3),4],[(1,2),(2,3),6])$	$[(4,0;((1,4),2),((1,2),3));(3,1;(1,3),(2,3))]$
	$((4\cdot 3,-1),0;[(3,4),(0,1),4],[(3,4),(1,3),4],[(1,2),(2,3),6])$	$[(4,0;((3,4),2),((1,2),3));(3,1;(1,3),(2,3))]$
$\mathcal{M}(2,6,3,-1)$	$((2\cdot 6,3,-1),0;[(1,4),(0,1),4],[(3,4),(5,6),4],[(0,1),(1,6),6])$	$[(4,0;((1,4),2),((1,2),3));(6,0;(1,6),(5,6),((1,2),2))]$
	$((2\cdot 6,3,-1),0;[(3,4),(0,1),4],[(1,4),(5,6),4],[(0,1),(1,6),6])$	$[(4,0;((3,4),2),((1,2),3));(6,0;(1,6),(5,6),((1,2),2))]$
$\mathbb{Z}_6 \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 6,-1),0;[(1,2),(0,1),2],[(1,2),(1,3),2],[(0,1),(1,2),2],[(0,1),(1,6),6])$	$[(2,1;((1,2),4);(6,0;((1,2),2),(1,6),(5,6))]$
	$((2\cdot 6,-1),0;[(1,2),(1,6),2],[(1,2),(1,2),2],[(0,1),(1,2),2],[(0,1),(1,6),6])$	$[(2,2,1;);(6,0;((1,2),2),(1,6),(5,6))]$
$\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$	$((4\cdot 4,-1),0;[(1,4),(0,1),4],[(0,1),(1,4),4],[(3,4),(1,4),4])^\dagger$	$[(4,0;((1,4),2),((1,2),3));(4,0;((1,4),2),((3,4),2))]$
	$((4\cdot 4,-1),0;[(1,4),(0,1),4],[(1,4),(1,4),4],[(1,2),(3,4),4])^\dagger$	$[(4,0;((1,4),2),((1,2),3));(4,1;((1,2),2))]$
	$((4\cdot 4,-1),0;[(3,4),(0,1),4],[(0,1),(1,4),4],[(1,4),(1,4),4])^\dagger$	$[(4,0;((3,4),2),((1,2),3));(4,0;((1,4),2),((3,4),2))]$
	$((4\cdot 4,-1),0;[(3,4),(0,1),4],[(3,4),(1,4),4],[(1,2),(3,4),4])^\dagger$	$[(4,0;((3,4),2),((1,2),3));(4,1;((1,2),2))]$
$\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$	$((2\cdot 8,5),0;[(1,2),(0,1),2],[(1,2),(7,8),8],[(0,1),(1,8),8])$	$[(2,1;((1,2),4));(8,0;(1,4),(1,8),(5,8))]$
	$((2\cdot 8,5),0;[(1,2),(0,1),2],[(1,2),(1,8),8],[(0,1),(7,8),8])$	$[(2,1;((1,2),4));(8,0;(3,4),(3,8),(7,8))]$
$\mathcal{M}(4,4,2,-1)$	$((4\cdot 4,2,-1),0;[(1,4),(1,4),2],[(1,8),(0,1),8],[(5,8),(3,4),8])$	$[(8,0;(1,4),(1,8),(5,8));(4,1;((1,2),2))]$
	$((4\cdot 4,2,-1),0;[(1,4),(1,4),2],[(3,8),(0,1),8],[(3,8),(3,4),8])$	$[(8,0;(3,4),(3,8),(7,8));(4,1;((1,2),2))]$
$\mathbb{Z}_7 \rtimes_2 \mathbb{Z}_3$	$((3\cdot 7,2),0;[(1,3),(0,1),3],[(2,3),(6,7),3],[(0,1),(1,7),7])$	$[(3,1;(1,3),(2,3));(7,0;(1,7),(2,7),(4,7))]$
	$((3\cdot 7,2),0;[(1,3),(0,1),3],[(2,3),(1,7),3],[(0,1),(6,7),7])$	$[(3,1;(1,3),(2,3));(7,0;(3,7),(6,7),(5,7))]$
$\mathbb{Z}_{12} \rtimes_5 \mathbb{Z}_2$	$((2\cdot 12,5),0;[(1,2),(0,1),2],[(1,2),(11,12),4],[(0,1),(1,12),12])$	$[(2,1;((1,2),4));(12,0;(1,12),(5,12),(1,2))]$
	$((2\cdot 12,5),0;[(1,2),(0,1),2],[(1,2),(5,12),4],[(0,1),(7,12),12])$	$[(2,1;((1,2),4));(12,0;(7,12),(11,12),(1,2))]$
	$((2\cdot 12,5),0;[(1,2),(1,6),2],[(1,2),(1,12),4],[(0,1),(1,12),12])$	$[(2,2,1;);(12,0;(1,12),(5,12),(1,2))]$
	$((2\cdot 12,5),0;[(1,2),(1,6),2],[(1,2),(7,12),4],[(0,1),(7,12),12])$	$[(2,2,1;);(12,0;(7,12),(11,12),(1,2))]$
$\mathcal{M}(2,4,2,-1)$	$((2\cdot 4,2,-1),1;[(0,1),(1,2),2])$	$[(4,1;((1,2),2));(4,1;((1,2),2))]$

Table 4.1: The weak conjugacy classes of finite non-abelian metacyclic subgroups of $\mathrm{Mod}(S_3)$. Note that each data set of type † is quaternionic, and therefore corresponds to the weak conjugacy action of a Q_8-action on S_2.(*The suffix refers to the multiplicity of the tuple in the split metacyclic data set.)

Group	**Weak conjugacy classes in** Mod(S_5)	**Cyclic factors** $[D_G; D_F]$
$\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 3,-1),1;[(0,1),(1,3),3]_2)^*$	$[(2,3,1;);(3,1;((1,3),2),((2,3),2))]$
	$((2\cdot 3,-1),0;[(1,2),(0,1),2]_4,[(0,1),(1,3),3],[(0,1),(2,3),3])$	$[(2,2;((1,2),4));(3,1;((1,3),2),((2,3),2))]$
$\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 4,-1),1;[(1,2),(0,1),2]_2)$	$[(2,2;((1,2),4));(4,2,1;)]$
	$((2\cdot 4,-1),1;[(1,2),(1,4),2]_2)$	$[(2,3,1;);(4,2,1;)]$
	$((2\cdot 4,-1),0;[(1,2),(0,1),2]_2,[(0,1),(1,2),2],[(0,1),(1,4),4]_2)$	$[(2,2;((1,2),4));(4,0;((1,2),2),((1,4),2),((3,4),2))]$
	$((2\cdot 4,-1),0;[(1,2),(1,4),2]_2,[(0,1),(1,2),2],[(0,1),(1,4),4]_2)$	$[(2,3,1;);(4,0;((1,2),2),((1,4),2),((3,4),2))]$
	$((2\cdot 4,-1),0;[(1,2),(0,1),2]_4,[(1,2),(1,4),2]_2)$	$[(2,1;((1,2),8));(4,2,1;)]$
	$((2\cdot 4,-1),0;[(1,2),(0,1),2]_2,[(1,2),(1,4),2]_4)$	$[(2,2;((1,2),4));(4,2,1;)]$
	$((2\cdot 4,-1),0;[(1,2),(0,1),2]_2,[(1,2),(1,4),2]_2,[(0,1),(1,2),2]_2)$	$[(2,2;((1,2),4));(4,1;((1,2),4))]$
$\mathbb{Z}_5 \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 5,-1),1;[(0,1),(1,5),5])$	$[(2,3,1;);(5,1;(1,5),(4,5))]$
	$((2\cdot 5,-1),1;[(0,1),(2,5),5])$	$[(2,3,1;);(5,1;(2,5),(3,5))]$
	$((2\cdot 5,-1),0;[(1,2),(0,1),2]_3,[(1,2),(4,5),2],[(0,1),(1,5),5])$	$[(2,2;((1,2),4));(5,1;(1,5),(4,5))]$
	$((2\cdot 5,-1),0;[(1,2),(0,1),2]_3,[(1,2),(3,5),2],[(0,1),(2,5),5])$	$[(2,2;((1,2),4));(5,1;(2,5),(3,5))]$
$\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4$	$((4\cdot 3,-1),1;[(0,1),(1,3),3])$	$[(4,2,1;);(3,1;((1,3),2),((2,3),2))]$
	$((4\cdot 3,-1),0;[(1,2),(0,1),2],[(0,1),(1,3),3],[(1,4),(0,1),4],[(1,4),(2,3),4])$	$[(4,0;((1,4),2),((1,2),5));(3,1;((1,3),2),((2,3),2))]$
	$((4\cdot 3,-1),0;[(1,2),(0,1),2],[(0,1),(1,3),3],[(3,4),(0,1),4],[(3,4),(2,3),4])$	$[(4,0;((3,4),2),((1,2),5));(3,1;((1,3),2),((2,3),2))]$
$\mathcal{M}(2,6,3,-1)$	$((2\cdot 6,3,-1),1;[(0,1),(1,3),3])$	$[(4,2,1;);(6,1;(1,3),(2,3))]$
	$((2\cdot 6,3,-1),0;[(0,1),(1,2),2],[(0,1),(1,3),3],[(1,4),(0,1),4],[(3,4),(1,6),4])$	$[(4,0;((1,4),2),((1,2),5));(6,0;(1,3),(2,3),((1,2),4))]$
	$((2\cdot 6,3,-1),0;[(0,1),(1,2),2],[(0,1),(1,3),3],[(3,4),(0,1),4],[(1,4),(1,6),4])$	$[(4,0;((3,4),2),((1,2),5));(6,0;(1,3),(2,3),((1,2),4))]$
$\mathbb{Z}_6 \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 6,-1),1;[(0,1),(1,3),3])$	$[(2,3,1;);(6,1;(1,3),(2,3))]$
	$((2\cdot 6,-1),0;[(1,2),(0,1),2]_2,[(0,1),(1,6),6],[(0,1),(5,6),6])$	$[(2,2;((1,2),4));(6,0;((1,6),2),((5,6),2))]$
	$((2\cdot 6,-1),0;[(1,2),(1,6),2]_2,[(0,1),(1,6),6],[(0,1),(5,6),6])$	$[(2,3,1;);(6,0;((1,6),2),((5,6),2))]$
	$((2\cdot 6,-1),0;[(1,2),(0,1),2],[(1,2),(2,3),2],[(1,2),(1,6),2]_2,[(0,1),(1,3),3])$	$[(2,2;((1,2),4));(6,1;(1,3),(2,3))]$
$\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$	$((4\cdot 4,-1),1;[(0,1),(1,2),2])$	$[(4,2,1;);(4,1;((1,2),4))]$
$\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$	$((2\cdot 8,5),1;[(0,1),(1,2),2])$	$[(2,3,1;);(8,1;((1,2),2))]$
	$((2\cdot 8,5),0;[(1,2),(1,4),4],[(0,1),(1,8),8],[(1,2),(1,8),8])$	$[(2,3,1;);(8,0;(1,2),(3,4),(1,8),(5,8))]$
	$((2\cdot 8,5),0;[(1,2),(1,4),4],[(1,2),(3,8),8],[(0,1),(3,8),8])$	$[(2,3,1;);(8,0;(1,2),(1,4),(3,8),(7,8))]$
$\mathcal{M}(4,4,2,-1)$	$((4\cdot 4,2,-1),1;[(0,1),(1,2),2])$	$[(8,1;((1,2),2));(4,1;((1,2),4))]$
	$((4\cdot 4,2,-1),0;[(0,1),(1,4),4],[(1,8),(0,1),8],[(7,8),(3,4),8])$	$[(8,0;(1,2),(3,4),(1,8),(5,8));(4,0;((1,2),2),((1,4),2),((3,4),2))]$
	$((4\cdot 4,2,-1),0;[(0,1),(1,4),4],[(3,8),(0,1),8],[(5,8),(3,4),8])$	$[(8,0;(1,2),(1,4),(3,8),(7,8));(4,0;((1,2),2),((1,4),2),((3,4),2))]$
$\mathbb{Z}_8 \rtimes_3 \mathbb{Z}_2$	$((2\cdot 8,3),0;[(1,2),(0,1),2],[(1,2),(1,4),2],[(1,2),(1,8),4],[(1,2),(3,8),4])$	$[(2,2;((1,2),4));(8,1;((1,2),2))]$
$\mathbb{Z}_8 \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 8,-1),0;[(1,2),(0,1),2]_2,[(1,2),(1,8),2],[(1,2),(5,8),2],[(0,1),(1,2),2])$	$[(2,2;((1,2),4));(8,1;((1,2),2))]$

Table 4.2: The weak conjugacy classes of finite non-abelian metacyclic subgroups of Mod(S_5).(*The suffix refers to the multiplicity of the tuple in the split metacyclic data set.)

Group	**Weak conjugacy classes in** $\mathrm{Mod}(S_5)$	**Cyclic factors** $[D_G; D_F]$
$\mathbb{Z}_5 \rtimes_{-1} \mathbb{Z}_4$	$((4\cdot 5,-1),0;[(1,4),(0,1),4],[(1,4),(1,5),4],[(1,2),(4,5),10])$	$[(4,0;((1,4),2),((1,2),5));(5,1;(2,5),(3,5))]$
	$((4\cdot 5,-1),0;[(1,4),(0,1),4],[(1,4),(2,5),4],[(1,2),(3,5),10])$	$[(4,0;((1,4),2),((1,2),5));(5,1;(1,5),(4,5))]$
	$((4\cdot 5,-1),0;[(3,4),(0,1),4],[(3,4),(1,5),4],[(1,2),(4,5),10])$	$[(4,0;((3,4),2),((1,2),5));(5,1;(2,5),(3,5))]$
	$((4\cdot 5,-1),0;[(3,4),(0,1),4],[(3,4),(2,5),4],[(1,2),(3,5),10])$	$[(4,0;((3,4),2),((1,2),5));(5,1;(1,5),(4,5))]$
$\mathcal{M}(2,10,5,-1)$	$((2\cdot 10,5,-1),0;[(1,4),(0,1),4],[(3,4),(9,10),4],[(0,1),(1,10),10])$	$[(4,0;((1,4),2),((1,2),5));(10,0;(1,10),(9,10),((1,2),2))]$
	$((2\cdot 10,5,-1),0;[(3,4),(0,1),4],[(1,4),(9,10),4],[(0,1),(1,10),10])$	$[(4,0;((3,4),2),((1,2),5));(10,0;(1,10),(9,10),((1,2),2))]$
	$((2\cdot 10,5,-1),0;[(3,4),(0,1),4],[(1,4),(7,10),4],[(0,1),(3,10),10])$	$[(4,0;((3,4),2),((1,2),5));(10,0;(3,10),(7,10),((1,2),2))]$
	$((2\cdot 10,5,-1),0;[(1,4),(0,1),4],[(3,4),(7,10),4],[(0,1),(3,10),10])$	$[(4,0;((1,4),2),((1,2),5));(10,0;(3,10),(7,10),((1,2),2))]$
$\mathbb{Z}_{10} \rtimes_{-1} \mathbb{Z}_2$	$((2\cdot 10,-1),0;[(1,2),(0,1),2],[(1,2),(1,5),2],[(0,1),(1,2),2],[(0,1),(3,10),10])$	$[(2,2;((1,2),4));(10,0;((1,2),2),(3,10),(7,10))]$
	$((2\cdot 10,-1),0;[(1,2),(0,1),2],[(1,2),(2,5),2],[(0,1),(1,2),2],[(0,1),(1,10),10])$	$[(2,2;((1,2),4));(10,0;((1,2),2),(1,10),(9,10))]$
	$((2\cdot 10,-1),0;[(1,2),(1,10),2],[(1,2),(3,10),2],[(0,1),(1,2),2],[(0,1),(3,10),10])$	$[(2,3,1;);(10,0;((1,2),2),(3,10),(7,10))]$
	$((2\cdot 10,-1),0;[(1,2),(1,10),2],[(1,2),(1,2),2],[(0,1),(1,2),2],[(0,1),(1,10),10])$	$[(2,3,1;);(10,0;((1,2),2),(1,10),(9,10))]$
$\mathbb{Z}_6 \rtimes_{-1} \mathbb{Z}_4$	$((4\cdot 6,-1),0;[(1,4),(0,1),4],[(3,4),(1,6),4],[(0,1),(5,6),6])$	$[(4,0;((1,4),2),((1,2),5));(6,0;((1,6),2),((5,6),2))]$
	$((4\cdot 6,-1),0;[(3,4),(0,1),4],[(1,4),(1,6),4],[(0,1),(5,6),6])$	$[(4,0;((3,4),2),((1,2),5));(6,0;((1,6),2),((5,6),2))]$
	$((4\cdot 6,-1),0;[(1,4),(0,1),4],[(1,4),(1,6),4],[(1,2),(5,6),6])$	$[(4,0;((1,4),2),((1,2),5));(6,1;(1,3),(2,3))]$
	$((4\cdot 6,-1),0;[(3,4),(0,1),4],[(3,4),(1,6),4],[(1,2),(5,6),6])$	$[(4,0;((3,4),2),((1,2),5));(6,1;(1,3),(2,3))]$
$\mathbb{Z}_{15} \rtimes_4 \mathbb{Z}_2$	$((2\cdot 15,4),0;[(1,2),(0,1),2],[(1,2),(14,15),6],[(0,1),(1,15),15])$	$[(2,2;((1,2),4));(15,0;(1,15),(4,15),(2,3))]$
	$((2\cdot 15,4),0;[(1,2),(0,1),2],[(1,2),(13,15),6],[(0,1),(2,15),15])$	$[(2,2;((1,2),4));(15,0;(2,15),(8,15),(1,3))]$
	$((2\cdot 15,4),0;[(1,2),(0,1),2],[(1,2),(8,15),6],[(0,1),(7,15),15])$	$[(2,2;((1,2),4));(15,0;(7,15),(13,15),(2,3))]$
	$((2\cdot 15,4),0;[(1,2),(0,1),2],[(1,2),(4,15),6],[(0,1),(11,15),15])$	$[(2,2;((1,2),4));(15,0;(11,15),(14,15),(1,3))]$
$\mathbb{Z}_5 \rtimes_{-1} \mathbb{Z}_6$	$((6\cdot 5,-1),0;[(1,2),(0,1),2],[(1,6),(1,5),6],[(1,3),(4,5),15])$	$[(6,0;(1,6),((1,3),2),(2,3),(1,2));(5,1;(2,5),(3,5))]$
	$((6\cdot 5,-1),0;[(1,2),(0,1),2],[(1,6),(3,5),6],[(1,3),(2,5),15])$	$[(6,0;(1,6),((1,3),2),(2,3),(1,2));(5,1;(1,5),(4,5))]$
	$((6\cdot 5,-1),0;[(1,2),(0,1),2],[(5,6),(1,5),6],[(2,3),(4,5),15])$	$[(6,0;(5,6),(1,3),((2,3),2),(1,2));(5,1;(2,5),(3,5))]$
	$((6\cdot 5,-1),0;[(1,2),(0,1),2],[(5,6),(3,5),6],[(2,3),(2,5),15])$	$[(6,0;(5,6),(1,3),((2,3),2),(1,2));(5,1;(1,5),(4,5))]$
$\mathbb{Z}_{20} \rtimes_9 \mathbb{Z}_2$	$((2\cdot 20,9),0;[(1,2),(0,1),2],[(1,2),(19,20),4],[(0,1),(1,20),20])$	$[(2,2;((1,2),4));(20,0;(1,20),(9,20),(1,2))]$
	$((2\cdot 20,9),0;[(1,2),(1,10),2],[(1,2),(1,20),4],[(0,1),(1,20),20])$	$[(2,3,1;);(20,0;(1,20),(9,20),(1,2))]$
	$((2\cdot 20,9),0;[(1,2),(0,1),2],[(1,2),(17,20),4],[(0,1),(3,20),20])$	$[(2,2;((1,2),4));(20,0;(3,20),(7,20),(1,2))]$
	$((2\cdot 20,9),0;[(1,2),(1,10),2],[(1,2),(19,20),4],[(0,1),(3,20),20])$	$[(2,3,1;);(20,0;(3,20),(7,20),(1,2))]$
	$((2\cdot 20,9),0;[(1,2),(0,1),2],[(1,2),(9,20),4],[(0,1),(11,20),20])$	$[(2,2;((1,2),4));(20,0;(11,20),(19,20),(1,2))]$
	$((2\cdot 20,9),0;[(1,2),(1,10),2],[(1,2),(11,20),4],[(0,1),(11,20),20])$	$[(2,3,1;);(20,0;(11,20),(19,20),(1,2))]$
	$((2\cdot 20,9),0;[(1,2),(0,1),2],[(1,2),(7,20),4],[(0,1),(13,20),20])$	$[(2,2;((1,2),4));(20,0;(13,20),(17,20),(1,2))]$
	$((2\cdot 20,9),0;[(1,2),(1,10),2],[(1,2),(9,20),4],[(0,1),(13,20),20])$	$[(2,3,1;);(20,0;(13,20),(17,20),(1,2))]$
$\mathcal{M}(2,4,2,-1)$	$((2\cdot 4,2,-1),0;[(1,4),(0,1),4]_2,[(0,1),(1,4),4]_2)$	$[(4,2;((1,4),2),((3,4),2),((1,2),2));(4,2;((1,4),2),((3,4),2),((1,2),2))]$
	$((2\cdot 4,2,-1),0;[(1,4),(0,1),4]_2,[(1,4),(1,4),4]_2)$	$[(4,2;((1,4),2),((3,4),2),((1,2),2));(4,1;((1,2),4))]$
	$((2\cdot 4,2,-1),0;[(0,1),(1,4),4]_2,[(1,4),(1,4),4]_2)$	$[(4,1;((1,2),4));(4,2;((1,4),2),((3,4),2),((1,2),2))]$

Continutation of Table 2.

Group	**Weak conjugacy classes in** $\mathrm{Mod}(S_{10})$	**Cyclic factors** $[D_G; D_F]$
$\mathcal{M}(2,4,2,-1)$	$((2\cdot 4,2,-1),0;[(0,1),(1,2),2],[(1,4),(0,1),4]_3,[(0,1),(1,4),4],[(1,4),(3,4),4])$	$[(4,1;(1,4),(3,4),((1,2),6));(4,0;((1,4),3),((3,4),3)((1,2),2))]$
	$((2\cdot 4,2,-1),0;[(0,1),(1,2),2],[(1,4),(0,1),4],[(0,1),(1,4),4]_3,[(3,4),(1,4),4])$	$[(4,0;((1,4),3),((3,4),3)((1,2),2));(4,1;(1,4),(3,4),((1,2),6))]$
	$((2\cdot 4,2,-1),0;[(0,1),(1,2),2],[(1,4),(0,1),4],[(0,1),(3,4),4],[(1,4),(1,4),4]_3)$	$[(4,1;(1,4),(3,4),((1,2),6));(4,1;(1,4),(3,4),((1,2),6))]$
	$((2\cdot 4,2,-1),0;[(0,1),(1,2),2]_4,[(1,4),(0,1),4],[(0,1),(1,4),4],[(3,4),(1,4),4])$	$[(4,0;(1,4),(3,4),((1,2),10));(4,0;(1,4),(3,4),((1,2),10))]$
	$((2\cdot 4,2,-1),1;[(1,4),(0,1),4],[(0,1),(1,4),4],[(1,4),(1,4),4])$	$[(4,2;(1,4),(3,4),((1,2),2));(4,2;(1,4),(3,4),((1,2),2))]$
$\mathcal{M}(2,8,4,-1)$	$((2\cdot 8,4,-1),0;[(1,4),(0,1),4],[(1,4),(1,8),4],[(0,1),(1,4),4],[(0,1),(1,8),8])$	$[(4,1;(1,4),(3,4),((1,2),6));(8,0;(1,8),(7,8),(1,4),(3,4),((1,2),2))]$
	$((2\cdot 8,4,-1),0;[(1,4),(0,1),4],[(1,4),(7,8),4],[(0,1),(1,4),4],[(0,1),(3,8),8])$	$[(4,1;(1,4),(3,4),((1,2),6));(8,0;(3,8),(5,8),(1,4),(3,4),((1,2),2))]$
$\mathcal{M}(2,12,6,7)$	$((2\cdot 12,6,7),0;[(1,4),(1,12),12],[(3,4),(1,3),12],[(0,1),(1,12),12])$	$[(4,2;(1,4),(3,4),((1,2),2));(12,0;(1,12),(7,12),((1,6),2))]$
	$((2\cdot 12,6,7),0;[(1,4),(1,6),12],[(3,4),(5,12),12],[(0,1),(5,12),12])$	$[(4,2;(1,4),(3,4),((1,2),2));(12,0;(5,12),(11,12),((5,6),2))]$
$\mathcal{M}(6,4,2,-1)$	$((6\cdot 4,2,-1),0;[(1,12),(0,1),12],[(1,12),(1,4),12],[(5,6),(3,4),12])$	$[(12,0;(1,12),(7,12),((1,6),2));(4,2;(1,4),(3,4),((1,2),2))]$
	$((6\cdot 4,2,-1),0;[(5,12),(0,1),12],[(5,12),(1,4),12],[(1,6),(3,4),12])$	$[(12,0;(5,12),(11,12),((5,6),2));(4,2;(1,4),(3,4),((1,2),2))]$
$\mathcal{M}(2,20,10,-1)$	$((2\cdot 20,10,-1),0;[(1,4),(0,1),4],[(1,4),(9,20),4],[(0,1),(1,20),20])$	$[(4,0;(1,4),(3,4),((1,2),10));(20,0;(1,20),(19,20),((1,2),2))]$
	$((2\cdot 20,10,-1),0;[(1,4),(0,1),4],[(1,4),(7,20),4],[(0,1),(3,20),20])$	$[(4,0;(1,4),(3,4),((1,2),10));(20,0;(3,20),(17,20),((1,2),2))]$
	$((2\cdot 20,10,-1),0;[(1,4),(0,1),4],[(1,4),(3,20),4],[(0,1),(7,20),20])$	$[(4,0;(1,4),(3,4),((1,2),10));(20,0;(7,20),(13,20),((1,2),2))]$
	$((2\cdot 20,10,-1),0;[(1,4),(0,1),4],[(1,4),(1,20),4],[(0,1),(9,20),20])$	$[(4,0;(1,4),(3,4),((1,2),10));(20,0;(9,20),(11,20),((1,2),2))]$

Table 4.3: The weak conjugacy classes of finite non-split metacyclic subgroups of $\mathrm{Mod}(S_{10})$.(*The suffix refers to the multiplicity of the tuple in the non-split metacyclic data set.)

Group	**Weak conjugacy classes in** $\mathrm{Mod}(S_{11})$	**Cyclic factors** $[D_G; D_F]$
$\mathcal{M}(2,12,6,-1)$	$((2\cdot 12,6,-1),1;[(0,1),(1,6),6])$	$[(4,3;((1,2),2));(12,1;(1,6),(5,6))]$
$\mathcal{M}(4,8,4,-1)$	$((4\cdot 8,4,-1),0;[(1,8),(0,1),8],[(7,8),(7,8),8],[(0,1),(1,8),8])$	$[(8,0;(1,8),(5,8),(1,4),((3,4),2),(1,2));(8,0;((1,8),2),((7,8),2)((1,2),2))]$
	$((4\cdot 8,4,-1),0;[(1,8),(1,8),8],[(7,8),(0,1),8],[(0,1),(1,8),8])$	$[(8,0;(3,8),(7,8),((1,4),2),(3,4),(1,2));(8,0;((1,8),2),((7,8),2)((1,2),2))]$
	$((4\cdot 8,4,-1),0;[(1,8),(0,1),8],[(7,8),(5,8),8],[(0,1),(3,8),8])$	$[(8,0;(1,8),(5,8),(1,4),((3,4),2),(1,2));(8,0;((3,8),2),((5,8),2)((1,2),2))]$
	$((4\cdot 8,4,-1),0;[(1,8),(3,8),8],[(7,8),(0,1),8],[(0,1),(3,8),8])$	$[(8,0;(3,8),(7,8),((1,4),2),(3,4),(1,2));(8,0;((3,8),2),((5,8),2)((1,2),2))]$
	$((4\cdot 8,4,-1),0;[(1,8),(1,8),8],[(5,8),(0,1),8],[(1,4),(1,8),8])$	$[(8,0;(1,8),(5,8),((1,4),3),(1,2));(8,1;(1,4),(3,4)((1,2),2))]$
	$((4\cdot 8,4,-1),0;[(1,8),(3,8),8],[(5,8),(0,1),8],[(1,4),(3,8),8])$	$[(8,0;(1,8),(5,8),((1,4),3),(1,2));(8,1;(1,4),(3,4)((1,2),2))]$
	$((4\cdot 8,4,-1),0;[(3,8),(1,8),8],[(3,8),(0,1),8],[(1,4),(1,8),8])$	$[(8,0;(3,8),(7,8),((3,4),3),(1,2));(8,1;(1,4),(3,4)((1,2),2))]$
	$((4\cdot 8,4,-1),0;[(3,8),(3,8),8],[(3,8),(0,1),8],[(1,4),(3,8),8])$	$[(8,0;(3,8),(7,8),((3,4),3),(1,2));(8,1;(1,4),(3,4)((1,2),2))]$
$\mathcal{M}(2,20,10,11)$	$((2\cdot 20,10,11),1;[(0,1),(1,2),2])$	$[(4,1;((1,2),10));(20,1;((1,2),2))]$
$\mathcal{M}(10,4,2,-1)$	$((10\cdot 4,2,-1),1;[(0,1),(1,2),2])$	$[(20,1;((1,2),2));(4,1;((1,2),10))]$

Table 4.4: The weak conjugacy classes of finite non-split metacyclic subgroups of $\mathrm{Mod}(S_{11})$ other than quaternions.(*The suffix refers to the multiplicity of the tuple in the non-split metacyclic data set.)

4.6 Geometric realizations of metacyclic actions

We begin this section with the following elementary lemma, which is crucial in the realization of metacyclic actions. This lemma is a direct generalization of [12, Lemma 6.1]. Here, note that the lemma can also be generalized to any finitely generated subgroup of $\text{Mod}(S_g)$.

Lemma 4.6.1. *Let $H = \langle F, G\rangle$ be a finite metacyclic subgroup of* $\text{Mod}(S_g)$*. Then*

$$\text{Fix}(H) = \text{Fix}(\langle F\rangle) \cap \text{Fix}(\langle G\rangle).$$

Proof. Suppose that $x \in \text{Fix}(H)$. Then $x \in \text{Fix}(\langle F\rangle)$ and $x \in \text{Fix}(\langle G\rangle)$, and so $x \in \text{Fix}(\langle F\rangle)\cap\text{Fix}(\langle G\rangle)$. Conversely, given $x \in \text{Fix}(\langle F\rangle)\cap\text{Fix}(\langle G\rangle)$, we have $F(x) = G(x) = x$. Hence $F^iG^j(x) = x$ for all i, j, that is, every element in H fixes x. Hence $x \in \text{Fix}(H)$. □

Now, we give an algorithm for obtaining the hyperbolic structures that realize finite metacyclic subgroups of $\text{Mod}(S_g)$ (up to weak conjugacy) as groups of isometries.

Step 1. Consider a weak conjugacy class represented by $(H, (\mathcal{G}, \mathcal{F}))$.

Step 2. Use Theorem 4.3.3 to determine the conjugacy classes D_F (resp. D_G) of the generators F (resp. G).

Step 3. We apply Lemma 4.6.1, and Theorems 2.3.1-2.3.2, to obtain the hyperbolic structures that realize H as a group of isometries.

We will now describe the geometric realizations of some metacyclic actions on S_3, S_5, S_{10} and S_{11} represented by the split and non-split metacyclic data sets listed in Tables 4.1, 4.2, 4.3 and 4.4 in Section 4.5. In particular, we will describe some realizations of split metacyclic actions on S_3 and S_5, which were not discussed in earlier sections.

It may be noted that the realizations of non-split metacyclic group actions on S_g are far more challenging as these groups are not realizable as isometry groups of $\mathbb{R}^3$. However, we will see later that the Proposition 5.2.6 enables us to realize the lifts of certain metacyclic actions under suitably chosen regular cyclic covers. Hence, by using Proposition 5.2.6 and Corollary 5.2.7, we describe nontrivial geometric realizations of some finite split metacyclic actions that are realized as lifts of non-split metacyclic actions on S_{10} and S_{11} under regular cyclic covers.

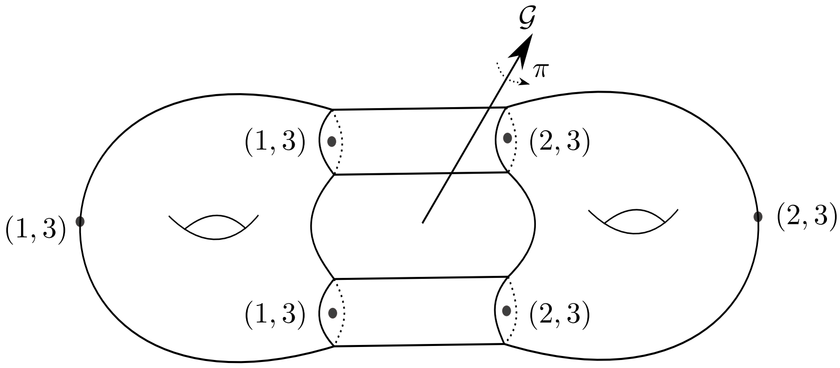

Figure 4.5: A realization of a D_6-action $\langle \mathcal{F}, \mathcal{G} \rangle$ on S_3, where $D_G = (2, 2, 1;)$ and $D_F = (3, 1; (1, 3), (2, 3))$. The action $\mathcal{F}$ is realized through two 1-compatibilities between two actions $\mathcal{F}'$ and $\mathcal{F}''$ on S_1 with $D_{F'} = (3, 0; ((1, 3), 3))$ and $D_{F''} = (3, 0; ((2, 3), 3)))$. The weak conjugacy class of $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$ is encoded by the first split metacyclic data set in Table 4.1.

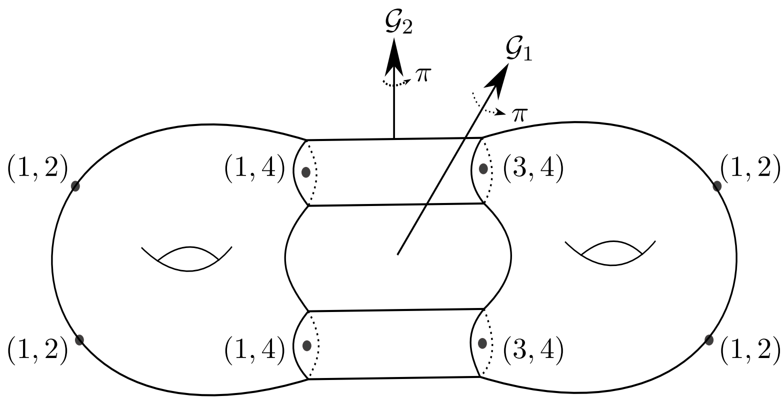

Figure 4.6: The realizations of two distinct D_8-actions $\langle \mathcal{F}, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}, \mathcal{G}_2 \rangle$ on S_3, where $D_F = (4, 1; ((1, 2), 2))$, $D_{G_1} = (2, 2, 1;)$, and $D_{G_2} = (2, 1; ((1, 2), 4)$. The action $\mathcal{F}$ is realized via two 1-compatibilities between two actions $\mathcal{F}'$ and $\mathcal{F}''$ on S_1, where $D_{F'} = (4, 0; ((1, 4), 2), (1, 2))$ and $D_{F''} = (4, 0; ((3, 4), 2), (1, 2))$. The weak conjugacy classes of $(\langle \mathcal{F}, \mathcal{G}_1 \rangle, (\mathcal{G}_1, \mathcal{F}))$ and $(\langle \mathcal{F}, \mathcal{G}_2 \rangle, (\mathcal{G}_2, \mathcal{F}))$ are encoded by split metacyclic data sets nos. 3 and 6, respectively, in Table 4.1.

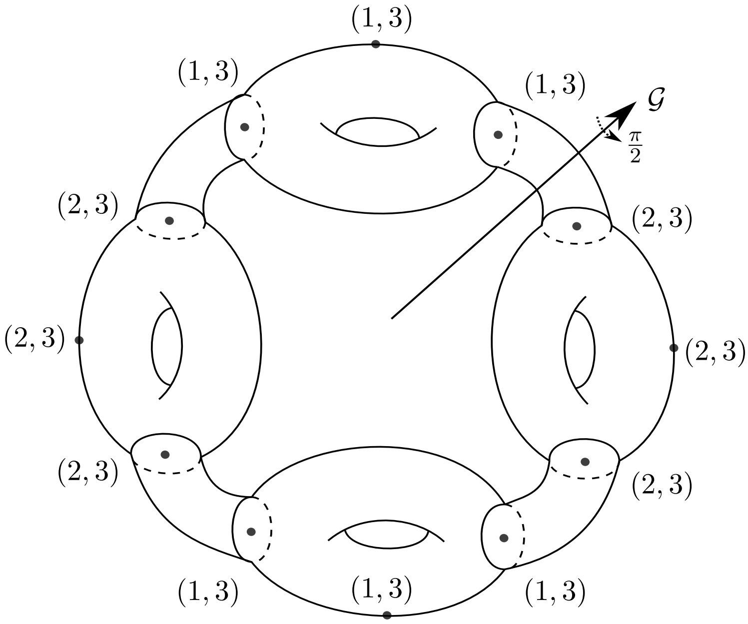

Figure 4.7: A realization of a $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4$-action $\langle \mathcal{F}, \mathcal{G} \rangle$ on S_5, where $D_G = (4, 2, 1;)$ and $D_F = (3, 1; ((1, 3), 2), ((2, 3), 2))$. The action $\mathcal{F}$ is realized via two 1-compatibilities between the action $\mathcal{F}'$ on two copies of S_2 with $D_{F'} = (3, 0; ((1, 3), 2), ((2, 3), 2))$. Furthermore, the action $\mathcal{F}'$ is realized by a 1-compatibility between the actions $\mathcal{F}''$ and $\mathcal{F}'''$ on S_1, where $D_{F''} = (3, 0; ((1, 3), 3))$ and $D_{F'''} = (3, 0; ((2, 3), 3))$. The weak conjugacy class of $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$ is encoded by the split metacyclic data set no. 14 in Table 4.2.

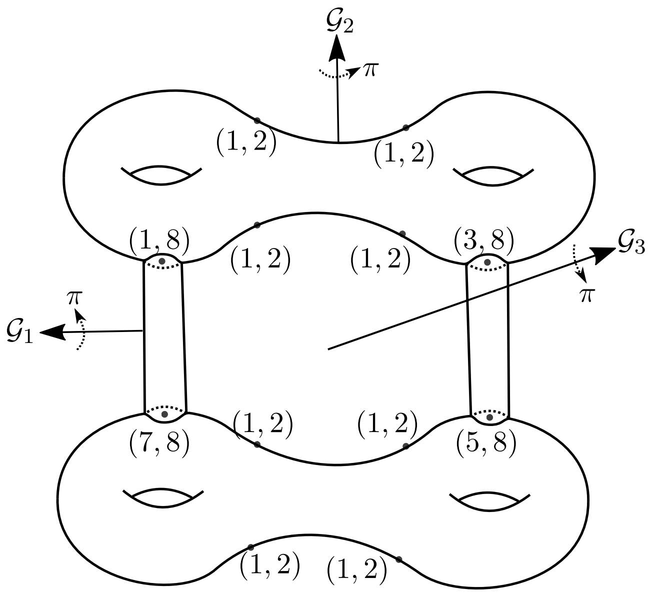

Figure 4.8: Realization of $\mathbb{Z}_8 \rtimes_{-1} \mathbb{Z}_2$-action $\langle \mathcal{F}, \mathcal{G}_1 \rangle$, $\mathbb{Z}_8 \rtimes_3 \mathbb{Z}_2$-action $\langle \mathcal{F}, \mathcal{G}_2 \rangle$ and $\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$-action $\langle \mathcal{F}, \mathcal{G}_3 \rangle$ on S_5, where $D_{G_1} = D_{G_2} = (2, 2; ((1, 2), 4))$, $D_{G_3} = (2, 3, 1;)$ and $D_F = (8, 1; ((1, 2), 2))$. The action $\mathcal{F}$ is realized via two 1-compatibilities between two actions $\mathcal{F}'$ and $\mathcal{F}''$ on S_2 where $D_{F'} = (8, 0; (1, 2), (1, 8), (3, 8))$ and $D_{F''} = (8, 0; (1, 2), (5, 8), (7, 8))$. The weak conjugacy class of $(\langle \mathcal{F}, \mathcal{G}_i \rangle, (\mathcal{G}_i, \mathcal{F}))$ $1 \leq i \leq 3$ is encoded by the split metacyclic data set nos. 26, 25, and 22, respectively, in Table 4.2.

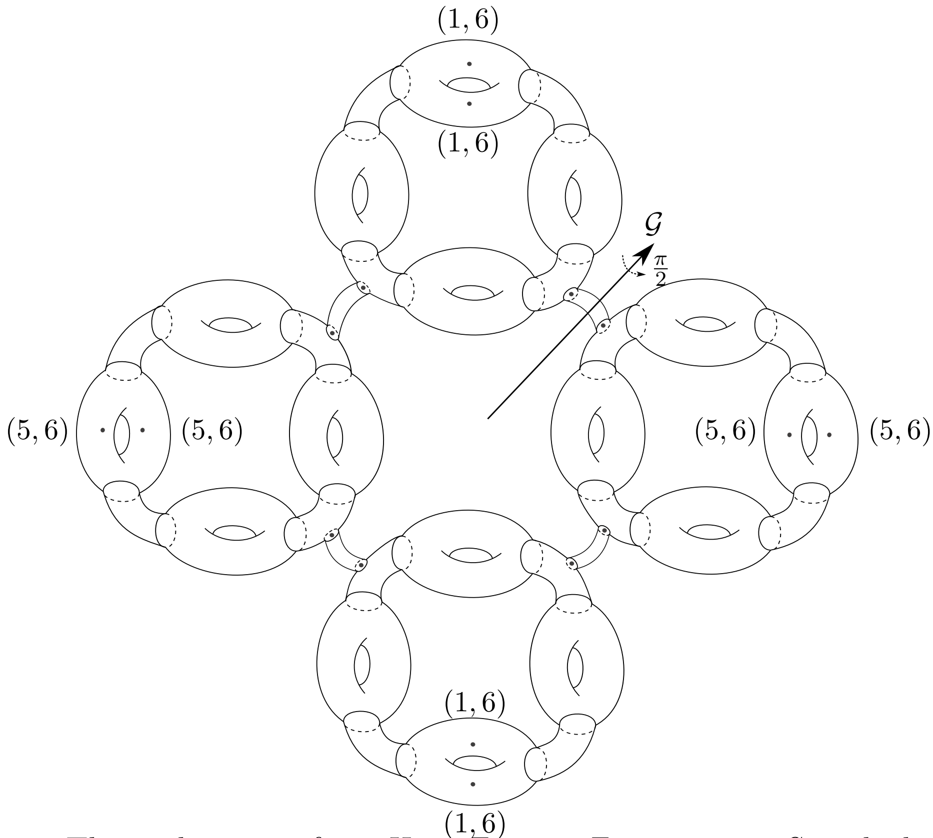

Figure 4.9: The realization of an $H = \mathbb{Z}_{12} \rtimes_{-1} \mathbb{Z}_4$-action on S_{21} which is the lift of a $\bar{H} = \mathrm{Dic}_6$-action on S_{11} under the regular cyclic cover p_2. Here, $H = \langle \mathcal{F}, \mathcal{G} \rangle$, where $D_G = (4, 6, 1;)$ and $D_F = (12, 1; ((1, 6), 2), ((5, 6), 2))$, and $\bar{H} = \langle \bar{\mathcal{F}}, \bar{\mathcal{G}} \rangle$, where $(\langle \bar{\mathcal{F}}, \bar{\mathcal{G}} \rangle, \bar{\mathcal{G}}, \bar{\mathcal{F}})$ is encoded by the first metacyclic data set in Table 4.4. Note that the $\mathcal{G}$ maps each orbit of the $\langle \mathcal{F} \rangle$-action of size 2 with local rotation angle $2\pi/6$ to an orbit with local rotation angle $10\pi/6$ (and vice versa).

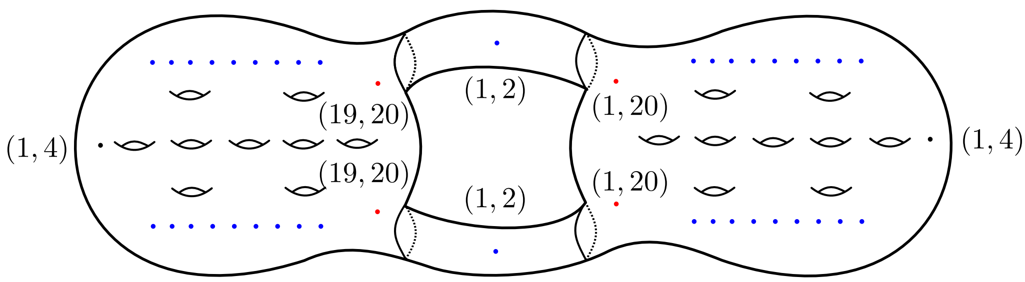

Figure 4.10: The realization of a $H = \mathbb{Z}_{20} \rtimes_{-1} \mathbb{Z}_4$-action on S_{19} which is the lift of a $\bar{H} = \mathrm{Dic}_{10}$-action on S_{10} under the regular cyclic cover p_2. Here, $H = \langle \mathcal{F}, \mathcal{G} \rangle$, where $D_G = (4, 0; ((1, 4), 2), ((1, 2), 19))$ and $D_F = (20, 0; ((1, 20), 2), ((19, 20), 2))$, and $\bar{H} = \langle \bar{\mathcal{F}}, \bar{\mathcal{G}} \rangle$, where $(\langle \bar{\mathcal{F}}, \bar{\mathcal{G}} \rangle, \bar{\mathcal{G}}, \bar{\mathcal{F}})$ is encoded by metacyclic data set no. 10 in Table 4.3. Note that the four fixed points of $\mathcal{F}$ (marked in red) form an orbit of size 4 under the $\langle \mathcal{G} \rangle$-action where each fixed point with local rotation $2\pi/20$ is mapped to fixed point with local rotation $38\pi/20$ (and vice versa). The point marked in blue are distinct size 2 orbits of the $\langle \mathcal{G} \rangle$-action, while the points marked in black are the fixed points of $\mathcal{G}$.

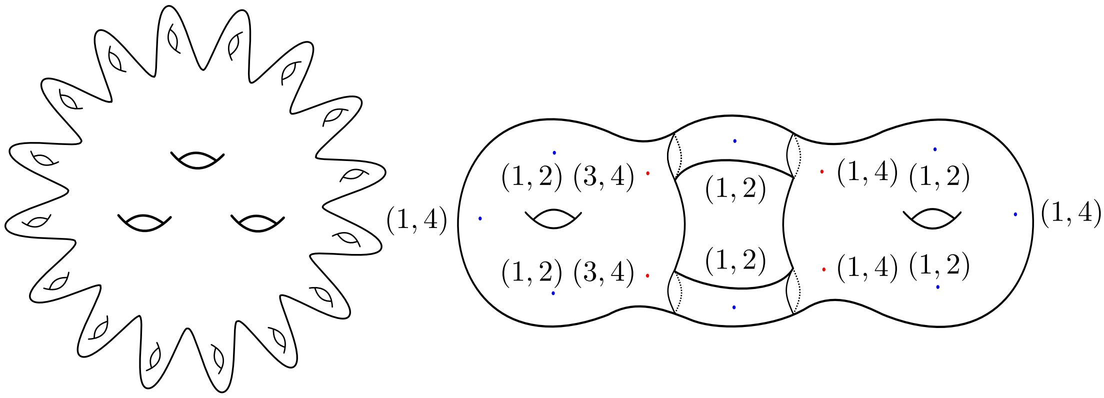

Figure 4.11: The realization of a $H = \mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$-action on S_{19} which is the lift of a $\bar{H} = Q_8$-action on S_{10} under the regular cyclic cover p_2. Here, $H = \langle \mathcal{F}, \mathcal{G} \rangle$, where $D_F = (4, 4; ((1, 4), 2), ((3, 4), 2))$ and $D_G = (4, 4; ((1, 4), 2), ((1, 2), 3))$, and $\bar{H} = \langle \bar{\mathcal{F}}, \bar{\mathcal{G}} \rangle$, where $(\langle \bar{\mathcal{F}}, \bar{\mathcal{G}} \rangle, \bar{\mathcal{G}}, \bar{\mathcal{F}})$ is encoded by metacyclic data set no. 5 in Table 4.3. Note that the H-action on S_{19} cyclically permutes the genera in the petals of the subfigure on the left. This H-action is induced by an analogous action of an $H' = \langle \mathcal{F}', \mathcal{G}' \rangle \cong \mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$ on S_3 (shown in the subfigure on the right) with $D_{F'} = (4, 0; ((1, 4), 2), ((3, 4), 2))$ and $D_{G'} = (4, 0; ((1, 4), 2), ((1, 2), 3))$. Note that the four fixed points of $\mathcal{F}'$ (marked in red) form an orbit of size 4 under the $\langle \mathcal{G}' \rangle$-action where each fixed point with local rotation $2\pi/4$ is mapped to fixed point with local rotation $6\pi/4$ (and vice versa). The remaining fixed (and orbit) points of the $\langle \mathcal{G}' \rangle$-action are marked in blue.

CHAPTER 5

LIFTABILITY UNDER REGULAR CYCLIC BRANCHED COVERS

In this chapter, we will derive several applications to our main theorem concerning the liftability viewpoint.

5.1 Liftability of torsion under finite cyclic covers

From the viewpoint of liftability, a metacyclic group $\langle \mathcal{F}, \mathcal{G} \rangle$ acts on S_g if and only if there exists $\bar{\mathcal{G}} \in \mathrm{Homeo}_k(D_{\langle \mathcal{F} \rangle})$ that lifts under the branched cover $S_g \to \mathcal{O}_{\langle \mathcal{F} \rangle}$ to $\mathcal{G}$. This is equivalent to requiring the existence of a short exact sequence:

$$1 \to \langle \mathcal{F} \rangle \to \langle \mathcal{F}, \mathcal{G} \rangle \to \langle \bar{\mathcal{G}} \rangle \to 1.$$

Let $S_{h,b}$ be the closed oriented surface of genus h with b marked points. Let $p : S_g \to S_{h,b}$ be a branched cover with finite deck-transformation group. Let $\mathrm{LMod}_p(S_{h,b})$ be the subgroup of $\mathrm{Mod}(S_{h,b})$ comprising all elements that have representatives that lift to homeomorphisms under p, and let $\mathrm{SMod}_p(S_g)$ be the subgroup of $\mathrm{Mod}(S_g)$ consisting of all elements represented by homeomorphisms that preserve the fibers under p (see [32]). The groups $\mathrm{LMod}_p(S_{h,b})$ (resp. $\mathrm{SMod}_p(S_g)$) are called the *liftable (resp. symmetric) mapping class groups* of p. Our main theorem can now be equivalently stated as follows.

Theorem 5.1.1 (Main theorem-Alternative version)**.** *Let $p : S_g \to S_{h,b}$ be an n-sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Then $\bar{G} \in \mathrm{LMod}_p(S_{h,b})$ lifts to a $G \in \mathrm{SMod}_p(S_g)$ if and only if there exists a metacyclic data set $\mathcal{D}$ of degree $u \cdot n$, twist factor k, amalgam r, and genus g such that $\mathcal{D}_1 = D_F$ and $\mathcal{D}_2 = D_G$.*

Thus, the main theorem also provides necessary and sufficient conditions under which periodic elements of mapping class groups lift under finite cyclic covers.

Remark 5.1.2. Given a metacyclic data set

$$\mathcal{D} = ((u \cdot n, r, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

encoding the weak conjugacy class represented by $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$, it follows from the Proposition 4.3.1 and the exact sequence

$$1 \to \langle \mathcal{F} \rangle \to \langle \mathcal{F}, \mathcal{G} \rangle \to \langle \bar{\mathcal{G}} \rangle \to 1.$$

that

$$D_{\bar{G}} = (u, g_0; (c'_{11}, n'_{11}), \ldots, (c'_{\ell 1}, n'_{\ell 1})),$$

where $n'_{i1} = \dfrac{u}{\gcd(c_{i1}\frac{m}{n_{i1}}, u)}$, and $c'_{i1}\dfrac{u}{n'_{i1}} \equiv c_{i1}\dfrac{m}{n_{i1}} \pmod u$. In fact, we can also recoverD_F (that encodes the $\langle \mathcal{F} \rangle$-action on S_g) using $D_{\bar{G}}$ in the following manner. From $\mathcal{D}$ and $D_{\bar{G}}$, we have

$$\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}) = (g_1; \underbrace{\frac{n_1}{n'_{11}}, \ldots, \frac{n_1}{n'_{11}}}_{\frac{u}{n'_{11}} \text{ times}}, \ldots, \underbrace{\frac{n_\ell}{n'_{\ell 1}}, \ldots, \frac{n_\ell}{n'_{\ell 1}}}_{\frac{u}{n'_{\ell 1}} \text{ times}})$$

where if $n_i/n'_{i1} = 1$, for some $1 \leq i \leq \ell$, then we exclude it from the signature, and $g_1 = g(D_{\bar{G}})$ is determined by Equation (2.2.2) of Definition 2.2.1. Thus, we get

$$D_F = \left(n, g_1; \left(d_{11}, \frac{n_1}{n'_{11}}\right), \ldots, \left(d_{1\frac{u}{n'_{11}}}, \frac{n_1}{n'_{11}}\right), \ldots, \left(d_{\ell 1}, \frac{n_\ell}{n'_{\ell 1}}\right), \ldots, \left(d_{\ell \frac{u}{n'_{11}}}, \frac{n_\ell}{n'_{\ell 1}}\right)\right),$$

where

$$\frac{d_{i1} n n'_{i1}}{n_i} \equiv b_i r + c_{i2}\frac{n}{n_{i2}} \sum_{j'=1}^{n'_{i1}} k^{c_{i1}\frac{m}{n_{i1}}(j'-1)} \pmod n,$$

$b_i \in \mathbb{N}$ such that $b_i u \equiv c_{i1}\frac{m}{n_{i1}} n'_{i1} \pmod m$, and

$$d_{ij_i} \equiv d_{i1} k^{(j_i - 1)} \pmod{\frac{n_i}{n'_{i1}}}\ 1 \leq i \leq l,\ 1 \leq j_i \leq \frac{u}{n'_{i1}}.$$

This leads us to the following corollary, which is an application of Theorem 5.1.1.

Corollary 5.1.3. *Let $p : S_{n(g-1)+1} \to S_g$ be an n-sheeted regular cover with deck transfor-*

mation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. *Suppose that there exists a* $\bar{G} \in \mathrm{LMod}_p(S_g)$ *of finite order with* $\mathcal{O}_{\langle \bar{\mathcal{G}} \rangle} \approx S_0$ *that lifts to a* $G \in \mathrm{SMod}_p(S_{n(g-1)+1})$. *Then* $H = \langle F, G \rangle$ *is a split metacyclic group.*

Proof. From Theorem 5.1.1, we have a metacyclic data set

$$\mathcal{D} = ((u \cdot n, r, k), 0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell])$$

of degree $u \cdot n$ with twist factor k, amalgam r, and genus $n(g-1)+1$. Following the notation in the proof of Proposition 4.3.1, let $\phi_H(\xi_i) = \mathcal{G}^{\gamma_i}\mathcal{F}^{\delta_i}$, where $\gamma_i = c_{i1}m/n_{i1}$, $\delta_i = c_{i2}n/n_{i2}$, and $\xi_i \in \pi_1^{orb}(\mathcal{O}_H)$ is the generator enclosing the cone point of order n_i. Now, as $\mathcal{F}$ generates a free action on $S_{n(g-1)+1}$, we have that $\langle \mathcal{G}^{\gamma_i}\mathcal{F}^{\delta_i} \rangle \cap \langle \mathcal{F} \rangle = \{id\}$. Hence, it follows that $\langle G^{\gamma_i}F^{\delta_i}, F \rangle$ is a split metacyclic group for all i, and consequently, $\langle G^{\gamma_i}, F \rangle$ is a split metacyclic group for all i.

Now, we claim that $\langle G^{\gamma_1}, \dots, G^{\gamma_\ell}, F \rangle$ is a split metacyclic group. We establish this claim by induction on ℓ. From the preceding argument, the statement holds for $\ell = 1$. For $\ell = 2$, we have to show that $\langle G^{\gamma_1}, G^{\gamma_2}, F \rangle$ is a split metacyclic group. We can write $\langle G^{\gamma_1}, G^{\gamma_2}, F \rangle = \langle G', F \rangle$, where $\langle G' \rangle = \langle G^{\gcd(\gamma_1,\gamma_2)} \rangle = \langle G^{\gamma_1}, G^{\gamma_2} \rangle$. Suppose we assume on the contrary that $(G')^a = F^b$, for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_n$ with $b \neq 0$. Then $\langle (G')^a \rangle \subseteq \langle G^{\gamma_1}, G^{\gamma_2} \rangle$, and so we have that $\langle (G')^{at} \rangle \subseteq \langle G^{\gamma_1} \rangle$ or $\langle G^{\gamma_2} \rangle$, for some t such that $(G')^{at} \neq 1$. Hence, it follows that $G^{\gamma_1 t_1} = (G')^{at} = F^{bt}$ or $G^{\gamma_2 t_2} = (G')^{at} = F^{bt}$, for some $t_1, t_2 \in \mathbb{Z}$, which contradicts the fact that $\langle G^{\gamma_i}, F \rangle$ is a split metacyclic group. Therefore, our claim holds true for $\ell = 2$.

Suppose we assume that our claim holds for $\ell - 1$. By similar arguments (as above), we have that $\langle G^{\gamma_1}, \dots, G^{\gamma_\ell}, F \rangle = \langle G'', G^{\gamma_\ell}, F \rangle$, where $\langle G'', F \rangle$ is split metacyclic group. So, it immediately follows from the case $\ell = 2$ that our claim holds for ℓ. Since ϕ_H is surjective, we have $\langle G^{\gamma_1}F^{\delta_1}, \dots, G^{\gamma_\ell}F^{\delta_\ell} \rangle = H$, and hence it follows that H is a split metacyclic group.

□

It is known that the bound on the order of a periodic mapping class $G \in \mathrm{Mod}(S_g)$ is $4g+2$ (see [49]) which is realized by the action $D_G = (4g+2, 0; (1,2), (1, 2g+1), (2g-1, 4g+2))$. This inspires the following corollary.

Corollary 5.1.4. *Let* $p : S_{n(g-1)+1} \to S_g$ *be a finite n-sheeted regular cover with deck* *transformation group* $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$*. If* n *is prime and* $(4g+2) \mid (n-1)$*, then there exists a* $\bar{G} \in \mathrm{LMod}_p(S_g)$ *with* $|\bar{G}| = 4g+2$*.*

Proof. Since $(4g+2) \mid (n-1)$, there exists a $k \in \mathbb{Z}_n^{\times}$ such that $|k| = 4g+2$. Let $G \in \mathrm{SMod}_p(S_{n(g-1)+1})$ be a lift of $\bar{G}$. Now from Proposition 4.3.1, it can be easily seen that the metacyclic data set $\mathcal{D} = ((n \cdot 4g+2, n, k), 0; [(1,2),(1,n),2], [(1,2g+1),(n-k^2,n),2g+1], [(2g-1,4g+2),(0,1),4g+2])$ represents the weak conjugacy class of $(\langle \mathcal{F}, \mathcal{G} \rangle, \mathcal{G}, \mathcal{F})$ with $\mathcal{D}_1 = D_F = (n,g,1;)$, $\mathcal{D}_2 = D_G = (4g+2, \frac{(n-1)(g-1)}{4g+2}; (1,2),(1,2g+1),(2g-1,4g+2))$ and $D_{\bar{G}} = (4g+2, 0; (1,2),(1,2g+1),(2g-1,4g+2))$. Hence, our assertion follows. □

5.2 Liftability of non-split metacylic actions under regular cyclic covers

Considering the fact that every non-split metacyclic group is a quotient of a split metacyclic group (Lemma 3.4.3), a natural question that arises is whether one can determine when a metacyclic action on S_g factor via a split metacyclic action. Equivalently, when does a metacyclic action on S_g lift under a regular cover to a split metacyclic action? In this section, we provide equivalent conditions for such a liftability to occur. In the following subsection, we begin by examining this problem for quaternionic actions.

5.2.1 Lifting generalized quaternionic actions

For $n \geq 2$, the generalized quaternion group $Q_{2^{n+1}}$ is a metacyclic group of order 2^{n+1} that admits the presentation

$$\langle x, y \mid x^{2^n} = y^4 = 1,\, x^{2^{n-1}} = y^2,\, y^{-1}xy = x^{-1} \rangle.$$

Remark 5.2.1. Let $\mathcal{D}$ be a split metacyclic data set of genus g, degree $4 \cdot 2^n$ and twist factor -1 (as in Definition 4.2.7) encoding a weak conjugacy class represented by $(H, (\mathcal{G}, \mathcal{F}))$. Suppose that $\mathcal{D}$ has the property that $[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j] = [(1,2),(1,2),2]$, for some $1 \leq j \leq \ell$. Then it follows from the proof of Proposition 4.3.1 that under the epimorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \to H$ which preserves the order of torsion elements, the tuple $[(1,2),(1,2),2]$ would correspond to an involution $\mathcal{G}^2\mathcal{F}^{2^{n-1}} \in H$ which defines a non-free action on S_g.

Remark 5.2.1 motivates the following definition.

Definition 5.2.2. A *quaternionic data set* is a split metacyclic data set of degree $4 \cdot 2^n$ that has the form

$$\mathcal{D} = ((4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

such that $[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j] \neq [(1,2), (1,2), 2]$, for $1 \leq j \leq \ell$.

Proposition 5.2.3. *For $g, n \geq 2$, quaternionic data sets of genus $2g-1$ and degree $4 \cdot 2^n$ correspond to $Q_{2^{n+1}}$-actions on S_g.*

Proof. Suppose that there exists an action of $H = Q_{2^{n+1}}$ on S_g. By Lemma 2.1.2, there exists an epimorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \to H$

$$\xi_i \xmapsto{\phi_H} y^{c_{i1} \frac{m}{n_{i1}}} x^{c_{i2} \frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell,$$

which is order-preserving on torsion elements. Let $H' = \mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4$. Since the canonical projection $q : H' \to H (\cong H'/\mathbb{Z}_2)$ preserves the order of torsion elements on $H' \setminus \ker q$, the map ϕ_H naturally factors via q. Thus, as there are exactly two possible choices for $\phi_H|_{\{\xi_i : 1 \leq i \leq \ell\}}$ that preserves the order, at least one of which yields an action H' on $S_{g'}$ (for some $g' > g$). A weak conjugacy class associated with this action is encoded by a split metacyclic data set of genus g' and degree $2^{n+2} = 4 \cdot 2^n$, which has one of the following forms

$$((4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell])$$

or

$$(4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \ldots, [(c'_{\ell 1}, n'_{\ell 1}), (c'_{\ell 2}, n'_{\ell 2}), n_\ell]),$$

where $c'_{\ell 1} \frac{4}{n'_{\ell 1}} \equiv c_{\ell 1} \frac{4}{n_{\ell 1}} + 2 \pmod 4$ and $c'_{\ell 2} \frac{2^n}{n'_{\ell 2}} \equiv c_{\ell 2} \frac{2^n}{n_{\ell 2}} + 2^{n-1} \pmod{2^n}$. Further, since $\ker q \cong \mathbb{Z}_2$ and q preserves the orders of all $x \in H' \setminus \ker q$, it follows that $\ker q$ acts freely on $S_{g'}$. Hence, it follows that $g' = 2g-1$ and further by Remark 5.2.1, both (possible) tuples cannot contain a triple of the type $[(1,2), (1,2), 2]$.

Conversely, if there exists a quaternionic data set $\mathcal{D}$ of genus $g' = 2g-1$ as in Definition 5.2.2. Then we obtain an epimorphism $\phi_{H'} : \pi_1^{orb}(\mathcal{O}_{H'}) \to H'$ which preserves the order of torsion elements, when composed with canonical projection $q : H' \to H$, yields an epimorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \to H$ which preserves the order of torsion elements, where

$\pi_1^{orb}(\mathcal{O}_{H'}) = \pi_1^{orb}(\mathcal{O}_H)$. Further, as $\mathcal{D}$ does not contain a triple of type $[(1,2),(1,2),2]$, $\ker q$ acts freely on $S_{g'}$, thereby yielding an action of $Q_{2^{n+1}}$ on S_g, where $g' = 2g-1$. □

Remark 5.2.4. A crucial step in the proof (of Proposition 5.2.3) is the establishment of the fact that the canonical projection $q : \mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4 \to Q_{2^{n+1}}$ is order-preserving on $(\mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4) \setminus \ker q$. However, it is interesting to note that this fact does not generalize to arbitrary metacyclic groups [23] arising as quotients of split metacyclic groups. This motivates the study of finite non-split metacyclic actions on surfaces, which we plan to undertake in future works.

Example 5.2.5. The split metacyclic data set in Example 4.3.4 is quaternionic. Hence, this represents the weak conjugacy class of an induced Q_8-action on S_3.

5.2.2 Lifting arbitrary non-split metacyclic groups

In the following proposition (which follows directly from Theorem 4.3.3), we provide necessary and sufficient conditions for lifting non-split metacyclic groups to split metacyclic groups under regular cyclic covers.

Proposition 5.2.6. *Let $p_\nu : S_{\nu(g-1)+1} \to S_g$ be a regular cyclic cover, and let $H = \langle F, G\rangle < \mathrm{Mod}(S_g)$ be a finite non-split metacyclic group such that $H \cong \mathcal{M}(u,n,r,k)$ and the weak conjugacy class $(H,(G,F))$ encoded by the data set*

$$\mathcal{D} = ((u \cdot n, r, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]).$$

Then H lifts under p_ν to a split metacyclic group $\tilde{H} = \langle \tilde{F}, \tilde{G}\rangle < \mathrm{Mod}(S_{v(g-1)+1})$ such that $\tilde{H} \cong \mathcal{M}(\nu u, n, n, k) \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_{\nu u}$ if and only if

(i) $\nu = n/r$ and

(ii) the weak conjugacy class $(\tilde{H},(\tilde{G},\tilde{F}))$ is encoded by the data set

$$\tilde{\mathcal{D}} = ((m \cdot n, n, k), g_0; [(c'_{11}, n'_{11}), (c'_{12}, n'_{12}), n_1], \dots, [(c'_{\ell 1}, n'_{\ell 1}), (c'_{\ell 2}, n'_{\ell 2}), n_\ell]),$$

where $m = u\frac{n}{r}$, $c'_{i1}\frac{m}{n'_{i1}} \equiv c_{i1}\frac{m}{n_{i1}} + a_i u \pmod{m}$ and $c'_{i2}\frac{n}{n'_{i2}} \equiv c_{i2}\frac{n}{n_{i2}} - a_i r \pmod{n}$, for some $a_i \in \mathbb{Z}$.

An immediate consequence of Proposition 5.2.6 is the following.

Corollary 5.2.7. *The actions on S_g of the metacyclic groups* Dic_n*,* $\mathrm{Dic}_n \times \mathbb{Z}_m$*, and* $\mathrm{Dic}_n \times \mathbb{Z}_m \times \mathbb{Z}_p$*, where n is even and m, p are odd with* $\gcd(p, n) = 1$*, factor via split metacyclic actions.*

Proposition 5.2.6 and Corollary 5.2.7 motivate the following conjecture.

Conjecture 5.2.8. *Every non-split metacyclic action on S_g lifts under a suitably chosen finite regular cyclic cover to a split metacyclic action.*

5.3 Lifting cyclic subgroups of mapping classes to metacyclic groups

For $n, g \geq 2$, let $p : S_{\tilde{g}} \to S_g$ be a covering map (that is possibly branched) with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$.

Remark 5.3.1. From the Birman-Hilden theory [6], we have the exact sequence

$$1 \to \langle F \rangle \to \mathrm{SMod}_p(S_{\tilde{g}}) \to \mathrm{LMod}_p(S_g) \to 1. \tag{B}$$

Let $G \in \mathrm{Mod}(S_g)$ be of finite order. Then $G \in \mathrm{LMod}_p(S_g)$ if and only if G has a lift $\tilde{G} \in \mathrm{SMod}_p(S_{\tilde{g}})$ of finite order so that the sequence (B) yields a sequence of the form

$$1 \to \langle F \rangle \to \langle F, \tilde{G} \rangle \to \langle G \rangle \to 1.$$

Thus, $G \in \mathrm{LMod}_p(S_g)$ if and only if for any lift $\tilde{G}$ of G, $\langle G \rangle$ lifts under p to a metacyclic group $\langle F, \tilde{G} \rangle$.

Proposition 5.3.2. *For $g, n \geq 2$, let $p : S_{n(g-1)+1} \to S_g$ be a regular cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Then any involution $G' \in \mathrm{Mod}(S_g)$ has a conjugate $G \in \mathrm{LMod}_p(S_g)$ with a lift $\tilde{G} \in \mathrm{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G} \rangle \cong D_{2n}$.*

Proof. Let $G' \in \mathrm{Mod}(S_g)$ be an involution. When $\mathcal{G}'$ generates a free action on S_g, it is easy to see that $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$ represents a weak conjugacy class in $\mathrm{Mod}(S_{n(g-1)+1})$ with $\langle F, \tilde{G} \rangle \cong D_{2n}$. Now, we assume that $\mathcal{G}'$ generates a non-free action with $D_{G'} = (2, g_0; ((1, 2), t))$. By Theorem 4.3.3 and Remark 5.3.1, it suffices to show that there exists a dihedral data set $\mathcal{D}$ of degree $2 \cdot n$ and genus $n(g-1)+1$ representing the weak conjugacy class of $(\langle F, \tilde{G}\rangle, (\tilde{G}, F))$. When $g_0 \geq 1$, we take $\mathcal{D}$ to be the tuple

$$((2 \cdot n, -1), g_0; \underbrace{[(1,2),(0,1),2], \ldots, [(1,2),(0,1),2]}_{t \text{ times}}),$$

and when $g_0 = 0$, $t \geq 4$, and so we take $\mathcal{D}$ to be the tuple

$$((2 \cdot n, -1), 0; \underbrace{[(1,2),(0,1),2], \ldots, [(1,2),(0,1),2]}_{t-2 \text{ times}},$$
$$[(1,2),(1,n),2], [(1,2),(1,n),2]).$$

It is an easy computation to check that $\mathcal{D}$ satisfies conditions (i)-(iv) of Definition 4.2.7 in both cases. When $g_0 = 0$, taking $v = 1$,

$$(p_1, \ldots, p_t) = (1, 0, \ldots, 0), \text{ and } (q_1, \ldots, q_t) = (0, \ldots, 0, 1, 1, 0)$$

we obtain condition (v). Moreover, when $g_0 = 1$, we take $v = 1$,

$$(p_1, \ldots, p_t) = (1, 0, \ldots, 0), \text{ and } (q_1, \ldots, q_t) = (0, \ldots, 0),$$

thereby verifying condition (vi). Thus, we have shown that $\mathcal{D}$ is a dihedral data set as desired. Finally, it follows from Theorem 4.3.3 that $\mathcal{D}$ encodes the weak conjugacy class of $(\langle F, \tilde{G}\rangle, (\tilde{G}, F))$. □

Note that the same $\mathbb{Z}_2$-action can lift to multiple non-isomorphic groups under a regular cyclic cover. We illustrate this phenomenon in the following example.

Example 5.3.3. Let $p : S_5 \to S_2$ be a regular 4-sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_4$ as illustrated in Figure 5.1 below. The involution $G \in \text{Mod}(S_2)$ with $D_G = (2, 1; (1,2), (1,2))$ has two distinct lifts $\tilde{G}_1, \tilde{G}_2 \in \text{SMod}_p(S_5)$ (as indicated) such that $\langle F, \tilde{G}_1\rangle \cong D_8$ and $\langle F, \tilde{G}_2\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Note that the weak conjugacy class of $(\langle \mathcal{F}, \tilde{\mathcal{G}}_1\rangle, (\tilde{\mathcal{G}}_1, \mathcal{F}))$ is represented by

$$((2 \cdot 4, -1), 1; [(1,2),(0,1),2], [(1,2),(0,1),2]).$$

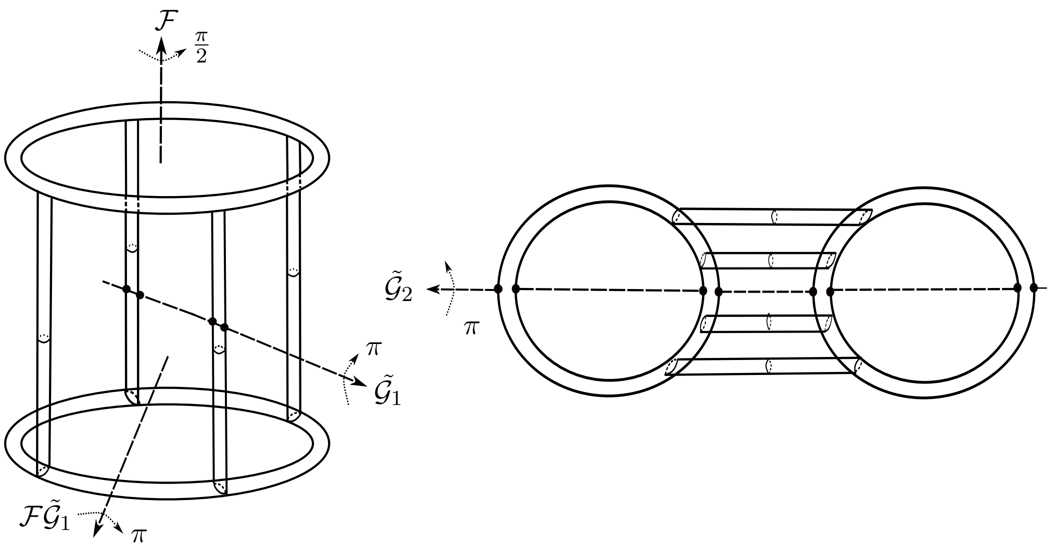

Figure 5.1: Two distinct lifts $\tilde{G}_1, \tilde{G}_2 \in \mathrm{SMod}(S_5)$ of an involution $G \in \mathrm{Mod}(S_2)$. Note that $\tilde{\mathcal{G}}_1$ has four fixed points, while $\tilde{\mathcal{G}}_2$ has eight fixed points.

The following proposition provides a sufficient condition for the liftability of $\mathbb{Z}_m$-actions whose corresponding orbifolds are spheres with a cone point of order m.

Proposition 5.3.4. *For $g, n \geq 2$, let $p : S_{n(g-1)+1} \to S_g$ be a regular n-sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Let $G' \in \mathrm{Mod}(S_g)$ be of order m such that $D_{G'} = (m, 0; (c_1, m_1), \ldots, (c_\ell, m_\ell))$ with $m_\ell = m$ (say). Then G' has a conjugate $G \in \mathrm{LMod}_p(S_g)$ with a lift $\tilde{G} \in \mathrm{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ if the following conditions hold.*

(a) There exists $a_1, \ldots, a_{\ell-1} \in \mathbb{Z}$, and $k \in \mathbb{Z}_n^\times$, $k^m \equiv 1 \pmod{n}$ such that

$$\sum_{i=1}^{\ell-1} a_i (k^{c_i \frac{m}{m_i}} - 1) \prod_{s=i+1}^{\ell-1} k^{c_s \frac{m}{m_s}} \equiv 0 \pmod{n}.$$

(b) For $1 \leq i \leq \ell - 1$, there exists $d_i, n_i \in \mathbb{Z}$ such that $\gcd(d_i, n_i) = 1$, $n_i \mid n$, $d_i \frac{n}{n_i} \equiv a_i (k^{c_i \frac{m}{m_i}} - 1) \pmod{n}$, and

$$\mathrm{lcm}(n_1, n_2, \ldots, n_{\ell-1}) = n.$$

Proof. By Theorem 4.3.3 and Remark 5.3.1, it suffices to show that the tuple

$$\mathcal{D} = ((m \cdot n, k), 0; [(c_1, m_1), (d_1, n_1), m_1], \ldots,$$
$$[(c_{\ell-1}, m_{\ell-1}), (d_{\ell-1}, n_{\ell-1}), m_{\ell-1}], [(c_\ell, m_\ell), (0, 1), m_\ell])$$

forms a split metacyclic data set of genus $n(g-1)+1$ that represents the weak conjugacy class of $(\langle F, \tilde{G}\rangle, (\tilde{G}, F))$ for some lift $\tilde{G}$ of G under p. It can be verified easily that $\mathcal{D}$ satisfies conditions (i)-(iii) of Definition 4.2.7, and further, condition (iv) follows from condition (a) in our hypothesis. By taking $v = 1$, $(p_1, \ldots, p_\ell) = (0, \ldots, 0, w)$ such that $wc_\ell \equiv 1 \pmod m$, we see that condition (v)(a) holds. Finally, condition (v)(b) follows from condition (b) in our hypothesis, and our assertion follows. □

Using similar arguments, we can show the following.

Proposition 5.3.5. *For $g, n \geq 2$, let $p : S_{n(g-1)+1} \to S_g$ be a regular n-sheeted cover with deck transformation group $\langle \mathcal{F}\rangle \cong \mathbb{Z}_n$. Let $G' \in \mathrm{Mod}(S_g)$ be of order m such that $D_{G'} = (m, 0; (c_1, m_1), \ldots, (c_\ell, m_\ell))$ with $m_i \neq m$, for $1 \leq i \leq \ell$. Then G' has a conjugate $G \in \mathrm{LMod}_p(S_g)$ with a lift $\tilde{G} \in \mathrm{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G}\rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ if the following conditions hold.*

(i) There exists $a_1, \ldots, a_\ell \in \mathbb{Z}$, and $k \in \mathbb{Z}_n^\times$, $k^m \equiv 1 \pmod n$ such that

$$\sum_{i=1}^{\ell} a_i(k^{c_i \frac{m}{m_i}} - 1) \prod_{s=i+1}^{\ell} k^{c_s \frac{m}{m_s}} \equiv 0 \pmod n.$$

(ii) There exists $(p_1, \ldots, p_{\ell v}), (q_1, \ldots, q_{\ell v}) \in \mathbb{Z}^{\ell v}$ and $v \in \mathbb{N}$ such that condition (v)(b) of Definition 4.2.7 holds, where for $1 \leq i \leq \ell$, we have

$$c_{i1}\frac{m}{n_{i1}} \equiv c_i \frac{m}{m_i} \pmod m \text{ and } c_{i2}\frac{n}{n_{i2}} \equiv a_i(k^{c_i \frac{m}{m_i}} - 1) \pmod n.$$

A consequence of Propositions 5.3.4-5.3.5 is the following.

Corollary 5.3.6. *For $g \geq 2$ and prime n, let $p : S_{n(g-1)+1} \to S_g$ be a regular n-sheeted cover with deck transformation group $\langle \mathcal{F}\rangle \cong \mathbb{Z}_n$. Let $G' \in \mathrm{Mod}(S_g)$ be of order m such that the genus of $\mathcal{O}_{\langle \mathcal{G}'\rangle}$ is zero. Then G' has a conjugate $G \in \mathrm{LMod}_p(S_g)$ with a lift $\tilde{G} \in \mathrm{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G}\rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ if there exists $k \in \mathbb{Z}_n^\times$ such that $|k| = m$.*

Proof. Let $D_{G'} = (m, 0; (c_1, m_1), \ldots, (c_\ell, m_\ell))$. First, let us assume (without loss of generality) that $m_\ell = m$. By choosing

$$(a_1, \ldots, a_{\ell-1}) = (0, \ldots, 0, 1, -(k^{c_{\ell-2}\frac{m}{m_{\ell-2}}} - 1) \cdot k^{c_{\ell-1}\frac{m}{m_{\ell-1}}} \cdot (k^{c_{\ell-1}\frac{m}{m_{\ell-1}}} - 1)^{-1}),$$

we see that condition (i) of Proposition 5.3.4 holds true. Moreover, since $|k| = m$, we have $\gcd((k^{c_{\ell-2}\frac{m}{m_{\ell-2}}} - 1), n) = 1$, and so condition (ii) also holds, and our assertion follows.

Similarly, for the case when each $m_i < m$ for $1 \leq i \leq \ell$, the result follows by taking

$$(a_1, \ldots, a_\ell) = (0, \ldots, 0, 1, -(k^{c_{\ell-1}\frac{m}{m_{\ell-1}}} - 1) \cdot k^{c_\ell \frac{m}{m_\ell}} \cdot (k^{c_\ell \frac{m}{m_\ell}} - 1)^{-1}),$$

and applying Proposition 5.3.5. □

CHAPTER 6

APPLICATIONS

In this chapter, we will derive several applications to our main theorem.

6.1 Bound on the order of a non-split metacyclic action

In this section, we derive a realizable bound for the order of a non-split metacyclic subgroup of $\mathrm{Mod}(S_g)$.

We recall $\mathrm{Dic}_n := \mathcal{M}(2, 2n, n, -1)$ the dicyclic group of order $4n$. We will now derive a realizable bound on the order of a finite non-split metacyclic subgroup of $\mathrm{Mod}(S_g)$.

Proposition 6.1.1. *Suppose that $H < \mathrm{Mod}(S_g)$ is a finite non-split metacyclic group. Then $|H| \leq 4g$ and this bound is realized when g is even and $H \cong \mathrm{Dic}_g$.*

Proof. We will show that if $H < \mathrm{Mod}(S_g)$ such that $|H| > 4g$, then H cannot be a non-split metacyclic group. If $\Gamma(\mathcal{O}_H) = (g_0; n_1, n_2, \ldots, n_\ell)$, then H satisfies the Riemann-Hurwitz equation:

$$\frac{2g-2}{|H|} = 2g_0 - 2 + \sum_{i=1}^{\ell} \left(1 - \frac{1}{n_i}\right).$$

When $|H| > 4g$, we have

$$2g_0 - 2 + \sum_{i=1}^{\ell} \left(1 - \frac{1}{n_i}\right) = \frac{2g-2}{|H|} < \frac{2g-2}{4g} = \frac{g-1}{2g} < \frac{1}{2}, \tag{6.1.2}$$

from which it follows that $g_0 = 0$ and $\ell = 3$ or 4.

From Proposition 3.4.5, if $g_0 = 0$, $\ell = 3$, and there is a cone point of prime order, then H cannot be a non-split metacyclic group. So, by Equation (6.1.2), when H is a non-split metacyclic group with $|H| > 4g$, the possible signatures for $\pi_1^{orb}(\mathcal{O}_H)$ are

$(0;2,2,3,3)$, $(0;2,2,3,4)$, $(0;2,2,3,5)$, $(0;2,2,2,n)$, $(0;4,4,n)$, $(0;4,6,6)$, $(0;4,6,8)$, $(0;4,6,9)$, and $(0;4,6,10)$, where $n < 2g$. We will now show that none of these signatures will arise from a non-split metacyclic action.

Assume that H is a metacyclic group. Then $H = \langle F, G\rangle$, where $\mathcal{F} \in \text{Homeo}^+(S_g)$ with $\mathcal{O}_{\langle \mathcal{F}\rangle} \approx S_{h,b}$ and $\bar{\mathcal{G}} \in \text{Homeo}_k(D_{\langle \mathcal{F}\rangle})$. From Remark 5.1.2, we have that $\Gamma(\mathcal{O}_{\langle \bar{\mathcal{G}}\rangle}) = (0; m_1, m_2, \ldots, m_\ell)$, where $m_i \mid n_i$. First, we will consider the case when $\ell = 4$. If $h \neq 0$, from Proposition 2.2.3, it follows that $\Gamma(\mathcal{O}_{\langle \bar{\mathcal{G}}\rangle})$ equals either $(0;2,2,3,3)$ or $(0;2,2,2,2)$. Thus, $\mathcal{F}$ either generates a free action or $\Gamma(\mathcal{O}_{\langle \mathcal{F}\rangle}) = (1; \frac{n}{2})$. But, by Proposition 2.2.3 and Corollary 5.1.3, we can see that neither of these possibilities occur when H is a non-split metacyclic group. If $h = 0$, then $\Gamma(\mathcal{O}_{\langle \bar{\mathcal{G}}\rangle}) = (0;u,u)$, where $u \in \{2,3\}$. Again, from Remark 5.1.2, we see that $\Gamma(\mathcal{O}_{\langle \mathcal{F}\rangle})$ equals one of $(0;2,2,2,2,2,2)$, $(0;3,3,3,3)$, $(0;3,3,4,4)$, $(0;2,2,2,3,3)$, $(0;3,3,5,5)$, $(0;2,2,n,n)$, or $(0;2,2,2,2,n/2)$. Hence, either H is a split metacyclic group or $|H| \leq 4g$. This completes our argument for $\ell = 4$.

Now, for $\ell = 3$, if $h \neq 0$, from Proposition 2.2.3, we get $\Gamma(\mathcal{O}_{\langle \bar{\mathcal{G}}\rangle})$ equals either $(0;2,4,4)$ or $(0;2,3,6)$. This implies that $\Gamma(\mathcal{O}_{\langle \mathcal{F}\rangle})$ equals one of $(1; \frac{n}{2}, \frac{n}{2})$, $(1;2,2,2,2,2)$, $(1;2,3,3)$, or $(1;2,2,2,3,3)$. Then Proposition 2.2.3 would imply that the final three signatures are not possible for cyclic actions. Moreover, by following the argument used in the proof of Corollary 5.1.3, we can conclude that the first signature corresponds to a split metacyclic action. If $h = 0$, then $\Gamma(\mathcal{O}_{\langle \bar{\mathcal{G}}\rangle}) = (0;u,u)$, where $u \in \{2,3,4,6\}$ and $\Gamma(\mathcal{O}_{\langle \mathcal{F}\rangle})$ equals one of signatures $(0;2,2,n,n)$, $(0;2,4,4,\frac{n}{2})$, $(0;n,n,n,n)$, $(0;4,4,4,4,\frac{n}{4})$, $(0;2,3,6,6)$, $(0;3,3,4,4)$, $(0;2,2,4,4,4)$, $(0;4,4,4,4,4,4)$, $(0;2,3,8,8)$, $(0;3,4,4,4)$, $(0;2,4,6,6)$, $(0;2,6,6,6,6)$, $(0;2,3,9,9)$, $(0;2,3,4,4,4)$, $(0;2,3,10,10)$, $(0;3,4,4,5)$, or $(0;2,5,6,6)$. By similar arguments as before, we can conclude that $\Gamma(\mathcal{O}_{\langle \mathcal{F}\rangle})$ equals either $(0;n,n,n,n)$ or $(0;2,2,n,n)$ (as other signatures either do not correspond to a cyclic action or to a metacyclic action such that $|H| > 4g$ on S_g). Furthermore, from the arguments in the proof of Corollary 5.1.3, it follows that the signature $(0;n,n,n,n)$ corresponds to a split metacyclic action. For the signature $(0;2,2,n,n)$, the fact that $|H| > 4g$ implies n is odd. Thus, we have $|H| = 4n$, where n is odd with $|\bar{\mathcal{G}}| = 2$, which would imply that H is a split metacyclic group. This concludes the argument for this case.

For the realization of the bound, when $H \cong \text{Dic}_g$ and g is even, we can see that data set

$$\mathcal{D} = ((2 \cdot 2g, g, -1), 0; [(1,4),(0,1),4],[(1,4),(1,2g),4],[(0,1),(2g-1,2g),2g])$$

represents the weak conjugacy class of $(H, (\mathcal{G}, \mathcal{F}))$. □

An immediate consequence of Proposition 6.1.1 is the following.

Corollary 6.1.3. *Suppose that $H < \text{Mod}(S_g)$ is a finite non-split metacyclic group. Then there exists no irreducible periodic mapping class in H.*

Proof. Since $|H| \leq 4g$ and H is non-split, we have $|F| \leq 2g$ for any $F \in H$. Our assertion now follows from the fact that the order of any irreducible periodic mapping class is at least $2g+1$. □

Corollary 6.1.3 further yields the following.

Corollary 6.1.4. *Suppose that $H = \langle F, G \rangle < \text{Mod}(S_g)$ is a finite non-split metacyclic group.*

(i) If $g = 2$, then $|F| \leq 2g$, $|G| \leq 2g$, and $|\bar{G}| \leq g$. Moreover, these bounds are realized when $H \cong Q_8$.

(ii) If $g > 2$, then $|F| \leq 2g$, $|G| \leq 2g-2$, and $|\bar{G}| \leq g-1$. Moreover, the bound on $|F|$ is realized when $H \cong \text{Dic}_g$, where g is even, while the bounds on $|G|$ and $|\bar{G}|$ are realized when $H \cong Q_8 \times \mathbb{Z}_{\frac{g-1}{2}}$, where $g \equiv 3 \pmod 4$.

Proof. From Corollary 6.1.3, we have that $|F|, |G| \leq 2g$. Also, as $|\bar{G}| < |G|$, from Lemma 4.1.2, we have $|\bar{G}| \leq g$. Hence, the assertion in (i) follows immediately from Proposition 6.1.1.

Furthermore, by Proposition 6.1.1, the bound $4g$ on $|H|$ is realized when $H \cong \text{Dic}_g$ and g is even. It is apparent that $|F| = 2g$ in H, which realizes the required bound in the first part of (ii). Moreover, from the proof of Proposition 6.1.1 it is apparent that Dic_g does not realize the bounds for $|G|$ and $|\bar{G}|$. However, it can be easily seen that the bounds on $|G|$ and $|\bar{G}|$ are realized when $H \cong Q_8 \times \mathbb{Z}_{\frac{g-1}{2}}$ with the weak conjugacy class $(H, (G, F))$ represented by the metacyclic data set

$$(((g-1) \cdot 4, 2, -1), 1; [(0,1),(1,2),2]).$$

□

We conclude this section with the following application of Corollary 6.1.3.

Corollary 6.1.5. *Let $p : S_g \to S_{0,3}$ be a finite n-sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$, where $D_F = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n_3))$. If $\bar{G} \in \mathrm{LMod}_p(S_{0,3})$ lifts to a $G \in \mathrm{SMod}_p(S_g)$, then $H = \langle F, G \rangle$ is a split metacyclic group such that either $H \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_2$ or $H \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_3$. Furthermore, a $G' \in \mathrm{Mod}(S_{0,3})$ of order m has a conjugate $G \in \mathrm{LMod}_p(S_{0,3})$ that lifts under p if and only if one of the following conditions hold.*

(a) $D_F = (n, 0; (c_1, n_1), (c_2, n), (c_2 k, n))$ for some $k \in \mathbb{Z}_n^\times$ such that $k^2 \equiv 1 \pmod{n}$.

(b) $D_F = (n, 0; (c_1, n), (c_1 k, n), (c_1 k^2, n))$ for some $k \in \mathbb{Z}_n^\times$ such that $k^3 \equiv 1 \pmod{n}$.

Proof. Since F is an irreducible mapping class, from Corollary 6.1.3, it follows that H is a split metacyclic group. For proving the second part of the corollary, suppose that $G' \in \mathrm{Mod}(S_{0,3})$ has a conjugate $G \in \mathrm{LMod}_p(S_{0,3})$ with a lift $\tilde{G} \in \mathrm{SMod}_p(S_g)$ such that $H = \langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$.

First, we claim that the n_i, for $1 \leq i \leq 3$, are not distinct. We assume on the contrary that the n_i, for $1 \leq i \leq 3$, are indeed distinct. Since $\mathcal{G}' \in \mathrm{Homeo}_k(D_{\langle \mathcal{F} \rangle})$ and $|\mathcal{G}'| > 1$, it has to fix all three cone points of $\mathcal{O}_{\langle \mathcal{F} \rangle}$, which contradicts the fact that any nontrivial automorphism of the sphere has exactly two fixed points. Thus, the following two cases arise.

Case 1: $n_2 = n_3 = n \neq n_1$. In this case, $\mathcal{G}'$ fixes the cone point, say of order n_1, and should permute the remaining 2 cone points of orders n_2 and n_3. This implies that D_F takes the form in condition (a) in our hypothesis (by Definition 4.1.1), and hence $H = \langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_2$.

Case 2: $n_i = n, 1 \leq i \leq 3$. In this case, if $\mathcal{G}'$ permutes all the three cone points cyclically, then D_F takes the form in condition (b) in our hypothesis, and hence $H \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_3$. Alternatively, $\mathcal{G}'$ could also fix a cone point of order n and permute the remaining 2 cone points, in which case, D_F will take the form in condition (a). Here, note that $H \cong \mathbb{Z}_{2n}$.

Conversely, let $D_F = (n, 0; (c_1, n_1), (c_2, n), (c_2 k, n))$ for some $k \in \mathbb{Z}_n^\times$ such that $k^2 \equiv 1 \pmod{n}$. Up to conjugacy, let $\mathcal{G}' \in \mathrm{Homeo}_k(D_{\langle \mathcal{F} \rangle})$ be an involution so that $\mathcal{G}'$ maps the cone point represented by (c_2, n) to the cone point represented by $(c_2 k, n)$. To prove our assertion, it would suffice to show the existence of an involution $\mathcal{G} \in \text{Homeo}^+(S_g)$ that induces $\mathcal{G}'$. This amounts to showing that there exists a split metacyclic data set $\mathcal{D}$ of degree $2 \cdot n$ with twist factor k encoding the weak conjugacy class $(H, (\mathcal{G}, \mathcal{F}))$ so that D_G has degree 2. Consider the tuple $((2 \cdot n, k), 0; [(1,2), (0,1), 2], [(1,2), (n - c_2, n), 2n_1], [(0,1), (c_2, n), n])$. A simple computation would reveal that conditions (i) - (iv) of Definition 4.2.7 hold true. Condition (v) is true by taking $v = 1$, $(p_1, p_2, p_3) = (1, 0, 0)$ and $(q_1, q_2, q_3) = (0, 0, w)$ such that $wc_2 \equiv 1 \pmod{n}$, which proves our claim.

For the case when $D_F = (n, 0; (c_1, n), (c_1 k, n), (c_1 k^2, n))$ for some $k \in \mathbb{Z}_n^\times$ such that $k^3 \equiv 1 \pmod{n}$, let $\mathcal{G}' \in \text{Homeo}_k(D_{\langle \mathcal{F} \rangle})$ be of order 3 so that for $1 \leq i \leq 2$, $\mathcal{G}'^i$ maps the cone point represented by (c_1, n) to the cone point represented by $(c_1 k^{3-i}, n)$. By similar argument as above, we can show that the tuple $((3 \cdot n, k), 0; [(1,3), (0,1), 3], [(2,3), (n - c_1, n), 3], [(0,1), (c_1, n), n])$ forms a split metacyclic data set of degree $3 \cdot n$ with twist factor k. □

Example 6.1.6. For $i = 1, 2$, consider the branched cover $p : S_3 \to \mathcal{O}_{\langle \mathcal{F}_i \rangle} (\approx S_{0,3})$, where $D_{F_1} = (8, 0; (1,4), (1,8), (5,8))$ and $D_{F_2} = (8, 0; (3,4), (3,8), (7,8))$. Then (up to conjugacy) the order 2 mapping class $G \in \text{LMod}_p(S_{0,3})$ represented by an automorphism $\mathcal{G} \in \text{Homeo}_5(D_{\langle \mathcal{F}_i \rangle})$, that permutes two cone points of order 8 and fixes order 4 cone point, lifts to a $\tilde{G} \in \text{SMod}_p(S_3)$ with $D_{\tilde{G}} = (2, 1; ((1,2), 4))$ such that $\langle F_i, \tilde{G} \rangle \cong \mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$. Moreover, the weak conjugacy class of $(\langle \mathcal{F}_i, \tilde{\mathcal{G}} \rangle, (\tilde{\mathcal{G}}, \mathcal{F}_i))$, for $i = 1, 2$, is encoded by

$$((2 \cdot 8, 5), 0; [(1,2), (0,1), 2], [(1,2), (7,8), 8], [(0,1), (1,8), 8]) \text{ and}$$
$$((2 \cdot 8, 5), 0; [(1,2), (0,1), 2], [(1,2), (1,8), 8], [(0,1), (7,8), 8]),$$

respectively.

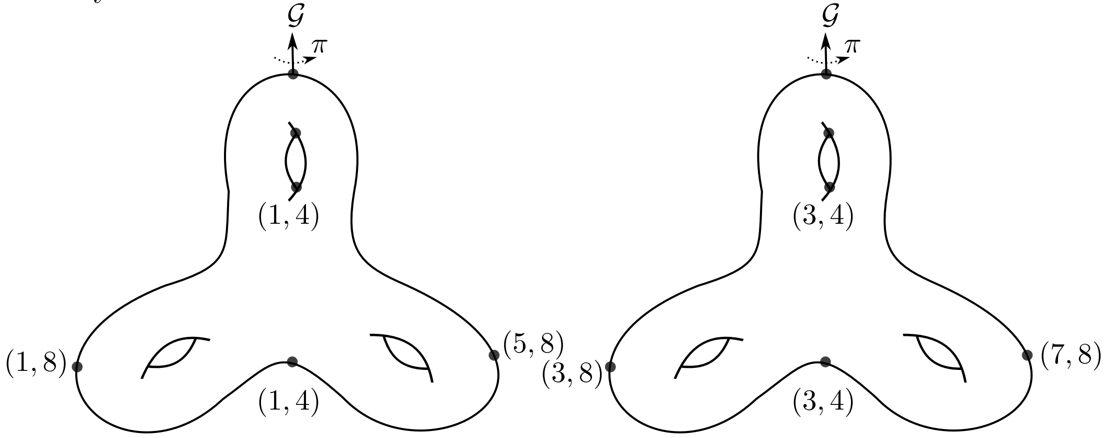

Figure 6.1: The realizations of two distinct $\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$-actions on S_3.

The geometric realization of these actions is illustrated in Figure 6.1 above, where for each i, the action $\mathcal{F}_i$ is realized by the rotation of a polygon of type $\mathcal{P}_{F_i}$ described in Theorem 2.3.1.

6.2 Infinite metacyclic subgroups of $\mathrm{Mod}(S_g)$

An *infinite metacyclic group* that is isomorphic to $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ admits a presentation of the form

$$\langle x, y \mid y^{2m} = 1, y^{-1}xy = x^{-1}\rangle. \tag{6.2.1}$$

In this section, we give an explicit construction of an infinite metacyclic subgroup isomorphic to $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ of $\mathrm{Mod}(S_g)$. Let $T_c \in \mathrm{Mod}(S_g)$ denote the left-handed Dehn twist about a simple closed curve c in S_g. A *root of* T_c *of degree* s is an $F \in \mathrm{Mod}(S_g)$ such that $F^s = T_c$. In the following lemma, by using some basic properties of Dehn twists [15, Chapter 3], we show that a root of Dehn twist cannot generate an infinite split metacyclic group that admits a presentation as in (6.2.1).

Lemma 6.2.2. *For $g \geq 2$, no root of T_c is a generator of any infinite split metacyclic subgroup of* $\mathrm{Mod}(S_g)$ *that is isomorphic to $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$.*

Proof. Let F be a root of T_c of degree s. Suppose we assume on the contrary that for some $g \geq 2$, there exists an infinite split metacyclic subgroup $H \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ of $\mathrm{Mod}(S_g)$ that admits the presentation

$$H = \langle F, G \mid G^{2m} = 1, G^{-1}FG = F^{-1}\rangle.$$

First, we consider the case when $s = 1$, that is, $F = T_c$. Then we have that

$$G^{-1}T_cG = T_c^{-1} \implies T_{G^{-1}(c)} = T_c^{-1},$$

which is impossible. Thus, we have that $H \neq \langle G, T_c\rangle$, which contradicts our assumption.

For $s > 1$, suppose that $H = \langle F, G\rangle$. Then the subgroup $\langle F^s, G\rangle$ of H would also be an infinite metacyclic group. Since $F^s = T_c$, this would contradict our conclusion in the previous case, and so our assertion follows. □

By a *multitwist* in $\mathrm{Mod}(S_g)$, we mean a finite product of powers of commuting Dehn twists. In view of Lemma 6.2.2, a natural question that arises is whether a multitwist in

$\mathrm{Mod}(S_g)$ can generate an infinite metacyclic group. In the following examples, we answer this question in the affirmative.

Example 6.2.3. Let $F' \in \mathrm{Mod}(S_2)$ be of order 3 with

$$D_{F'} = (3, 0; ((1, 3), 2), ((2, 3), 2)).$$

First, we note that $\mathcal{F}'$ has four fixed points on S_2. Further, it induces a local rotation angle of $2\pi/3$ around two of these points (corresponding to the two $(1, 3)$ pairs in $D_{F'}$) and rotation angle of $4\pi/3$ around the remaining two points (corresponding to the two $(2, 3)$ pairs in $D_{F'}$), as indicated in Figure 6.2. Considering this action on two distinct copies of S_2, we remove invariant disks around a distinguished $(1, 3)$-type fixed point and a distinguished $(2, 3)$-type fixed point in each of the two copies. We now attach two annuli connecting the resulting boundary components across the two surfaces so that:

(a) each annulus connects a pair of boundary components where the induced rotation angle is the same, as shown in Figure 6.2 below, and further,

(b) the annulus connecting the boundary components with rotation $4\pi/3$ (with the nonseparating curve c) has a $1/3^{rd}$ twist, while the other (with the nonseparating curve d) has a $-1/3^{rd}$ twist.

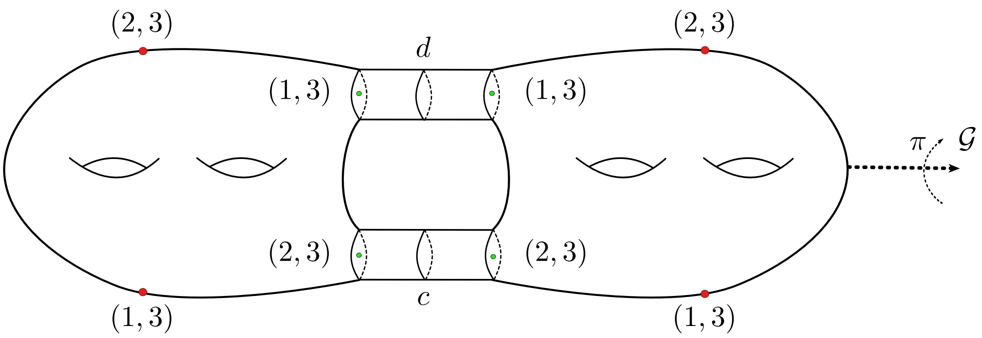

Figure 6.2: Realization of an infinite dihedral subgroup of $\mathrm{Mod}(S_5)$.

Thus, by applying the theory developed in [43], we obtain an $F \in \mathrm{Mod}(S_5)$, which is a root of the bounding pair map $T_c T_d^{-1}$ of degree 3. Now, we consider the hyperelliptic involution $G \in \mathrm{Mod}(S_5)$ with $D_G = (2, 0; ((1, 2), 12))$ (also indicated in Figure 6.2). By our construction, it follows that $G^{-1}FG = F^{-1}$, and so we have $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_2$.

Example 6.2.4. Let $F' \in \text{Mod}(S_5)$ be of order 3 with

$$D_{F'} = (3, 1; ((1,3), 2), ((2,3), 2)).$$

First, we note that $\mathcal{F}'$ has four fixed points on S_5. Furthermore, it induces a local rotation angle of $2\pi/3$ around two of these points (corresponding to the two $(1,3)$ pairs in $D_{F'}$) and rotation angle of $4\pi/3$ around the remaining two points (corresponding to the two $(2,3)$ pairs in $D_{F'}$), as indicated in Figure 6.3. Considering this action on two distinct copies of S_5, we remove invariant disks around all fixed point in each of the two copies.

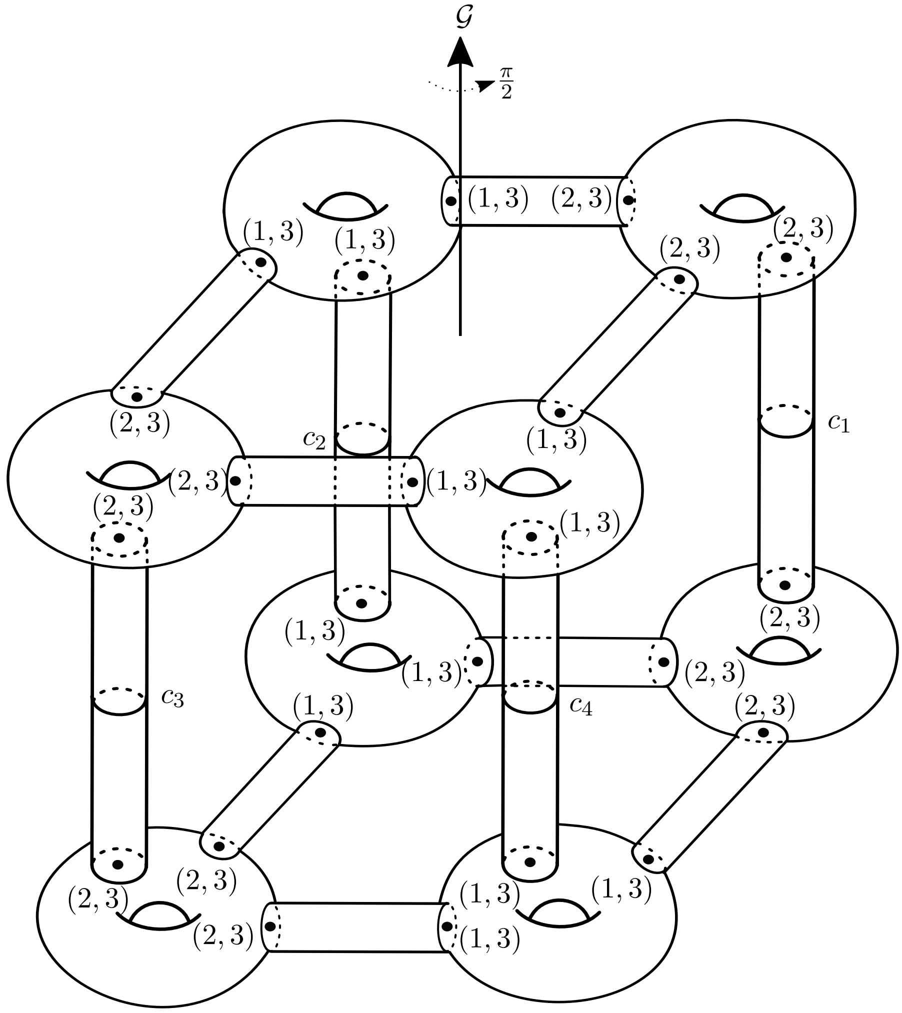

Figure 6.3: Realization of an infinite metacyclic subgroup of $\text{Mod}(S_{13})$.

We now attach four annuli connecting the resulting boundary components across the two surfaces so that:

(a) each annulus connects a pair of boundary components where the induced rotation angle is the same, as shown in Figure 6.3 below, and further,

(b) the annulus connecting the boundary components with rotation $4\pi/3$ (with the nonseparating curve c_1 and c_3) has a $1/3^{rd}$ twist, while the other (with the nonseparating curve c_2 and c_4) has a $-1/3^{rd}$ twist.

Thus, by applying the theory developed in [43], we obtain an $F \in \mathrm{Mod}(S_{13})$, which is a root of the multitwist $T_{c_1}T_{c_2}^{-1}T_{c_3}T_{c_4}^{-1}$ of degree 3. Now, we consider a $G \in \mathrm{Mod}(S_{13})$ with $D_G = (4, 4, 1;)$ (also indicated in Figure 6.3). By our construction, as $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4 \cong \langle F', G' \rangle \leq \mathrm{Mod}(S_5)$, where $D_{G'} = (4, 2, 1;)$, it follows that $G^{-1}FG = F^{-1}$, and so we have $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_4$.

Generalizing the above all constructions in Example 6.2.3 and Example 6.2.4, we have the following.

Proposition 6.2.5. *For $i = 1, 2$, let $H_i = \langle F_i, G_i \rangle \leq \mathrm{Mod}(S_{g_i})$ with $H_i \cong \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_{2m}$, such that the weak conjugacy class $(H_i, (G_i, F_i))$ is represented by a split metacyclic data set $\mathcal{D}_{H_i}$ containing a tuple $[(0, 1), (a_i, n), n]$. Then there exists an infinite metacyclic subgroup of $\mathrm{Mod}(S_{g_1+g_2+2m-1})$ isomorphic to $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ that is generated by a periodic mapping class of order $2m$ and a root of a multitwist of degree n.*

Proof. As $\mathcal{D}_{H_i}$ contains a tuple $[(0, 1), (a_i, n), n]$, by Proposition 4.3.1, we have

$$D_{F_1} = (n, g_0; (c_1, n_1), \ldots, (c_s, n_s), \underbrace{(a_1, n), (n - a_1, n), \ldots, (a_1, n), (n - a_1, n)}_{m \text{ times}})$$

and

$$D_{F_2} = (n, g_0'; (c_1', n_1'), \ldots, (c_t', n_t'), \underbrace{(a_2, n), (n - a_2, n), \ldots, (a_2, n), (n - a_2, n)}_{m \text{ times}}).$$

Taking inspiration from the theory developed in [41, 43] and Examples 6.2.3-6.2.4, we glue $2m$ annuli connecting the boundary components resulting from removing invariant disks around the orbit points corresponding to the pairs (a_1, n) and

$$\begin{cases} (a_2, n), & \text{if } a_2 \neq n - a_1, \text{ or} \\ (n - a_2, n), & \text{if } a_2 = n - a_1. \end{cases}$$

This yields a degree-n root F of a multitwist of the form

$$\begin{cases} \prod_{i=1}^{2m} T_{c_i}^{(-1)^{i+1}(a_1^{-1}+a_2^{-1})}, & \text{if } a_2 \neq n-a_1, \text{ or} \\ \prod_{i=1}^{2m} T_{c_i}^{(-1)^{i+1}(a_1^{-1}+(n-a_2)^{-1})}, & \text{if } a_2 = n-a_1, \end{cases}$$

where $a_i a_i^{-1} \equiv 1 \pmod{n}$ and $a_1^{-1} + a_2^{-1} \in \mathbb{Z}_n$. By considering the action $\mathcal{G}$ obtained by performing a $2m$-compatibility on $\mathcal{G}_1$ and $\mathcal{G}_2$ (see Section 2.3), we see that $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$, as desired. □

The group for $m = 1$ in the presentation of the infinite split metacyclic group of the type in the Equation (6.2.1) is known as the *infinite dihedral group*. Here is the corollary, which directly follows from Proposition 6.2.5.

Corollary 6.2.6. *For $g \geq 5$, there exists an infinite dihedral subgroup of* $\mathrm{Mod}(S_g)$ *that is generated by an involution and a root of a bounding pair map of degree* 3.

REFERENCES

[1] L. Bers. Deformations and moduli of Riemann surfaces with nodes and signatures. *Math. Scand.*, 36:12–16, 1975. Collection of articles dedicated to Werner Fenchel on his 70th birthday.

[2] A. Bhattacharya, S. Parsad, and K. Rajeevsarathy. Geometric realizations of cyclic actions on surfaces-ii. *arXiv preprint arXiv:1803.00328v3*, 2018.

[3] J. S. Birman. Mapping class groups and their relationship to braid groups. *Comm. Pure Appl. Math.*, 22:213–238, 1969.

[4] J. S. Birman and H. M. Hilden. On the mapping class groups of closed surfaces as covering spaces. In *Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, pages 81–115. Ann. of Math. Studies, No. 66, 1971.

[5] J. S. Birman and H. M. Hilden. Isotopies of homeomorphisms of Riemann surfaces and a theorem about Artin's braid group. *Bull. Amer. Math. Soc.*, 78:1002–1004, 1972.

[6] J. S. Birman and H. M. Hilden. On isotopies of homeomorphisms of Riemann surfaces. *Ann. of Math. (2)*, 97:424–439, 1973.

[7] O. V. Bogopolski. Classifying the actions of finite groups on orientable surfaces of genus 4 [translation of *proceedings of the institute of mathematics, 30* (Russian), 48–69, Izdat. Ross. Akad. Nauk, Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1996]. volume 7, pages 9–38, 1997. Siberian Advances in Mathematics.

[8] T. Breuer. *Characters and automorphism groups of compact Riemann surfaces*, volume 280 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.

[9] S. A. Broughton. Classifying finite group actions on surfaces of low genus. *J. Pure Appl. Algebra*, 69(3):233–270, 1991.

[10] S. A. Broughton and A. Wootton. Finite abelian subgroups of the mapping class group. *Algebr. Geom. Topol.*, 7:1651–1697, 2007.

[11] E. Bujalance, F. J. Cirre, J. M. Gamboa, and G. Gromadzki. On compact Riemann surfaces with dihedral groups of automorphisms. *Math. Proc. Cambridge Philos. Soc.*, 134(3):465–477, 2003.

[12] N. K. Dhanwani and K. Rajeevsarathy. Commuting conjugates of finite-order mapping classes. *Geometriae Dedicata*, 2020.

[13] N. K. Dhanwani, K. Rajeevsarathy, and A. Sanghi. Split metacyclic actions on surfaces. *New York J. Math.*, 28:617–649, 2022.

[14] T. Dokchitser. *GroupNames*. available at: https://people.maths.bris.ac.uk/~matyd/GroupNames/MC.html, consulted on 27/08/2022.

[15] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[16] W. Fenchel. Estensioni di gruppi discontinui e trasformazioni periodiche delle superficie. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)*, 5:326–329, 1948.

[17] W. Fenchel. Remarks on finite groups of mapping classes. *Mat. Tidsskr. B*, 1950:90–95, 1950.

[18] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.12.0*, 2022. available at: https://www.gap-system.org.

[19] J. Gilman. Structures of elliptic irreducible subgroups of the modular group. *Proc. London Math. Soc. (3)*, 47(1):27–42, 1983.

[20] W. J. Harvey. Cyclic groups of automorphisms of a compact Riemann surface. *Quart. J. Math. Oxford Ser. (2)*, 17:86–97, 1966.

[21] W. J. Harvey. On branch loci in Teichmüller space. *Trans. Amer. Math. Soc.*, 153:387–399, 1971.

[22] C. E. Hempel. *Metacyclic groups*. 1998. Thesis (Ph.D.)–The Australian National University. Available at: "https://openresearch-repository.anu.edu.au/bitstream/1885/144742/2/b20383460_Hempel_E_J_C.pdf".

[23] C. E. Hempel. Metacyclic groups. *Comm. Algebra*, 28(8):3865–3897, 2000.

[24] A. Hurwitz. Beweis der Transcendenz der Zahl e. *Math. Ann.*, 43(2-3):220–221, 1893.

[25] S. Katok. *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.

[26] S. P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

[27] H. Kimura. Classification of automorphism groups, up to topological equivalence, of compact Riemann surfaces of genus 4. *J. Algebra*, 264(1):26–54, 2003.

[28] S. Kravetz. On the geometry of Teichmüller spaces and the structure of their modular groups. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 278:35, 1959.

[29] A. M. Macbeath and H. Wilkie. *Discontinuous groups and birational transformations:[Summer School], Queen's College Dundee, University of St. Andrews*. [Department of Math.], Queen's College, 1961.

[30] C. Maclachlan. Abelian groups of automorphisms of compact Riemann surfaces. *Proc. London Math. Soc. (3)*, 15:699–712, 1965.

[31] D. Margalit and S. Schleimer. Dehn twists have roots. *Geom. Topol.*, 13(3):1495–1497, 2009.

[32] D. Margalit and R. R. Winarski. Braids groups and mapping class groups: the Birman-Hilden theory. *Bull. Lond. Math. Soc.*, 53(3):643–659, 2021.

[33] H. Masur. On a class of geodesics in Teichmüller space. *Ann. of Math. (2)*, 102(2):205–221, 1975.

[34] D. McCullough and K. Rajeevsarathy. Roots of Dehn twists. *Geometriae Dedicata*, 151:397–409, 2011. 10.1007/s10711-010-9541-4.

[35] J. Nielsen. Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen. III. *Acta Math.*, 58(1):87–167, 1932.

[36] J. Nielsen. *Die Struktur periodischer Transformationen von Flächen*, volume 15. Levin & Munksgaard, 1937.

[37] J. Nielsen. Abbildungsklassen endlicher Ordnung. *Acta Math.*, 75:23–115, 1943.

[38] S. Parsad, K. Rajeevsarathy, and B. Sanki. Geometric realizations of cyclic actions on surfaces. *J. Topol. Anal.*, 11(4):929–964, 2019.

[39] J. Paulhus. *Branching data for curves up to genus 48.* available at: https://paulhus.math.grinnell.edu/monodromy.html, consulted on 27/08/2022.

[40] K. Rajeevsarathy. Roots of dehn twists about separating curves. *Journal of the Australian Mathematical Society*, 95:266–288, 10 2013.

[41] K. Rajeevsarathy. Fractional powers of Dehn twists about nonseparating curves. *Glasg. Math. J.*, 56(1):197–210, 2014.

[42] K. Rajeevsarathy and A. Sanghi. Metacyclic actions on surfaces. *arXiv preprint arXiv:2201.09602*, 2022.

[43] K. Rajeevsarathy and P. Vaidyanathan. Roots of Dehn twists about multicurves. *Glasg. Math. J.*, 60(3):555–583, 2018.

[44] A. Schweizer. Metacyclic groups as automorphism groups of compact Riemann surfaces. *Geom. Dedicata*, 190:185–197, 2017.

[45] J.-P. Serre. *Trees.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[46] W. P. Thurston. *The Geometry and Topology of Three-Manifolds.* notes available at: "http://www.msri.org/communications/books/gt3m/PDF/13.pdf".

[47] W. P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

[48] T. W. Tucker. Finite groups acting on surfaces and the genus of a group. *J. Combin. Theory Ser. B*, 34(1):82–98, 1983.

[49] A. Wiman. Ueber die hyperelliptischen curven und diejenigen vom geschlechte p= 3, welche eindeutigen transformationen in sich zulassen" and "ueber die algebraischen curven von den geschlechtern p= 4, 5 und 6 welche eindeutigen transformationen in sich besitzen". *Svenska Vetenskaps-Akademiens Hadlingar, Stockholm*, 96, 1895.

[50] H. Zieschang. On decompositions of discontinuous groups of the plane. *Math. Z.*, 151(2):165–188, 1976.

Ph.D. Thesis Apeksha Sanghi 2022