INJECTIONS OF MAPPING CLASS GROUPS

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This is to certify that **Aniket Hinge**, BS-MS (Dual Degree) student in Department of Mathematics, has completed bonafide work on the thesis entitled **'Injections of Mapping Class Groups'** under my supervision and guidance.

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ABSTRACT

It was known that the mapping class group of a surface is finitely generated, but a presentation for this group was not known for many surfaces. To find the presentation of the mapping class group of the genus 2 surface, Birman and Hilden [6] used the covering map from the genus 2 surface to the 6-punctured sphere to construct a homomorphism from the mapping class group of the genus 2 surface to the 6-punctured sphere. They realized that this theory could be generalized in various ways, and they wrote a series of papers on the subject, culminating in the paper *On Isotopies of Homeomorphisms of Riemann Surfaces*, published in the Annals of Mathematics in 1973 [6].

In this thesis, we study the theory developed by Birman and Hilden. In particular, we study the two main theorems from their paper. We also study the construction of injective homomorphisms between the mapping class groups of closed surfaces induced by the covering maps from the paper of Javier Aramayona, Christopher J. Leininger, and Juan Souto [1].

Contents

Chapter 1

Preliminaries

The universal cover of hyperbolic surfaces is the upper half-plane. We can use properties of the isometry group of the upper half-plane to get important results about hyperbolic surfaces. In this chapter, we state some results about the hyperbolic surfaces, which will be used extensively throughout this thesis. These results can be found in [15].

1.1 Surfaces

Definition 1.1. A *surface* S is a second countable Hausdorff space such that every point in S has a neighborhood that is homeomorphic to an open set in $\mathbb{R}^2$.

From here on, we will only consider connected and orientable surfaces. We say a surface is *closed* if it is compact and does not have a boundary. The following result classifies all the orientable surfaces topologically.

Theorem 1.2 (Classification of surfaces)**.** *Any closed, connected, and orientable surface is homeomorphic to either the sphere or the connected sum of g tori for*

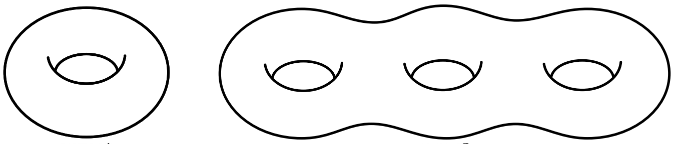

Figure 1.1: Surfaces of genus 1 and 3.

some positive integer $g > 0$.

Suppose that S_g denotes the closed surface of genus $g \geq 1$ and S_0 denotes the 2-sphere. The punctured surface obtained by removing n points from the interior of the surface S_g will be denoted by $S_{g,n}$. Similarly, the surface obtained by removing b open disks from the interior of the surface with pairwise disjoint boundaries will be denoted by S_g^b.

Definition 1.3. The *Euler characteristic* of a finite-dimensional CW-complex K is

$$\chi(K) = \sum_{i=0}^{n} (-1)^i \#(i - cell),$$

where n is the dimension of K.

The Euler characteristic is a homotopy invariant (homotopic spaces have the same Euler characteristic). The closed surface S_g as a CW-complex can be constructed from a 0-cell, $2g$-many 1-cells, and a 2-cell. Note that removing a disc introduces one more 1-cell in CW-complex. Since removing a disk and removing a point from a surface yields homotopically equivalent surfaces, we must have,

$$\chi(S_{g,n}^b) = 2 - 2g - n - b.$$

If $p : \tilde{S} \to S$ is a k-sheeted cover, then a n-cell in S lifts to k n-cells in $\tilde{S}$. Thus, we have the following relation.

$$\chi(\tilde{S}) = k\chi(S).$$

For a covering map, a continuous function $f : Y \to B$ can be lifted to a continuous function $\tilde{f} : Y \to E$ if it satisfies the lifting criterion.

Proposition 1.4. *Let $p : E \to B$ be a covering map with a base point $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map, with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to an unique map $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)).$$

We recall the following result from covering space theory [13] which will be used extensively in the following chapters.

Proposition 1.5. *Let $h, k : X \to Y$ be 2 homotopic maps such that $h(x_0) = y_1$ and $k(x_0) = y_2$. Suppose h_* and k_* are the homomorphisms of fundamental groups induced by h and k. Then we have $k_* = \hat{\alpha} \circ h_*$, where α is the path from y_1 to y_2 induced by the homotopy between h and k.*

We also need the following fact in the exposition.

Theorem 1.6 (Isotopy extension property [9])**.** *Let S be any surface. If $F : S_1 \times [0, 1] \to S$ is a smooth isotopy of simple closed curves, then there is an isotopy $H : S \times [0, 1] \to S$ so that $H|_{S\times 0}$ is the identity and $H|_{F(S1\times 0)\times I} = F$.*

1.2 Isometries of upper-half plane

Let $\mathbb{H}$ be the space of all complex numbers with the positive imaginary part. We define a Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$. The *upper half-plane* model is the space $\mathbb{H}$ endowed with this metric. The orientation preserving isometries of $\mathbb{H}$ are given by

$$\mathrm{Isom}^+(\mathbb{H}) = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R}, ad-bc=1 \right\}.$$

The group $\mathrm{Isom}^+(\mathbb{H})$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. The one-point compactification $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$ is known as the *boundary at infinity* of $\mathbb{H}$ and is denoted by $\partial\mathbb{H}$. The union $\mathbb{H} \cup \partial\mathbb{H}$ is denoted by $\overline{\mathbb{H}}$. Also, any isometry $f : \mathbb{H} \to \mathbb{H}$ extends uniquely to an homeomorphism $\bar{f} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$.

If $f \in \mathrm{Isom}^+(\mathbb{H})$ is not the identity map, then $\bar{f}$ has at most 2 fixed points. The isometries of $\mathbb{H}$ can be classified in the following way based on the number of fixed points and their location on $\overline{\mathbb{H}}$.

(i) **Elliptic:** If $\bar{f}$ has exactly one fixed point in $\mathbb{H}$, then f is said to be elliptic. It can be seen as a rotation of $\mathbb{H}$ about the fixed point.

(ii) **Parabolic:** If $\bar{f}$ has only one fixed point and the point belongs to $\partial\mathbb{H}$, then f is called parabolic. We can conjugate f with an element of $\mathrm{Isom}^+(\mathbb{H})$ to obtain a translation of $\mathbb{H}$ of the form $z \to z \pm 1$.

(iii) **Hyperbolic:** If $\bar{f}$ has 2 fixed points, both in $\partial\mathbb{H}$, then f is called hyperbolic. The map f translates the unique geodesic joining the 2 fixed points. The geodesic is called the *axis* of f.

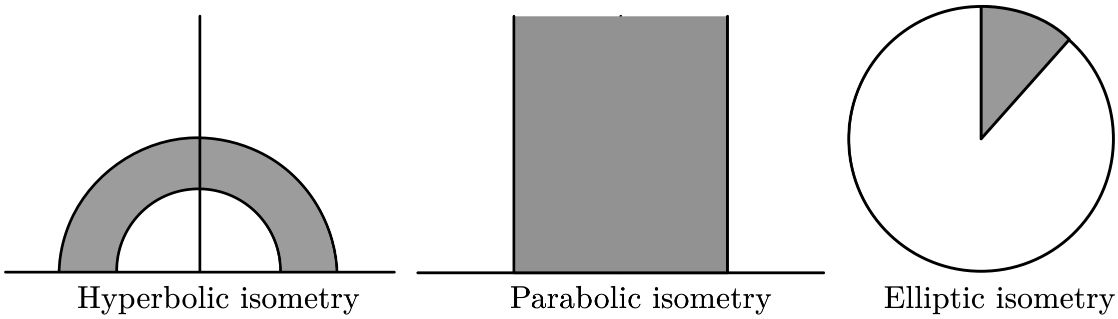

Figure 1.2: Dirichlet domain, for hyperbolic, parabolic, and elliptic isometries.

1.3 Hyperbolic surfaces

Definition 1.7. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Definition 1.8. If G is a Fuchsian group, then an open subset $R \subset \mathbb{H}$ is called a *fundamental domain* for G if the following hold.

(i) $gR \cap R = \phi$, for all $g \in G - \{id\}$ and

(ii) for every $z \in \mathbb{H}$, there exists $g \in G$ such that $gz \in \bar{R}$.

We note that a fundamental domain is not unique. The Dirichlet domain gives us a method of constructing a fundamental domain for a given Fuchsian group.

Definition 1.9. Let G be a Fuchsian group and $z_0 \in \mathbb{H}$ such that $\tilde{z}_0$ is not fixed by any non-identity element of G. The *Dirichlet domain* with center z_0 is defined as

$$R_{z_0} = \bigcup_{g \in G - \{id\}} \{z \in \mathbb{H} : d_{\mathbb{H}}(z, z_0) < d_{\mathbb{H}}(z, gz_0)\}.$$

Example 1.10. Let $G = \langle T \rangle$ be a cyclic Fuchsian group. The Dirichlet domain for G, depending on the type of T up to conjugacy, is described in Figure 1.2.

Definition 1.11. A surface endowed with a complete, finite-area Riemannian metric of constant curvature -1 is called a *hyperbolic surface.* If the surface has a non-empty boundary, then the geodesics in ∂S must be geodesics in S.

If a Fuchsian group G acts freely on $\mathbb{H}$, then we have an induced hyperbolic structure on the orbit space $\mathbb{H}/G$.

Theorem 1.12. *Let S be a complete hyperbolic surface. Then S is isometric to $\mathbb{H}/\Gamma$, where Γ is a Fuchsian group acting freely on $\mathbb{H}$.*

Since $\mathbb{H}$ is simply connected and Γ is the deck group of the quotient map $\mathbb{H} \to \mathbb{H}/\Gamma$, we have the following corollary.

Corollary 1.12.1. *The fundamental group of a hyperbolic surface is isomorphic to a discrete subgroup of isometries of $\mathbb{H}$.*

Remark 1.13. Since the deck group of cover $\mathbb{H} \to S$, where S is a hyperbolic surface, acts freely on $\mathbb{H}$, $\pi_1(S)$ can only have parabolic or hyperbolic isometries, and hence $\pi_1(S)$ is torsion-free.

Proposition 1.14. *The center of the fundamental group of a hyperbolic surface is trivial.*

Proof. We identify the fundamental group as a discrete subgroup of isometries of $\mathbb{H}$. The center of any group is abelian, and since it is a subgroup of a discrete group, it is also discrete. Therefore, it is generated by a parabolic or hyperbolic isometry. The center's generator is either parabolic or hyperbolic because the fundamental group is torsion-free. The Dirichlet domain corresponding to these isometries has infinite area. Hence, the center must be trivial. □

Proposition 1.15. *Let S be a hyperbolic surface and $g \in \pi_1(S)$. If $h^n = k^n$ for some $h, k \in \pi_1(S)$ and positive integer n, then $h = k$.*

Proof. We identify $\pi_1(S)$ with a Fuchsian group. Let $\langle h, k \rangle$ be the discrete subgroup of $\pi_1(S)$. We observe that h and k have the same fixed points as that of g. Thus, the subgroup $\langle h, k \rangle$ is abelian and hence cyclic. Thus, $h = k$. □

Chapter 2

Mapping Class Groups

In this chapter, we study mapping class groups of surfaces. After defining the mapping class group of a surface, we explicitly compute it for some surfaces. We also state a classical theorem relating the mapping class group to the automorphisms of the fundamental group of a surface. This chapter is based on [9, Chapter 2].

2.1 Definition of mapping class groups

Let $S = S^b_{g,n}$ be an orientable surface of genus g with n punctures and b boundary components. Let $\mathrm{Homeo}^+(S, \partial S)$ be the group of orientation-preserving self-homeomorphisms of S that restricts to identity on ∂S. The group $\mathrm{Homeo}^+(S, \partial S)$ can be endowed with compact-open topology. Let $\mathrm{Homeo}^+_0(S, \partial S)$ denote the path component of $\mathrm{Homeo}^+(S, \partial S)$ containing the identity element.

Since for any topological group, the path component of identity is a normal subgroup, we have $\mathrm{Homeo}^+_0(S, \partial S) \lhd \mathrm{Homeo}^+(S, \partial S)$.

Definition 2.1. The *mapping class group* of surface S, denoted by $\mathrm{Mod}(S)$, is defined as

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0^+(S, \partial S). \tag{2.1}$$

Definition 2.2. We say that 2 homeomorphisms $f, g : X \to Y$ are *isotopic* if there exists a continuous map $H : X \times [0,1] \to Y$ such that the following conditions hold.

(i) $H(x, 0) = f(x)$ for all $x \in X$,

(ii) $H(x, 1) = g(x)$ for all $x \in X$, and

(iii) the function $H_t : X \to Y$ defined as $H_t(x) = H(x, t)$ is a homeomorphism for all values of t.

If f and g are isotopic, we will write $f \simeq g$.

2.1.1 Mapping class groups and isotopy classes

For $h \in \mathrm{Homeo}_0^+(S, \partial S)$, there is a path in $\mathrm{Homeo}^+(S, \partial S)$ from the identity to h, that is an isotopy between h and identity fixing ∂S pointwise. Conversely, if there exists an isotopy fixing ∂S pointwise exists between h and the identity, then it defines a path between h and identity. Thus, the set $\mathrm{Homeo}_0^+(S, \partial S)$ can also be seen as the set of all homeomorphisms isotopic to the identity map via an isotopy that fixes ∂S pointwise. Therefore, the mapping class group of a surface can also be defined as the group of isotopy classes of orientation-preserving homeomorphisms of S that fixes ∂S pointwise.

2.1.2 Equivalent definitions of mapping class groups

There are other possible ways of defining the mapping class group, for which we will need the following facts.

Theorem 2.3 (Baer)**.** *If 2 orientation-preserving diffeomorphisms of a compact surface S are homotopic relative to ∂S, then they are smoothly isotopic relative to ∂S.*

Baer first proved the theorem in [3, 4] for closed surfaces, and for general surfaces (not necessarily compact, and also possibly with boundary), it was proved by Epstein in [8].

Theorem 2.4 (Munkres [12])**.** *Every homeomorphism of a compact surface S (relative to ∂S) is isotopic to a diffeomorphism of S (relative to ∂S).*

Let $\mathrm{Diff}^+(S, \partial S)$ be the group of orientation-preserving diffeomorphisms of S that are identity on the boundary, equipped with the compact-open topology. Let $\mathrm{Diff}_0(S, \partial S)$ be the path-component containing the identity and let $\mathrm{Homeo}^+(S, \partial S)/\mathrm{homotopy}$ be the group of homotopy classes of orientation-preserving homeomorphisms. From Theorems 2.3-2.4, we have the following corollary.

Corollary 2.4.1. *The following groups are isomorphic.*

(i) $\mathrm{Mod}(S)$*,*

(ii) $\mathrm{Diff}^+(S, \partial S)/\mathrm{Diff}_0(S, \partial S)$*, and*

(iii) $\mathrm{Homeo}^+(S, \partial S)/\mathrm{homotopy}$*.*

The equivalent definitions allow us to freely switch between homeomorphisms and diffeomorphisms or between isotopy and homotopy. In general, to exploit the geometrical properties of surfaces, we will use diffeomorphisms, and to use topological properties, we will use homeomorphisms.

Definition 2.5. Let $\mathrm{Homeo}(S_{g,n})$ be the topological group of all self-homeomorphisms of $S_{g,n}$ and $\mathrm{Homeo}_0(S_{g,n})$ be the component of the identity element in topological group $\mathrm{Homeo}(S_{g,n})$. Then we can define the *extended mapping class group* of $S_{g,n}$ as,

$$\overline{\mathrm{Mod}}(S_{g,n}) = \mathrm{Homeo}(S_{g,n})/\mathrm{Homeo}_0(S_{g,n}).$$

2.1.3 Punctures vs. marked points

If $S = S_{g,n}$ is a punctured surface, then it is more convenient to think of the punctures as marked points on the surface S_g. A surface S with n marked points will be denoted by $(S_g, \mathcal{B})$, where $\mathcal{B}$ is the set of n marked points.

Let $\mathrm{Homeo}(S, \mathcal{B})$ be the group of homeomorphisms of S, which preserves $\mathcal{B}$. Since S is Hausdorff and $S^0 := S - \mathcal{B}$ is dense in S, any homeomorphism in $\mathrm{Homeo}(S^0)$ extends uniquely to a homeomorphism in $\mathrm{Homeo}(S, \mathcal{B})$. Conversely, any homeomorphism in $\mathrm{Homeo}(S, \mathcal{B})$ restricts to a homeomorphism of S^0. Thus, the group $\mathrm{Homeo}(S, \mathcal{B})$ is isomorphic to $\mathrm{Homeo}(S^0)$. Now, it follows that $\mathrm{Mod}(S, \mathcal{B}) \cong \mathrm{Mod}(S^0)$. Note that the isotopies in $\mathrm{Mod}(S, \mathcal{B})$ preserve $\mathcal{B}$.

2.2 Computation of mapping class groups

In this section, we will compute the mapping class group of some basic surfaces like disk, annulus, and sphere with at most 3 punctures.

2.2.1 Mapping class group of the closed disk

Proposition 2.6 (Alexander Lemma)**.** *The mapping class group of a closed disk, i.e.,* $\mathrm{Mod}(D^2)$*, is trivial.*

Proof. Let $f : D^2 \to D^2$ be a homeomorphism that restricts to the identity on ∂D^2. Then

$$f_t(x) = \begin{cases} (1-t)f(x/(1-t)), & 0 \leq |x| < 1-t, \\ x, & 1-t \leq |x| \leq 1 \end{cases} \tag{2.2}$$

defines an isotopy between f and the identity map fixing boundary pointwise. □

Since every homeomorphism f_t in Equation (2.2) fixes origin, the same isotopy will also work for a punctured disk (disk with the center removed). Thus, $\mathrm{Mod}(S^1_{0,1})$ is also trivial.

2.2.2 Mapping class group of a punctured sphere

Lemma 2.7. *Let* α, β *be simple arcs on the 3-punctured sphere with distinct endpoints. If* α, β *have the same endpoints, then they are isotopic.*

Proof. Since the once punctured sphere is homeomorphic to the plane, we can take α, β to be arcs between 2 points in the plane. If α, β intersect, then there exists an innermost disk bounded by subarcs of α, β. Then α can be isotoped across this disk to reduce the intersection number until α, β are disjoint. Moreover, their union is an embedded circle in the plane, which bounds a disk. Hence, they are isotopic. □

Proposition 2.8. *The mapping class group of a 3-punctured sphere is isomorphic to* Σ_3*, the symmetric group on 3 letters.*

Proof. Suppose that $h \in \mathrm{Homeo}^+(S_0, \{1,2,3\})$, then h induces a permutation of marked points (labelled 1,2, and 3). Since isotopic homeomorphism will induce the same permutation of marked points, the mapping class $[h]$ corresponding to h will induce a permutation of marked points. We define a map $\psi : \mathrm{Mod}(S_{0,3}) \to \Sigma_3$ which sends an element $[h]$ of $\mathrm{Mod}(S_{0,3})$ to the permutation induced on the three marked points. It is clear that ψ is a homomorphism. Since Möbius transformations act triply-transitively on the Riemann sphere, ψ is surjective.

We want to show that ψ is injective. Suppose that $[h] \in \mathrm{Mod}(S_{0,3})$ and h fixes 3 marked points. Let α be an arc connecting two different marked points. Then $h(\alpha)$ and α are the arcs with same endpoints. By Lemma 2.7, $h(\alpha)$ and α are isotopic. By Theorem 1.6, h is isotopic to a homeomorphism fixing α pointwise, and so without loss of generality, we can assume that h fixes α pointwise.

Now cutting the surface along α gives us a disk with one marked point. Since h is orientation-preserving, it induces an orientation-preserving homeomorphism h' of the disk, which restricts to the identity map on the boundary. Hence, by Alexander lemma, h' is isotopic to identity, and thus h is isotopic to identity. □

Corollary 2.8.1. *For $n = 1, 2, 3$, we have $\mathrm{Mod}(S_{0,n}) \cong \Sigma_n$ for $1 \le n \le 3$, where S_n is the symmetric group on n letters.*

Proof. We take map $\psi : \mathrm{Mod}(S_{0,n}) \to \Sigma_n$ defined in the proof of Proposition 2.8. It follows that ψ is a surjective homomorphism. To prove injectivity, suppose that $h : S_{0,n} \to S_{0,n}$ fixes punctures. Since the group of orientation-preserving Möbius transformations of S_0, i.e., $\mathrm{PSL}_2(\mathbb{C})$ acts transitively on the S_0, we take a $\phi \in \mathrm{PSL}_2(\mathbb{C})$ such that $h \circ \phi$ fixes 3 points. Then h is isotopic to $h \circ \phi$, as $\mathrm{PSL}_2(\mathbb{C})$ is connected. From Proposition 2.8, we have $h \circ \phi$ is isotopic to identity. Therefore, ψ is injective. □

Corollary 2.8.2. *The mapping class group of the sphere is trivial.*

Proof. If h is a homeomorphism of a sphere, then it is isotopic to a homeomorphism fixing one point. Thus, h restricts to a homeomorphism of the complex plane. Since $\mathrm{Mod}(S_{0,1})$ is trivial, we have h is isotopic to identity map, and therefore $\mathrm{Mod}(S_0)$ is trivial. □

2.2.3 Mapping class group of the annulus

Proposition 2.9. *The mapping class group of annulus $A = [0,1] \times S^1$ is infinite cyclic.*

Proof. The universal cover of A is $\tilde{A} = [0,1] \times \mathbb{R}$, with the covering map $p : \tilde{A} \to \tilde{A}$ given by $p(x,y) = (x, e^{2\pi i y})$. Let $\phi : A \to A$ be a homeomorphism of the annulus that restricts to the identity on $\partial A = \{0,1\} \times S^1$. Now $\phi \circ p : \tilde{A} \to A$ has a unique lift $\tilde{\phi} : \tilde{A} \to \tilde{A}$ such that $\tilde{\phi}$ fixes origin. Let $\tilde{\phi}_1$ denote the restriction of $\tilde{\phi}$ to $\{1\} \times \mathbb{R}$,

$$p|_{\{1\}\times S^1} = p|_{\{1\}\times S^1}\tilde{\phi}_1.$$

Now observe that $p|_{\{1\}\times S^1}$ is the universal covering map from $\mathbb{R}$ to S^1 and $\tilde{\phi}_1$ is a deck transformation of this cover. Therefore, $\tilde{\phi}_1$ is a translation by some integer n.

The homotopy lifting criterion (Proposition 1.4) implies that modifying ϕ by a homotopy change n continuously, so n is constant for homotopic maps (since $\mathbb{Z}$ is discrete). Thus, we have a well-defined map $\rho : \mathrm{Mod}(A) \to \mathbb{Z}$ given by $\rho([\phi]) = \tilde{\phi}_1(0)$. It follows from the uniqueness of the lift that ρ is a homomorphism.

First, we will prove that the homomorphism ρ is surjective. For $n \in \mathbb{Z}$, let

$\phi : A \to A$ be the homeomorphism given by

$$\phi(x, e^{2\pi iy}) = (x, e^{2\pi i(nx+y)}).$$

Then ϕ lift to $\tilde{\phi}$ given by $\tilde{\phi}(x, y) = (x, nx + y)$. Then $\tilde{\phi}$ translate $\{1\} \times \mathbb{R}$ by n. This proves that ρ is a surjective.

Now we will show that ρ is injective. Assume that $\phi : A \to A$ is in kernel of ρ. Then $\tilde{\phi}$ fixes (0,1). To prove injectivity, we must show that ϕ is isotopic to the identity map. Let α be the arc defined as $\alpha(t) = (t, 1)$. Then α and $\phi(\alpha)$ are isotopic by an isotopy leaving ∂A pointwise fixed. Using the fact that cutting A along α gives us a disk, one obtains a homotopy between ϕ and the identity. Hence, ρ is an isomorphism, and therefore, $\text{Mod}(A) \cong \mathbb{Z}$. □

2.3 Dehn twist

Dehn twists are the simplest infinite-order mapping classes that can be studied by looking at their action on closed curves on the surface. It is known [7] that finitely many Dehn twists generate the mapping class group of a compact surface.

Let $A = S^1 \times [0, 1]$ be the annulus. Let $T : A \to A$ be the twist map A given by

$$T(\theta, t) = (\theta + 2\pi t, t). \tag{2.3}$$

observe that the map T is orientation-preserving and restricts to identity on ∂A.

Definition 2.10. Let α be a simple closed curve in S. Let N be a regular neighborhood of α and choose an orientation-preserving homeomorphism $\phi : A \to N$. We obtain a homeomorphism $T_\alpha : S \to S$, called a *Dehn twist* about α, defined as

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x), & x \in N, \\ x, & \text{otherwise.} \end{cases} \tag{2.4}$$

Even though T_α depends on the choice of N and ϕ, the isotopy class of T_α does not depend on these choices. Moreover, for two isotopic simple closed curves α and β the homoemorphisms T_α and T_β belongs to same isotopy class. Therefore, T_a represents an element of $\mathrm{Mod}(S)$, where a is the isotopy class of α.

2.4 Birman exact sequence

For a punctured surface S, let (S, x) denote the surface S marked at point x in the interior of S. If $[f] \in \mathrm{Mod}(S, x)$, then f is a homeomorphism of S which fixes x. Thus, we can define a natural homomorphism $Forget : \mathrm{Mod}(S, x) \to \mathrm{Mod}(S)$ by $Forget([f]) = [f]$. Here, for the sake of simplicity, we resort to the abuse of notation. Note that the homotopies in the $\mathrm{Mod}(S, x)$ fix the point x while the homotopies in $\mathrm{Mod}(S)$ may not fix the point x.

For $\alpha \in \pi_1(S, x)$, let a be the representative of the homotopy class α. Since the loop a is an isotopy of points from x to itself, by isotopy extension property, we get an isotopy from identity to a homeomorphism f_a of S fixing x. The isotopy class $[f_a]$ defines a mapping class in $\mathrm{Mod}(S, x)$, which lies in the kernel of $Forget$ map. Suppose $[f]$ is in the kernel of $Forget$. We have a loop a traced by x under the isotopy from f to identity, and thus we get $[a] \in \pi_1(S, x)$. Let $Push : \pi_1(S, x) \to \mathrm{Mod}(S, x)$ be the map $\alpha \mapsto [f_a]$. It can be shown that $Push$ is well-defined [5]. Hence, we get the following short exact sequence known as the Birman-exact sequence.

Theorem 2.11. *The sequence of groups and their homomorphisms,*

$$1 \to \pi_1(S,x) \xrightarrow{Push} \mathrm{Mod}(S,x) \xrightarrow{Forget} \mathrm{Mod}(S). \tag{2.5}$$

is a short exact sequence.

We note that if S is a closed surface, then $\mathrm{Mod}(S,x) = \mathrm{Mod}(S^o)$, where $S^o = S \setminus \{x\}$.

2.5 Dehn-Nielsen-Baer Theorem

The Dehn-Nielsen-Baer Theorem gives us a way to relate the automorphism group of $\pi_1(S)$ to $\mathrm{Mod}(S)$. Let S be the closed surface of genus g. Let

$$\mathrm{Inn}(G) = \{\phi \in \mathrm{Aut}(G) | \phi(x) = gxg^{-1}, \forall x \in G, \text{ for some } g \in G\},$$

be the group of inner automorphisms. The *outer automorphism* group of G is defined as,

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G).$$

Let f be an homeomorphism of S, then it induces an isomorphism $f_* : \pi_1(S,x) \to \pi_1(S, f(x))$, where x is the base point. If γ is a path from x to $f(x)$, then γ induces an isomorphism $\hat{\gamma} : \pi_1(S, f(x)) \to \pi_1(S,x)$. Thus, $\hat{\gamma} \circ f_*$ is an automorphism of $\pi_1(S,x)$. We observe that the automorphism $\hat{\gamma} \circ f_*$ differ by an inner automorphism for a different choice of path. Therefore, f_* represents a well-defined element in $\mathrm{Out}(\pi_1(S,x))$, will be denoted by f_* for simplicity.

Furthermore, the automorphisms induced by 2 homotopic homeomorphisms differ by an inner automorphism. Hence, there is a well-defined homomorphism $\sigma : \overline{\mathrm{Mod}}(S) \to \mathrm{Out}(\pi_1(S, x))$ which maps the homotopy class $[f]$ to the outer automorphism f_*, where $\overline{\mathrm{Mod}}(S)$ is the extended mapping class group of S.

Theorem 2.12 (Dehn-Nielsen-Baer Theorem)**.** *For $g \geq 1$, the homomorphism $\sigma : \overline{\mathrm{Mod}}(S_g) \to \mathrm{Out}(\pi_1(S_g))$ is an isomorphism.*

The original proof of the surjectivity of σ is due to Dehn [7], although Nielsen was the first to publish a proof [14].

Chapter 3

Birman–Hilden Theory

In 1970s, Birman and Hilden wrote several papers on the problem of relating mapping class group of a surface and its covering space, concluding in [6]. The aim of this chapter is to understand the main theorems of their paper [6].

3.1 Introduction

Let $p : \tilde{S} \to S$ be a finite-sheeted, possibly branched, covering map branched at finitely many points $\mathcal{B} \subset S \setminus \partial S$ with deck group $D < \mathrm{Homeo}^+(\tilde{S})$. A homeomorphism $f \in \mathrm{Homeo}^+(\tilde{S})$ is said to be *fiber-preserving* if $p(x) = p(x')$, implies $p\tilde{f}(x) = p\tilde{f}(x')$ for every $x, \tilde{x} \in \tilde{S}$. We denote the subgroup consisting of all fiber-preserving homeomorphisms by $\mathrm{SHomeo}(\tilde{S})$.

A homeomorphism $f \in \mathrm{Homeo}^+(S, \mathcal{B})$ is said to be *liftable*, if there exists a hoemomorphisms $\tilde{f} \in \mathrm{Homeo}^+(\tilde{S})$ such that $p\tilde{f}(x) = fp(x)$ for every $x \in \tilde{S}$. We denote the subgroup consisting of all liftable homeomorphisms by $\mathrm{LHomeo}(S, \mathcal{B})$. we note that any lift $\tilde{f}$ of a liftable homeomorphism f must be fiber-preserving.

Proposition 3.1. *There exists a surjective homomorphism* $\phi : \mathrm{SHomeo}(\tilde{S}) \to$

LHomeo$(S, \mathcal{B})$*, with ker* $\phi = D$*. Hence, we have the following short exact sequence.*

$$1 \to D \to \mathrm{SHomeo}(\tilde{S}) \to \mathrm{LHomeo}(S, \mathcal{B}) \to 1. \tag{3.1}$$

Proof. Given $\tilde{f} \in \mathrm{SHomeo}(\tilde{S})$, we define $\phi(\tilde{f}) = f$, where f is the projection of $\tilde{f}$ defined as $f(x) = p(\tilde{f}(p^{-1}(x)))$. This function is well-defined since $\tilde{f}$ is fiber-preserving. Also, f is a homeomorphism because $\tilde{f}$ is a homeomorphism. Therefore, $f \in \mathrm{LHomeo}(S, \mathcal{B})$ and the map ϕ is well-defined.

The map ϕ is a surjective homomorphism since a lift of $f \in \mathrm{LHomeo}(S, \mathcal{B})$ is a fiber-preserving homeomorphism. A homeomorphism $\tilde{f}$ is in $\ker \phi$ if and only if $p(\tilde{f}(p^{-1}(x))) = x$ that is $p(\tilde{f}(x)) = p(x)$. Thus, we have $\ker \phi = D$. □

Remark 3.2 (Branched vs Unbranched Cover)**.** Let $p : \tilde{S} \to S$ be a finite-sheeted, branched cover branched at $\mathcal{B} \subset S$. For $\tilde{S}^o = \tilde{S} \setminus p^{-1}(\mathcal{B})$ and $S^o = S \setminus \mathcal{B}$ the restriction of p to $\tilde{S}^o$ is an unbranched cover $p^o : \tilde{S}^o \to S^o$. Since $\mathrm{Homeo}^+(S, \mathcal{B}) \cong \mathrm{Homeo}^+(S^o)$ and any fiber-preserving homeomorphism of $\tilde{S}$ preserves the set $p^{-1}(\mathcal{B})$, there is an isomorphism $\mathrm{SHomeo}(\tilde{S}) \cong \mathrm{SHomeo}(\tilde{S}^o)$. It follows that $\mathrm{LHomeo}(S, \mathcal{B}) \cong \mathrm{LHomeo}(S^o)$.

Proposition 3.3. *If the covering map* $p : \tilde{S} \to S$ *is regular, then* $\mathrm{SHomeo}(\tilde{S})$ *is the normalizer of* D *in* $\mathrm{Homeo}^+(\tilde{S})$*, where* D *is the deck group.*

Proof. For $\tilde{f} \in \mathrm{SHomeo}(\tilde{S})$ and $h \in D$, we have $p(\tilde{f} \circ h \circ \tilde{f}^{-1}) = p$ and thus $\tilde{f}$ belongs to normalizer of D. Now let $\tilde{f}$ be in normalizer of D and suppose $p(x) = p(y)$ for $x, y \in \tilde{S}$. Then there exists a deck transformation h such that $h(x) = y$. But, there exists a deck transformation $h' \in D$ such that $\tilde{f}h(x) = h'\tilde{f}(x)$. Thus

$\tilde{f}(y) = h'\tilde{f}(x)$ which implies that $p\tilde{f}(y) = p\tilde{f}(x)$. Therefore, $\tilde{f}$ is fiber-preserving. Hence, SHomeo($\tilde{S}$) is the normalizer of D. □

Proposition 3.4. *A homeomorphism f lifts to $\tilde{f}$ if and only if for every $x \in S$, we have $f_*(p_*(\pi_1(\tilde{S}, \tilde{x}))) = p_*(\pi_1(\tilde{S}, \tilde{x}'))$, where $\tilde{x} \in p^{-1}(x)$ and $\tilde{x}' \in p^{-1}(f(x))$.*

The proof of Proposition 3.4 follows from the fact that both f and f^{-1} satisfy the lifting criterion (Proposition 1.4).

Definition 3.5. A finite sheeted, possibly branched covering map $p : \tilde{S} \to S$ is said to have the *Birman–Hilden property* if any fiber-preserving homeomorphism isotopic to identity is isotopic through fiber-preserving homeomorphisms.

Example 3.6 (Example of a cover with the Birman–Hilden property)**.** The hyperelliptic involution of S_2 induces a 2-sheeted branched cover $p : S_2 \to S_{0,6}$ (see Figure 3.1). Since the cover is regular and the hyperelliptic involution fixes the branch points, it follows from Theorem 3.8 that p has the Birman–Hilden property.

Example 3.7 (Example of a cover without the Birman–Hilden property)**.** Consider the 2-sheeted branched covering of $S_{0,2}$ by sphere S_0 induced by π rotation about an axis of S_0 (see Figure 3.2). Let $\tilde{f}$ be the fiber-preserving homeomorphism of S_0 isotopic to identity exchanging the branch points. Since its projection f exchanges the 2 punctures of $S_{0,2}$, f is not isotopic to identity, and thus $\tilde{f}$ is not fiber-isotopic to identity.

Now a natural question to ask is when does a cover have the Birman–Hilden property? In the following sections, we give a sufficient condition for a cover to have the Birman–Hilden property.

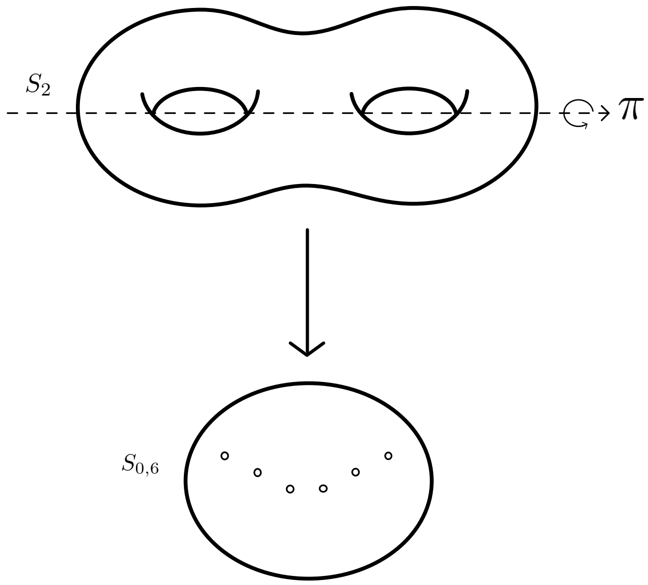

Figure 3.1: A 2-sheeted branched cover from S_2 and $S_{0,6}$ with the Birman–Hilden property.

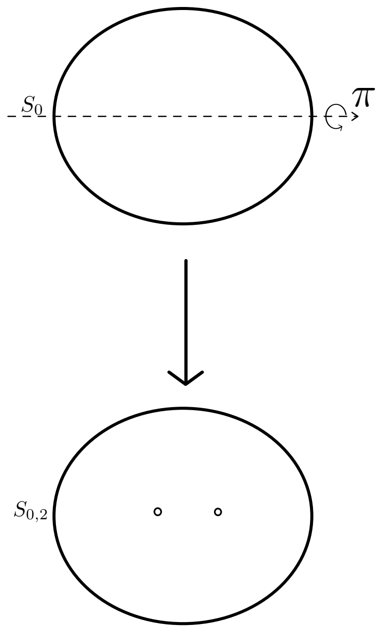

Figure 3.2: A covering map from S_0 to $S_{0,2}$ without the Birman–Hilden property.

3.2 Statement of the theorems

Let $\tilde{S}$ and S be orientable surfaces. Following are the main theorems of [6].

Theorem 3.8. *Let $p : \tilde{S} \to S$ be a regular covering map, either branched or unbranched, with a finite deck group and a finite number of branch points. Suppose that the deck transformations leave each branch point fixed. In the case of branched covering, assume that $\tilde{S}$ is not homeomorphic to a sphere or torus. Then $p : \tilde{S} \to S$ has the Birman–Hilden property.*

Theorem 3.9. *Let $p : \tilde{S} \to S$ be a regular, possibly branched covering map with, at most, finitely many branch points. Let the deck group be finite and solvable. Then $p : \tilde{S} \to S$ has the Birman–Hilden property.*

We note that Theorem 3.8 implies that all regular unbranched covers have the Birman–Hilden property. On the other hand, Theorem 3.9 implies that all regular solvable branched covers have the Birman–Hilden property. Thus, Theorems 3.8-3.9 can be stated together as follows.

Theorem 3.10 ([11])**.** *Let $p : \tilde{S} \to S$ be a finite sheeted, regular branched cover. Let $\tilde{S}$ be a hyperbolic surface. Assume that p is either unbranched or solvable. Then p has the Birman–Hilden property.*

3.3 Birman–Hilden property for regular unbranched covers

In this section, we prove Theorem 3.8 via a sequence of lemmas. Without loss of generality, we can assume that $\tilde{S}$ and S are Riemann surfaces. It is known that if $p : \tilde{S} \to S$ is a covering map and S is a Riemann surface, then there is a unique conformal structure on $\tilde{S}$ such that p is analytic. Since p is analytic, we can assume that deck transformations are also analytic. In what follows, we assume that $\chi(\tilde{S}) < 0$ unless stated otherwise. From the Proposition 1.14 and Theorem 1.12, it follows that $\tilde{S}$ has the following 2 properties.

(i) The universal covering surface of $\tilde{S}$ is $\mathbb{H}$.

(ii) The center of $\pi_1(\tilde{S})$ is trivial.

Lemma 3.11. *Let f be a non-trivial analytic homeomorphism of $\tilde{S}$. Suppose $f(x) = x$ and f_* be the induced automorphism of $\pi_1(\tilde{S}, x)$. Then f_* leaves no element of $\pi_1(\tilde{S}, x)$ fixed except the identity element.*

Proof. Let $[\gamma] \in \pi_1(\tilde{S}, x)$ be a non-trivial loop such that $f(\gamma) \simeq \gamma$ (here $\simeq$ denotes path-homotopic). Consider universal covering q of $\tilde{S}$ such that lift of f is an analytic homeomorphism $\tilde{f} : \mathbb{H} \to \mathbb{H}$, so that, $\tilde{f} \in \mathrm{Isom}^+(\mathbb{H})$. Assuming $q(\tilde{x}) = x$ and $\tilde{f}(y) = \tilde{x}$, then there exists a deck transformation h such that $h(y) = \tilde{x}$. Composing $\tilde{f}$ with h, we get another lift of f that fixes $\tilde{x}$. Thus, without loss of generality, we can assume that $\tilde{f}(\tilde{x}) = \tilde{x}$.

If $\tilde{\gamma}$ is the lift of the loop γ, then $\tilde{f}(\tilde{\gamma})$ is a lift of $f(\gamma)$. Since γ and $f(\gamma)$ are path-homotopic, $\tilde{f}(\tilde{\gamma})$ and $\tilde{\gamma}$ are also path-homotopic. Thus, if $\tilde{y}$ is the end point of $\tilde{\gamma}$, then $\tilde{y}$ is also the end point of $\tilde{f}(\tilde{\gamma})$, that is, $\tilde{f}(\tilde{y}) = \tilde{y}$. Therefore, $\tilde{f}$ is an isometry of $\mathbb{H}$ with 2 fixed points in the interior of $\mathbb{H}$. Hence, $\tilde{f}$ must be identity which implies that f is identity. □

Lemma 3.12. *Let f be a fiber-preserving homeomorphism of $\tilde{S}$, which is isotopic to identity. Then f commutes with all the deck transformations.*

Proof. Let h be a deck transformation and $r = f \circ h \circ f^{-1} \circ h^{-1}$. We show that r is identity. Since f is fiber-preserving, by Proposition 3.3 it follows that $f \circ h \circ f^{-1}$ is a deck transformation. Thus, r is also a deck transformation. Since f is isotopic to identity, we have $h \circ g^{-1} \circ h^{-1}$ is isotopic to identity, and therefore, r is isotopic to identity.

Since r is a deck transformation, it is analytic and fixes branch points. Thus, $r_* = \hat{\alpha}$, where α is a loop based at a branch point b. Therefore, r_* is an automorphism of $\pi_1(\tilde{S}, b)$ which fixes loop α. Hence, by Lemma 3.11, r must be the identity map. □

Let $x_1, x_2, \dots, x_n \in S$ be the branch points. By our hypothesis, deck transformations fix each pre-image of the branch point. Assume that $\tilde{x}_1, \tilde{y}_1 \in \tilde{S}$ such that $p(\tilde{x}_1) = p(\tilde{y}_1)$. Since for regular covers, there exists a deck transformation that maps $\tilde{x}_1 \to \tilde{y}_1$, we must have $\tilde{x}_1 = \tilde{y}_1$. Thus, the pre-image of each branch point is a single point. Let us denote the corresponding pre-images by $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$.

Lemma 3.13. *Let f be a fiber-preserving homeomorphism of $\tilde{S}$ which is isotopic to the identity map via an isotopy $H(t, x)$. Then*

(i) $f(\tilde{x}_i) = \tilde{x}_i$, for all i, and

(ii) $H(t, \tilde{x}_i)$ is nullhomotopic in $\pi_1(\tilde{S}, \tilde{x}_1)$, for all i.

Proof. Suppose $f(\tilde{x}_i) = \tilde{x}_j$, for some $i \neq j$. Let γ be a loop based at $\tilde{x}_i$ and h be a non-trivial deck transformation. Then by Lemma 3.12, we have $f(h(\gamma)) = h(f(\gamma))$. Now, let β be a path from $\tilde{x}_i$ to $\tilde{x}_j$ defined as $\beta(s) = H(s, \tilde{x}_i)$. Then

$$\gamma \simeq \beta f(\gamma) \beta^{-1}, \tag{3.2}$$

and from Equation (3.2), we get

$$h(\gamma) \simeq h(\beta)h(f(\gamma))h(\beta)^{-1} \simeq h(\beta)f(h(\gamma))h(\beta)^{-1}. \tag{3.3}$$

Since γ is an arbitrary loop based at $\tilde{x}_i$ and $t(\gamma)$ is also a loop based at $\tilde{x}_i$, we have

$$h(\gamma) \simeq \beta f(h(\gamma))\beta^{-1}. \tag{3.4}$$

From Equations (3.3-3.4), we get

$$\beta^{-1}h(\beta)f(h(\gamma))h(\beta)^{-1}\beta \simeq f(h(\gamma)). \tag{3.5}$$

As γ was arbitrary and f, h are homeomorphisms, $\beta^{-1}h(\beta)$ commutes with all elements of $\pi_1(\tilde{S}, \tilde{x}_j)$. Observe that $\pi_1(\tilde{S})$ has a trivial center, therefore $\beta^{-1}h(\beta)$ is nullhomotopic, and so we have $h(\beta) \simeq \beta$. Now let $\tilde{h}$ be the lift of h to $\mathbb{H}$ under the universal cover. By composing $\tilde{h}$ with a deck transformation, if required, we can assume that $\tilde{h}(\tilde{y}_i) = \tilde{y}_i$, for some $\tilde{y}_i$ in the fiber of $\tilde{x}_i$. Let $\tilde{y}_j$ be the endpoint of the lift of β starting at $\tilde{y}_i$. Since $h(\beta)$ is path-homotopic to β by lifting the homotopy, their lifts are also path-homotopic, and therefore, $\tilde{h}(\tilde{y}_j) = \tilde{y}_j$. Thus, $\tilde{h}$ has 2 fixed points and so $\tilde{h}$ must be identity. Since h was non-trivial, we must have $f(\tilde{x}_i) = \tilde{x}_i$. Hence, the first claim is proved.

Let h be a non-trivial deck transformation. Then following the similar arguments as above, we get $t(\beta) \simeq \beta$, where β is a closed loop based at $\tilde{x}_1$. By Lemma 3.11 β must be nullhomotopic, and hence $H(t, \tilde{x}_i)$ is nullhomotopic in $\pi_1(\tilde{S}, \tilde{x}_1)$. □

The following lemma follows from the simplicial approximation theorem [2] and Theorem 1.6.

Lemma 3.14. *Let P be a point in a piecewise-linear manifold X without boundary. Let $\beta(s)$ be a curve in X homotopic to 0 in $\pi_1(X,P)$. There is an isotopy $K:[0,1]\times X \to X$ such that $K(0,x)=K(1,x)=x$ where the map $K_s : X \to X, K_s(x) = K(s,x)$ has the compact support and $K_s(P)=\beta(s)$.*

Lemma 3.15. *Let f be a fiber-preserving homeomorphism of $\tilde{S}$, isotopic to identity via $H:[0,1]\times\tilde{S}\to\tilde{S}$. Then f is isotopic to identity via $\bar{H}:[0,1]\times\tilde{S}\to\tilde{S}$ such that $\bar{H}(s,\tilde{x}_i)=\tilde{x}_i$, for $1\le i\le n$ and $s\in[0,1]$.*

Proof. By Lemma 3.13 $f(\tilde{x}_1)=\tilde{x}_1$ and $\beta(s)=H(s,\tilde{x}_1)$ is nullhomotopic in $\pi_1(\tilde{S},\tilde{x}_1)$. Now, by Lemma 3.14 there is an isotopy $K:[0,1]\times\tilde{S}\to\tilde{S}$ with $K(0,x)=K(1,x)=x$ and $K_s(\tilde{x}_1)=\beta(s)$. Let $G_s=K_s^{-1}\circ H_s$, where K_s is a map from $\tilde{S}\to\tilde{S}$ and $K_s(x)=K(s,x)$ (H_s is defined similarly). Then, $G_s(\tilde{x}_1)=\tilde{x}_1$ for all $s\in[0,1]$ and $G:[0,1]\times\tilde{S}\to\tilde{S}$ is an isotopy of f to identity. We consider the covering map $\bar{p}:\tilde{S}-\tilde{x}_1\to S-x_1$ and homeomorphism $f|_{\tilde{S}-\tilde{x}_1}$.

Since $\tilde{S}-\tilde{x}_1$ is hyperbolic, deck transformations of $\bar{p}$ fixes branch points and $f|_{\tilde{S}-\tilde{x}_1}$ is isotopic to identity via the isotopy $G|_{[0,1]\times\tilde{S}-\tilde{x}_1}$. Thus, we can repeat the argument for each of $\tilde{x}_2,\tilde{x}_3,\dots,\tilde{x}_n$ to obtain an isotopy with the required properties. □

Lemma 3.16. *Let $q:\tilde{S}\to S$ be a regular, unbranched cover. Let $\tilde{f}:\tilde{S}\to\tilde{S}$ be a fiber-preserving homeomorphism isotopic to identity. Assume that the centralizer of $q_*\pi_1(\tilde{S})$ in $\pi_1(S)$ is trivial. Then $\tilde{f}$ is isotopic to identity via fiber-preserving homeomorphisms.*

Proof. Let f be the projection of $\tilde{f}$ to S. Suppose that $\tilde{x}\in\tilde{S}$ and $x\in S$ such that $q(\tilde{x})=x$. Let $\tilde{\beta}(s)=H(s,\tilde{x})$ be the path from $\tilde{x}$ to $\tilde{f}(\tilde{x})$, where

$H(s,y):[0,1]\times\tilde{S}\to\tilde{S}$ is an isotopy of $\tilde{f}$ to identity. The projection $\beta(s)=q(\tilde{\beta}(s))$ is a path from x to $f(x)$. Suppose that $\tilde{\gamma}$ is a loop based at $\tilde{x}$ and $\gamma=q(\tilde{\gamma})$. Since f is a homeomorphism, we have an automorphism f_* of $\pi_1(S,x)$ defined as $f_*([\gamma])=[\beta f(\gamma)\beta^{-1}]$. If $[\gamma]\in q_*\pi_1(\tilde{S},\tilde{x})$, then

$$f_*([\gamma])=[\beta q(\tilde{f}(\tilde{\gamma}))\beta^{-1}]=[q(\tilde{\beta}\tilde{f}(\tilde{\gamma})\tilde{\beta}^{-1})].$$

Since $\tilde{f}$ is isotopic to identity, we have $\tilde{\gamma}\simeq\tilde{\beta}\tilde{f}(\tilde{\gamma})\tilde{\beta}^{-1}$. Therefore, $f_*([\gamma])=[\gamma]$ for all $[\gamma]\in q_*\pi_1(\tilde{S},\tilde{x})$. We choose some $[\alpha]\in q_*\pi_1(\tilde{S},\tilde{x})$ and $[\delta]\in\pi_1(S,x)$. Since q is regular, $[\delta\alpha\delta^{-1}]\in q_*\pi_1(\tilde{S},\tilde{x})$. Therefore,

$$[\delta\alpha\delta^{-1}]=f_*([\delta\alpha\delta^{-1}])=f_*([\delta])[\alpha]f_*([\delta])^{-1}. \tag{3.6}$$

From Equation (3.6), it follows that $f_*([\delta])^{-1}[\delta]$ belongs to the centralizer of $\pi_1(S)$, and therefore, is trivial. Hence, $f_*([\delta])=[\delta]$ for any $[\delta]\in\pi_1(S,x)$. By the Dehn-Nielsen-Baer Theorem, it follows that f is isotopic to identity, and so $\tilde{f}$ is isotopic to identity through fiber-preserving homeomorphisms. □

Proof of Theorem 3.8. First, we consider the case when p is unbranched. Assume that S is not homeomorphic to $S_{0,0},S_{0,1},S_{0,2}$, and $S_{1,0}$. We show that the hypothesis of Lemma 3.16 is satisfied, by showing that H has a trivial centralizer in $G=\pi_1(S)$, where $H=q_*(\pi_1(\tilde{S}))$. The fundamental group G is either free or is 1-relator group. If G is free, then G must have rank at least 2 (as S is not homeomorpic to $S_{0,0},S_{0,1}$ or $S_{0,2}$). Hence, any subgroup of G has a trivial centralizer.

If G is 1-relator, then G has a trivial center. Therefore, H has also trivial center. Since the deck group is finite, H is of finite index in G. Suppose that

$\alpha \in C_G(H) - 1$, where $C_G(H)$ denotes the centralizer of H in G. Let $[\alpha]$ be a coset of H in G. Since H is of finite index in G, we have $[\alpha]^\lambda = H$ for some $\lambda \in \mathbb{Z}$. Therefore, $\alpha^\lambda \in H$, and since $\alpha^\lambda \in C_G(H)$ we must have α^λ in the center of H. Thus, $\alpha^\lambda = 1$. Since G is torsion-free, we have $\alpha = 1$. Hence, $C_G(H)$ is trivial.

If S is homeomorphic to $S_{0,0}$ or $S_{0,1}$, then the conclusion holds trivially, as $\mathrm{Mod}(S_{0,0})$ and $\mathrm{Mod}(S_{0,1})$ are trivial. Now assume that $S \approx S_{0,2}$. Let $\tilde{f}$ be a fiber-preserving homeomorphism isotopic to identity and f be its projection onto $\mathrm{Mod}(S_{0,2})$. Since $\mathrm{Mod}(S_{0,2}) \cong \mathbb{Z}_2$, f is either isotopic to identity (we are done in that case) or f is isotopic to a homeomorphism that exchange punctures. In the second case, $\tilde{f}$ must also exchange punctures, so that it can not be isotopic to identity.

For $S_{1,0}$, as in the proof of Lemma 3.16, we may assume f_* restricted to $p_*(\pi_1(\tilde{S}))$ is identity. We have that any automorphism of $\mathbb{Z} \oplus \mathbb{Z}$, whose restriction to a subgroup of finite index is identity, is itself identity. Since $\pi_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}$, it follows from the Dehn-Nielsen-Baer theorem that $\tilde{f}$ is isotopic to identity.

Now assume that p is branched. If $\chi(S) < 0$, then by Lemma 3.15, we can replace the branched covering with the associated unbranched covering. Hence, the conclusion follows from the previous case. Since the cases $\tilde{S} = S_{0,0}$ and $\tilde{S} = S_{1,0}$ are excluded by hypothesis, to complete the proof, we show that the conclusion holds when $\tilde{S} \approx S_{0,2}$ or $S_{0,1}$. We note that a deck transformation for the cover of $S_{0,2}$ is a rotation, and the identity is the only rotation that fixes a point other than the origin. Hence, the case is trivial. If there is a non-trivial deck transformation of $S_{0,1}$, then the deck transformation can have at most one fixed point (since it is an isometry of plane), which we can assume to be the origin. Thus, the deck group is a finite group of rotations which is cyclic. Thus, $S \approx S_{0,1}$.

The proof of the theorem follows from the fact that any homeomorphism of $S_{0,1}$ is isotopic to the identity. □

3.4 Birman–Hilden property for solvable deck groups

Lemma 3.17. *Let $p : \tilde{S} \to S$ be a regular, finite sheeted, branched covering of Riemann surfaces, with at least one branch point, where $\tilde{S}$ is either the torus or sphere. Assume that the deck transformations leave the branch points fixed. If $\tilde{f} : \tilde{S} \to \tilde{S}$ is a fiber-preserving homeomorphism isotopic to identity, then the induced homeomorphism $f : S \to S$ is also isotopic to identity.*

Proof. If $\tilde{S}$ is a sphere, then $\tilde{S}$ can only cover a sphere (as S is orientable). In this case, the result holds trivially. Now we assume that $\tilde{S}$ is the torus. We shall think of $\tilde{S}$ as $\mathbb{C}$ quotient with the subgroup of isometries isomorphic to $\mathbb{Z}^2$. Without loss of generality, we assume that the origin is a branch point. The deck transformations of p can be lifted to a Möbious transformations of $\mathbb{C}$, leaving the lattice invariant and fixing origin. These Möbious transformations are of the form $T(z) = e^{i\theta}z$, where $\theta \in \{\pi/2, \pi, 3\pi/2\}$. Observe that the quotient of $\tilde{S}$ by the cyclic group of isometries generated by any of the above isometry T is homeomorphic to the sphere. Since any homeomorphism of a sphere is isotopic to identity, the assertion follows. □

Remark 3.18. If $p : \tilde{S} \to S$ is a covering map, then S is homeomorphic to the orbit space $\tilde{S}/G$, where G is the deck group. Suppose that $H \lhd G$. The covering $q : \tilde{S} \to \tilde{S}/H$ is regular as $\tilde{S}/H$ is the orbit space of the $H-$action on $\tilde{S}$. Let

$r : \tilde{S}/H \to \tilde{S}/G$ be the covering map obtained by the action of G/H on $\tilde{S}$. Since $H \lhd G$, r is regular. Suppose that q and r have the Birman–Hilden property. Let $\tilde{f} : \tilde{S} \to \tilde{S}$ be fiber-preserving homeomorphism for covering map p isotopic to identity. Then the projection $f_1 : \tilde{S}/H \to \tilde{S}/H$ under q is isotopic to identity. Since f_1 preserves fibers of r, the projection $f_2 : \tilde{S}/G \to \tilde{S}/G$ of f_1 under r is isotopic to the identity. Since f_2 is the projection of $\tilde{f}$ under p, it follows that p has the Birman–Hilden property.

Proof of Theorem 3.9. First, we assume that the deck group G is cyclic of prime order. By the orbit-stabilizer theorem, the number of elements in each orbit divides the number of elements in G. Thus, p must be one-to-one on the branch points. Therefore, every deck transformation leaves the branch point fixed. Thus, the result follows from Theorem 3.8 and Lemma 3.17.

Now we assume that G is a finite solvable group. Then G can be factored through cyclic groups of prime order. The factoring of G induces a factoring of covering map p under composition. The result follows from Remark 3.18 and an inductive argument. □

3.5 Restatement of the theorem in terms of mapping class groups

In this section, we relate the Birman–Hilden property to mapping class groups. This section is based on [11].

Definition 3.19. Let $p : \tilde{S} \to S$ be a covering map. Then the subgroup of $\mathrm{Mod}(\tilde{S})$ consisting of mapping classes represented by fiber-preserving homeomorphisms is known as the *symmetric mapping class group*, denoted by $\mathrm{SMod}(\tilde{S})$.

Definition 3.20. Let $p : \tilde{S} \to S$ be a covering map and $\mathcal{B}$ be the set of branch points. Then the subgroup of $\mathrm{Mod}(S, \mathcal{B})$ consisting of mapping classes represented by liftable homeomorphisms is known as *liftable mapping class group*, denoted by $\mathrm{LMod}(S, \mathcal{B})$.

Let K be the subgroup of $\mathrm{SMod}(\tilde{S})$ consisting of the homotopy classes of the deck transformations of the covering $p : \tilde{S} \to S$. Now we can restate the above theorems in terms of mapping class groups using the following proposition.

Proposition 3.21 ([11]). *Let $p : \tilde{S} \to S$ be a finite sheeted, branched covering map with the set of branched points $\mathcal{B}$, where $\tilde{S}$ is a hyperbolic surface without boundary. Then the following are equivalent.*

(i) The map p has the Birman–Hilden property.

(ii) There exists a surjective homomorphism from $\mathrm{SMod}(\tilde{S}) \to \mathrm{LMod}(S, \mathcal{B})$, with kernel K.

Proof. Let p has the Birman–Hilden property. Define $\Phi : \mathrm{SMod}(\tilde{S}) \to \mathrm{LMod}(S, \mathcal{B})$ as $\Phi([\tilde{f}]) = [f]$, where f is the projection of $\tilde{f}$ under p. The map Φ is well-defined as p has the Birman–Hilden property. By definition, it is clear that Φ is a surjective homomorphism. Since deck transformations are lifts of identity, it follows that $\ker \Phi$ is K.

Conversely, if there exists a surjective homomorphism from $\mathrm{SMod}(\tilde{S}) \to \mathrm{LMod}(S, \mathcal{B})$, then given a fiber-preserving homeomorphism $\tilde{f}$ isotopic to identity, its projection f is also isotopic to identity. By lifting this isotopy, we get an isotopy through fiber-preserving homeomorphisms. Hence, p has the Birman–Hilden property. □

The following is immediate from Theorem 3.10 and Proposition 3.21.

Corollary 3.21.1. *Let $p : \tilde{S} \to S$ be a finite sheeted, regular branched covering map with the set of branched points $\mathcal{B}$, where $\tilde{S}$ is a hyperbolic surface. Assume that p is either unbranched or solvable. Then we have the following short exact sequence*

$$1 \to K \to \mathrm{SMod}(\tilde{S}) \to \mathrm{LMod}(S, \mathcal{B}) \to 1. \tag{3.7}$$

Chapter 4

Injections of Mapping Class Groups

In their paper [1], Aramayona, Leininger, and Souto constructed injective homomorphisms between the mapping class groups using the covering between surfaces. The construction of these homomorphisms between the mapping class groups uses the Birman–Hilden property of the covers. This chapter is based on their paper [1].

4.1 Introduction

Throughout this chapter, for the surface $S = S_{g,n}$ without boundary, we assume that $3g+n \geq 5$. To keep better track of marked points, we denote $S_{g,n}$ by $(S, \mathcal{B})$, where $S = S_g$ and $\mathcal{B} \subset S$ is a set of n marked points.

We say a cover is irregular if it is not regular. Let $p : \tilde{S} \to S$ be a k-sheeted, unbranched, possibly irregular cover and $\tilde{\mathcal{B}} = p^{-1}(\mathcal{B})$ a set of kn marked points of $\tilde{S}$. Let $\mathrm{LHomeo}(S, \mathcal{B})$ be the set of all liftable homeomorphisms of S which preserves $\mathcal{B}$. Let $\mathrm{SHomeo}(\tilde{S}, \tilde{\mathcal{B}})$ be the set of all fiber-preserving homeomorphisms of $\tilde{S}$ which preserves $\tilde{\mathcal{B}}$.

Remark 4.1. For a finite-sheeted unbranched cover $p : \tilde{S} \to S$, recall that there is a homomorphism $\phi : \mathrm{SHomeo}(\tilde{S}, \tilde{\mathcal{B}}) \to \mathrm{LHomeo}(S, \mathcal{B})$, with $\ker \phi \cong D$. In other words, we have the following short exact sequence

$$1 \to D \to \mathrm{SHomeo}(\tilde{S}, \tilde{\mathcal{B}}) \to \mathrm{LHomeo}(S, \mathcal{B}). \tag{4.1}$$

4.2 Birman–Hilden property for unbranched covers

The work of Birman–Hilden was mostly restricted to regular covers. Since we will be using irregular covers, we first show that all unbranched covers have the Birman–Hilden property.

Proposition 4.2. *Let $p : \tilde{S} \to S$ be a finite sheeted unbranched covering. If $\tilde{f} \in \mathrm{SHomeo}(\tilde{S}, \tilde{\mathcal{B}}) \cap \mathrm{Homeo}_0(\tilde{S}, \tilde{\mathcal{B}})$, then $\phi(\tilde{f}) \in \mathrm{Homeo}_0(S, \mathcal{B})$. Moreover, the restriction of ϕ to $\mathrm{SHomeo}(\tilde{S}, \tilde{\mathcal{B}}) \cap \mathrm{Homeo}_0(\tilde{S}, \tilde{\mathcal{B}})$ is an isomorphism.*

Proof. For $S^0 = S - \mathcal{B}$ and $\tilde{S}^0 = \tilde{S} - \tilde{\mathcal{B}}$, p induces a cover $q : \tilde{S}^0 \to S^0$. Let x and $\tilde{x}$ be the base points of S^0 and $\tilde{S}^0$ respectively with $q(\tilde{x}) = x$. Then $q_* : \pi_1(\tilde{S}^0, \tilde{x}) \to \pi_1(S^0, x)$ is a homomorphism and we have an isomorphism from $\pi_1(\tilde{S}^0, \tilde{x}) \to \pi_1(\tilde{S}^0, \tilde{f}(\tilde{x}))$ given by $[\eta] \mapsto [\tilde{f}(\eta)]$. For a path τ from $\tilde{x}$ to $\tilde{f}(\tilde{x})$, there is an isomorphism $\hat{\tau} : \pi_1(\tilde{S}^0, \tilde{f}(\tilde{x})) \to \pi_1(\tilde{S}^0, \tilde{x})$ given by $[\eta] \mapsto [\tau\eta\tau^{-1}]$. Thus, we get an automorphism $\tilde{f}_* : \pi_1(\tilde{S}^0, \tilde{x}) \to \pi_1(\tilde{S}^0, \tilde{x})$ by composing the two maps. Similarly, for $f = \phi(\tilde{f})$ we have an automorphism $f_* : \pi_1(S^0, x) \to \pi_1(S^0, x)$ by composing the isomorphism induced by f and the isomorphism induced by the path $q(\tau)$.

Since $q\tilde{f} = fq$, we have

$$q_* \circ \tilde{f}_*(\eta) = q_*([\tau\tilde{f}(\eta)\tau^{-1}]) = q([\tau])q([\tilde{f}(\eta)])q([\tau])^{-1} = f_* \circ q_*(\eta). \qquad (4.2)$$

As $\tilde{f}$ is isotopic to identity, we have $[f(\eta)] = [\alpha\eta\alpha^{-1}]$, where α is a path from $\tilde{f}(\tilde{x})$ to $\tilde{x}$ induced by the isotopy between $\tilde{f}$ and identity. Thus, $\tilde{f}_*$ is given by $\tilde{f}_*([\eta]) = [\tilde{\gamma}\eta\tilde{\gamma}^{-1}]$, where $\tilde{\gamma}$ is a loop based at $\tilde{x}$.

Let ψ be an automorphism of $\pi_1(S^0, x)$ defined as $\psi([\eta]) = [\gamma^{-1}f_*(\eta)\gamma]$, where $\gamma = q(\tilde{\gamma})$ is a loop based at x. We observe that ψ is identity on the subgroup $q_*(\pi_1(\tilde{S}^0, \tilde{x}))$. Since the deck group is finite, $q_*(\pi_1(\tilde{S}^0, \tilde{x}))$ has finite index in $\pi_1(S^0, x)$. Let $H = q_*(\pi_1(\tilde{S}^0, \tilde{x}))$ and $G = \pi_1(S^0, x)$. Suppose gH is a coset of H in G. Since $|G/H|$ is finite, $g^n \in H$ for some $n \in \mathbb{Z}$. For $[\eta] \in \pi_1(S, x)$, we have $[\eta^m] \in H$. Since ψ is identity on H, we have $\psi([\eta]^m) = [\eta]^m$, that is, $[(\gamma^{-1}f_*([\eta])\gamma)]^m = [\eta]^m$. By Proposition 1.15, we have $[\gamma^{-1}f_*([\eta])\gamma] = [\eta]$. Therefore, ψ is identity, and f_* is an inner automorphism. Hence, by the Dehn-Neilsen-Baer theorem, f is isotopic to identity.

The surjectivity of ϕ follows from Proposition 1.4. Without loss of generality, we assume that q is a Riemannian cover. Any $\tilde{f} \in \ker\ \phi$ is a lift of the identity (which is an isometry). Then it follows that $\tilde{f}$ is an isometry of $\tilde{S}^0$. Since $\tilde{f}$ is an isometry of a hyperbolic surface isotopic to identity the lift of $\tilde{f}$ an isometry of $\mathbb{H}$ which is at a bounded distance from identity, and hence $\tilde{f}$ is identity. Therefore, ϕ is an isomorphism. $\square$

Let $\overline{\text{SMod}}(\tilde{S}, \tilde{\mathcal{B}})$ be the subgroup of the extended mapping class group $\overline{\text{Mod}}(\tilde{S}, \tilde{\mathcal{B}})$

consisting of mapping classes represented by fiber-preserving homeomorphisms and $\overline{\mathrm{LMod}}(S, \mathcal{B})$ be the subgroup of $\overline{\mathrm{Mod}}(S, \mathcal{B})$ consisting of mapping classes represented by liftable homeomorphisms.

Corollary 4.2.1. *The sequence 4.1 descends to a short exact sequence.*

$$1 \to K \to \overline{\mathrm{SMod}}(\tilde{S}, \tilde{\mathcal{B}}) \to \overline{\mathrm{LMod}}(S, \mathcal{B}) \to 1. \tag{4.3}$$

Proof. Define $\Phi : \overline{\mathrm{SMod}}(\tilde{S}, \tilde{\mathcal{B}}) \to \overline{\mathrm{LMod}}(S, \mathcal{B})$ by $\Phi([\tilde{f}] = [\phi(\tilde{f})])$. If $\tilde{f}_1, \tilde{f}_2 \in \mathrm{SHomeo}(\tilde{S}, \tilde{\mathcal{B}})$ and $\tilde{f}_1$ is isotopic to $\tilde{f}_2$, then $\tilde{f}_1 \circ \tilde{f}_2^{-1}$ is isotopic to identity and $\tilde{f}_1 \circ \tilde{f}_2^{-1} \in \mathrm{SHomeo}(\tilde{S}, \tilde{\mathcal{B}})$. Thus, $f_1 \circ f_2$ is isotopic to identity by Proposition 4.2, where $f_1 = \phi(\tilde{f}_1)$ and $f_2 = \phi(\tilde{f}_2)$. Since f_1 is isotopic to f_2, Φ is well defined. Since ϕ is an isomorphism, it follows that the sequence in (4.3) short exact sequence exists. □

4.3 Injections between mapping class groups of punctured surfaces

In this section, we study how one can use the Birman–Hilden property of covers to construct injective homomorphisms between the mapping class groups of surfaces.

Proposition 4.3. *Given a finite sheeted, unbranched covering $p : \tilde{S} \to S$, there is an injective homomorphism $\overline{\mathrm{Mod}}(S, \mathcal{B}) \to \overline{\mathrm{Mod}}(\tilde{S}, \tilde{\mathcal{B}})$ obtained by lifting mapping classes in $\overline{\mathrm{Mod}}(S, \mathcal{B})$ to $\overline{\mathrm{Mod}}(\tilde{S}, \tilde{\mathcal{B}})$, provided the following conditions hold.*

(i) The conjugacy class of $p_*(\pi_1(\tilde{S}))$ in $\pi_1(S)$ is invariant by the action of $\overline{\mathrm{Mod}}(S, \mathcal{B})$.

(ii) The following sequence splits

$$1 \to K \to \overline{\mathrm{SMod}}(\tilde{S}, \tilde{\mathcal{B}}) \to \overline{\mathrm{LMod}}(S, \mathcal{B}) \to 1.$$

Proof. Let f be a homeomorphism of $(S, \mathcal{B})$. Then as in the proof of Proposition 4.2, we define an automorphism f_* of $\pi_1(S, x)$ by composing the isomorphism induced by f and isomorphism induced by a path from $f(x)$ to x. This automorphism is well-defined up to inner automorphisms of $\pi_1(S, x)$, since for different choices of paths, the isomorphisms induced by paths differ by inner automorphisms. Therefore, f_* maps a finite index subgroup of $\pi_1(S, x)$ to a conjugate subgroup of finite index. Thus, the group $\mathrm{Homeo}(S, \mathcal{B})$ acts on the set of conjugacy classes of finite index subgroups of $\pi_1(S)$. Note that this action does not depend on marked points.

This action of $\mathrm{Homeo}(S, \mathcal{B})$ descends to the action of $\overline{\mathrm{Mod}}(S, \mathcal{B})$ on the set of conjugacy classes of finite index subgroups of $\pi_1(S)$, as the automorphism induced by homotopic maps differ by inner automorphisms. By Proposition 3.4, condition *(i)* is equivalent to $\overline{\mathrm{Mod}}(S, \mathcal{B}) = \overline{\mathrm{LMod}}(S, \mathcal{B})$. Thus, we have the following short exact sequence

$$1 \to K \to \overline{\mathrm{SMod}}(\tilde{S}, \tilde{\mathcal{B}}) \to \overline{\mathrm{Mod}}(S, \mathcal{B}) \to 1.$$

By condition *(ii)*, the above short exact sequence splits, that is, there exists an injective homomorphism from $\overline{\mathrm{Mod}}(S, \mathcal{B}) \to \overline{\mathrm{Mod}}(\tilde{S}, \tilde{\mathcal{B}})$. □

From Proposition 4.3, we can construct injective homomorphisms between mapping class groups of punctured surfaces. The following corollary illustrates this construction.

Corollary 4.3.1. *For $g \geq 2$, let S be a surface of genus g with a single marked point $\{z\}$. Let $p : \tilde{S} \to S$ be a k-sheeted characteristic cover, that is, $p_*(\pi_1(\tilde{S}))$ is a characteristic subgroup of $\pi_1(S)$. Let $\mathcal{Z} = p^{-1}(z)$ be the set of k marked points in $\tilde{S}$. Then there is an injective homomorphism from $\overline{\text{Mod}}(S, z) \to \overline{\text{Mod}}(\tilde{S}, \mathcal{Z})$.*

Proof. We show that the cover p satisfies the hypothesis of Proposition 4.3. Since any automorphism of $\pi_1(S)$ fixes $p_*(\pi_1(\tilde{S}))$, condition *(i)* is satisfied. Since a characteristic subgroup is normal, the deck group acts transitively on $\mathcal{Z}$. Therefore, for any $f \in \text{Homeo}(S, z)$, by composing with a deck transformation if necessary, there is a lift of f which fixes $\tilde{z}$, for some fixed $\tilde{z} \in \mathcal{Z}$. This induces a homomorphism $\text{Homeo}(S, z) \to \text{SHomeo}(\tilde{S}, \mathcal{Z})$. Hence, the short exact sequence (4.3) splits. Therefore, p induces an injective homomorphism $\overline{\text{Mod}}(S, z) \to \overline{\text{Mod}}(\tilde{S}, \mathcal{Z})$. □

4.4 Injections between the mapping class groups of closed surfaces

In this section, we study an application of Proposition 4.3 to construct injective homomorphisms between mapping class groups of closed surfaces. From now on, we assume that S is a closed surface of genus $g \geq 2$.

Proposition 4.4. *For a finite group G, let $\rho : \pi_1(S) \to G$ be a surjective homomorphism with characteristic kernel. Suppose that $H \subset G$ is a subgroup such that*

(i) $N_G(H) = H$, *and*

(ii) $\text{Aut}(G)\ H = \text{Inn}(G)\ H$,

where $N_G(H)$ *is the normalizer of* H *in* G *and* $\mathrm{Aut}(G)$ *and* $\mathrm{Inn}(G)$ *are the group of automorphisms and inner automorphisms of* G*, respectively. Let* $p : \tilde{S} \to S$ *be a cover corresponding to* $\rho^{-1}(H)$*. Then* p *induces an injective homomorphism* $\overline{\mathrm{Mod}}(S) \to \overline{\mathrm{Mod}}(\tilde{S})$*.*

Proof. We show that p satisfies the hypothesis of Proposition 4.3. Let $X = \pi_1(S)$ and $X_0 = \rho^{-1}(H)$. We know that the deck group is isomorphic to $N_X(X_0)/X_0$. For $x \in N_X(X_0)$, we have $\rho(xX_0x^{-1}) = \rho(X_0)$. Since $N_G(H) = H$, it follows that $\rho(x) \in H$, so that, $N_X(X_0) = X_0$. Hence, the deck group of p is trivial. Thus, $\Phi : \overline{\mathrm{SMod}}(\tilde{S}) \to \overline{\mathrm{LMod}}(S)$ is an isomorphism, and therefore, the short exact sequence in (4.3) splits.

If $\mathrm{Aut}(X)\ X_0 = \mathrm{Inn}(X)\ X_0$, then by the Dehn-Nielsen-Baer theorem, conjugacy class of $\pi_1(\tilde{S})$ is invariant by the action of $\overline{\mathrm{Mod}}(S)$. Hence, it suffices to show that $\mathrm{Aut}(X)\ X_0 = \mathrm{Inn}(X)\ X_0$. For $\sigma \in \mathrm{Aut}(X)$. Define $\tau : G \to G$ as $\tau(g) = \rho\sigma\rho^{-1}(g)$. Since σ preserves $\ker \rho$ which is characteristic, the map τ is well-defined. Moreover, τ is an automorphism as σ is an automorphism. It follows that $\rho \circ \sigma = \tau \circ \rho$.

Since $\mathrm{Aut}(G)\ H = \mathrm{Inn}(G)\ H$, there is a $g \in G$ such that $\tau(H) = gHg^{-1}$. For, $x \in \rho^{-1}(g)$, we have

$$\sigma(X_0) = \rho^{-1}(\tau(H)) = x\rho^{-1}(H)x^{-1} = xX_0x^{-1}. \tag{4.4}$$

Thus, $\mathrm{Aut}(X)\ X_0 = \mathrm{Inn}(X)\ X_0$. Hence, by Proposition 4.3, p induces an injective homomorphism $\overline{\mathrm{Mod}}(S) \to \overline{\mathrm{Mod}}(\tilde{S})$. □

Now we show that there exists a finite group G with a self-normalizing group H such that $\mathrm{Aut}(G)\ H = \mathrm{Inn}(G)\ H$.

Proposition 4.5. *Let $\mathcal{S} = \Sigma_3 \times ... \times \Sigma_3$ be the k-fold product of Σ_3, where Σ_3 is the symmetric group on 3 letters. Let $r_j : \mathcal{S} \to \Sigma_3$ be the projection onto the j^{th} factor. Let G be a subgroup of $\mathcal{S}$ such that $r_j(G) = \Sigma_3$ for all $1 \le j \le k$. If H is a Sylow 2-subgroup of G, then H is a proper self-normalizing subgroup with* $\mathrm{Aut}(G)\ H = \mathrm{Inn}(G)\ H$.

Proof. Since $|\Sigma_3| = 6$ and r_j is surjective, it follows that 3 divides $|G|$, and thus any Sylow 2-subgroup of G is proper. For any $\tau \in \mathrm{Aut}(G)$, $\tau(H)$ is also a Sylow 2-subgroup of G. Let p be a prime. Since the Sylow p-subgroups of a finite group are conjugate, we have $\mathrm{Aut}(G)\ H = \mathrm{Inn}(G)\ H$. Since the Sylow p-subgroups are maximal p-subgroups of $\mathcal{S}$ and a Sylow p-subgroup of G is a p-subgroup of $\mathcal{S}$, every Sylow p-subgroup of G is contained in some Sylow p-subgroup of $\mathcal{S}$. Hence, every Sylow p-subgroup of G is the intersection of a Sylow p-subgroup $\mathcal{S}$ with G.

Since $|\Sigma_3| = 2 \times 3$, all order 2 subgroup of Σ_3 are Sylow 2-subgroups of Σ_3. Since order 2 subgroups are generated by order 2 elements. A Sylow 2-subgroup $P < \mathcal{S}$ must be $P = \{(x_1, \dots, x_k) | x_j = X_j \text{ or } x_j = 1\} = \langle X_1 \rangle \times ... \times \langle X_k \rangle$, where $X_1, \dots, X_k \in \Sigma_3$ are of order 2, and 1 is the identity in Σ_3. Let P be such a Sylow 2-subgroup with $H = P \cap G$. For $y = (y_1, \dots, y_k) \in N_G(H)$. We now show that $y \in H$. Since $y \in G$, it suffices to show that $y \in P$. For every $1 \le j \le k$, we claim that there exists $h = (h_1, \dots, h_k) \in H$ so that $h_j = X_j$. Since r_j is surjective , there exists $g \in G$ such that $r_j(g) = X_j$. Since g is order 2, it is in some Sylow 2-subgroup of G. Therefore, there exists a conjugate h of g in H. Since $r_j(g) = X_j$, h_j is non-trivial, $h_j = X_j$. As $yhy^{-1} \in H$, we have $y_j X_j y_j^{-1}$ or $y_j X_j y_j^{-1} = 1$. The second case is not possible, since $X_j \neq 1$, so y_j is in the centralizer of X_j in Σ_3 which is $\{1, X_j\}$. Since this is true for all j, we have $y_j = X_j$ or $y_j = 1$ for all j. Therefore, $y \in P$, and it follows that $N_G(H) = H$. $\square$

4.5 Proof of the Main Theorem

Now, we state and prove the main theorem of [1].

Theorem 4.6. *For every $g \geq 2$ there exists a $g' > g$ and an injective homomorphism $\phi : \overline{\text{Mod}}(S_g) \to \overline{\text{Mod}}(S_{g'})$.*

Proof. Let $\text{Hom}(X, \Sigma_3)$ be the group of all homomorphisms $X \to \Sigma_3$, where $X = \pi_1(S_g)$. The group $\text{Aut}(X)$ acts on $\text{Hom}(X, \Sigma_3)$. It is known that there exists a surjective homomorphism $X \to \Sigma_3$. Let $\{\rho_1, \dots, \rho_k\}$ be the orbit of such a surjective homomorphism under the action of $\text{Aut}(X)$. Define $\rho : X \to \mathcal{S}$ as $\rho = \rho_1 \times \dots \times \rho_k$. For $G = \rho(X) < \mathcal{S}$, we have ρ is surjective onto G. Moreover, since ρ_i is surjective, the projection of G onto j^{th} component is $r_j(G) = \rho_j(X) = \Sigma_3$. By Proposition 4.5, G has a proper subgroup which satisfies the hypothesis of Proposition 4.4. We claim that the kernel of ρ is characteristic. Then by Proposition 4.4, the result follows. Let K denote the kernel of ρ. For $\sigma \in \text{Aut}(X)$ and $k \in K$, we have $\rho_i(k) = 1$ for all i. Further, since $\rho_i \circ \sigma$ lies in the orbit $\{\rho_1, \dots, \rho_k\}$, we must have $\rho_i \circ \sigma = \rho_j$ for some j. Thus, $\rho_i(\sigma(k)) = 1$, for all i. Hence, it follows that K is characteristic. □

4.6 An alternative construction of injective

In this section, we describe an alternate construction of a group G and a $H < G$, satisfying the hypothesis of Proposition 4.4. For this construction, we need the following result of Hall [10]

Lemma 4.7. *Assume that X is any group, that $Q_1, \dots, Q_k$ are finite, non-abelian simple groups, and that $\rho_i : X \to Q_i$ is an epimorphism for $1 \leq i \leq k$.*

If ρ_i and ρ_j do not differ by an isomorphism $Q_i \to Q_j$ for any $i \neq j$, then $\rho = \rho_1 \times \ldots \times \rho_k : X \to Q_i \times \ldots \times Q_j$ is surjective.

For $p \geq 5$, let $A = \mathrm{PSL}_2(\mathbb{F}_p)$ be a finite, non-abelian, simple group. For $X = \pi_1(S)$, let $\rho_0 : X \to A$ be an epimorphism. We note that $\mathrm{Aut}(X)$ and $\mathrm{Aut}(A)$ both act on $\mathrm{Hom}(X, A)$. Let $\{\rho_1, \ldots, \rho_k\}$ be a maximal collection of elements of an $\mathrm{Aut}(X)$-orbit no two of which are in the same $\mathrm{Aut}(A)$-orbit.

Let G be a k-fold product of $\mathrm{PSL}_2(\mathbb{F}_p)$. Define $\rho : X \to G$ as $\rho = \rho_1 \times \ldots \times \rho_k$. For $\phi \in \mathrm{Aut}(X)$ and $1 \leq i \leq k$, $\rho_i \circ \phi$ must lie in a $\mathrm{Aut}(X)$-orbit. By maximality, either $\rho_i \circ \phi \in \{\rho_1, \ldots, \rho_k\}$ or $\rho_i \circ \phi$ differs by an automorphism from ρ_j for some $\tau \in \mathrm{Aut}(A)$ and $1 \leq j \leq k$. Thus, for $\phi \in \mathrm{Aut}(X)$ and $1 \leq i \leq k$, there exists $\tau \in \mathrm{Aut}(A)$ and $1 \leq j \leq k$ such that $\rho_i \circ \phi = \tau \circ \rho_j$. Further, if $x \in \ker \rho$, then $\rho_i(\phi(x)) = \tau(\rho_j(x)) = 0$, that is, $\phi(x) \in \ker \rho$ for all $\phi \in \mathrm{Aut}(X)$. Hence, $\ker \rho$ is characteristic in X. Moreover, Lemma 4.7 implies that ρ is surjective.

Now we construct $H < G$ such that the hypothesis of Proposition 4.4 holds. Consider the subgroup H_0 of upper triangular matrices in A and set $H = H_0 \times \ldots \times H_0$. Since $N_A(H_0) = H_0$, it follows that $N_G(H) = H$. Since G is a product of nonabelian finite simple groups, $\mathrm{Aut}(G)$ acts on G via automorphisms in each factor up to a permutation. Since $\mathrm{Aut}(A)\ H_0 = \mathrm{Inn}(A)\ H_0$, it follows that $\mathrm{Aut}(G)\ H = \mathrm{Inn}(G)\ H$.

4.7 Constructing injections using the Birman Exact Sequence

Let S and $\tilde{S}$ be closed surfaces of genus g and g', respectively such that $p : \tilde{S} \to S$ is a characteristic cover. Let $z \in S$ and $\tilde{z} \in \tilde{S}$ be such that $p(\tilde{z}) = z$. We denote

$\pi_1(S,z)$ and $\pi_1(\tilde{S},\tilde{z})$ by X and $\widetilde{X}$, respectively. Then we have the homomorphism $p_* : \widetilde{X} \to X$ induced by p. The Birman exact sequence is given by

$$1 \to X \to \overline{\mathrm{Mod}}(S,z) \to \overline{\mathrm{Mod}}(S) \to 1. \tag{4.5}$$

For $[f] \in \overline{\mathrm{Mod}}(S,z)$ we get an automorphism $f_* : \pi_1(S,z) \to \pi_1(S,z)$. If f' is isotopic to f relative to z, then the path induced by this isotopy from $f(z) = z$ to $f'(z) = z$ is null-homotopic. Therefore, we must have $f'_* = f_*$. Thus, an element of $\overline{\mathrm{Mod}}(S,z)$ induces an automorphism of X. Since X acts on itself via inner automorphisms, we have the following commutative diagram of short exact sequences.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & X & \longrightarrow & \overline{\mathrm{Mod}}(S,z) & \longrightarrow & \overline{\mathrm{Mod}}(S) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
1 & \longrightarrow & \mathrm{Inn}(X) & \longrightarrow & \mathrm{Aut}(X) & \longrightarrow & \mathrm{Out}(X) & \longrightarrow & 1,
\end{array}$$

Where α, β, and γ are obvious maps. Since $\chi(S) < 0$, the center of X is trivial, therefore α is an isomorphism. By the Dehn-Nielsen-Baer Theorem γ is also an isomorphism. Hence, by the Five Lemma, it follows that β is an isomorphism. Similarly, we have $\overline{\mathrm{Mod}}(\tilde{S},\tilde{z}) \cong \mathrm{Aut}(\widetilde{X})$.

Now, we construct an injective homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(\widetilde{X})$. For $\omega \in \mathrm{Aut}(X)$, define $\alpha : \mathrm{Aut}(X) \to \mathrm{Aut}(\widetilde{X})$ as $\alpha(\omega) = \tilde{\omega}$, where $\tilde{\omega}$ is defined as

$$\tilde{\omega}(\tilde{\gamma}) = p_*^{-1}(\omega(p_*(\tilde{\gamma}))), \tilde{\gamma} \in \widetilde{X}. \tag{4.6}$$

Since $p_*(\widetilde{X})$ is a characteristic in X, we have $\omega(p_*(\widetilde{X})) = p_*(\widetilde{X})$, where $\omega \in \mathrm{Aut}(X)$. Moreover, since p_* is an isomorphism onto $p_*(\widetilde{X})$, $\tilde{\omega}$ is a well-defined element of $\mathrm{Aut}(\widetilde{X})$.

Lemma 4.8. *The homomorphism* $\alpha : \mathrm{Aut}(X) \to \mathrm{Aut}(\widetilde{X})$ *is injective.*

Proof. Assume that $\omega \in \ker \alpha$, that is, $\tilde{\omega}(\tilde{\gamma}) = \tilde{\gamma}$ for all $\tilde{\gamma} \in \widetilde{X}$. Therefore, $\omega(\gamma) = \gamma$, where $\gamma = p_*(\tilde{\gamma})$. Thus, ω restricts to identity on the finite index subgroup $p_*(\widetilde{X})$. Now it follows from Proposition 1.15 that ω is identity. □

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