KNOTS, 3-MANIFOLDS AND THE LICKORISH WALLACE THEOREM

A REPORT

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by

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April 2018

CERTIFICATE

This is to certify that **M V Ajay Kumar Nair**, BS-MS (Dual Degree) student in Department of Mathematics has completed bonafide work on the dissertation entitled **'Knots, 3-manifolds and the Lickorish Wallace theorem'** under my supervision and guidance.

April 2018 **Dr. Kashyap Rajeevsarathy**
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Committee Member	**Signature**	**Date**

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M V Ajay Kumar Nair

ABSTRACT

The purpose of this project is to understand the proof of Lickorish-Wallace Theorem. We begin with studying some aspects of knot theory and prove the existence and uniqueness of prime factorisation of knots. We go on to understand the Jones polynomial and establish the fact that it is a knot invariant. We study surface homeomorphisms and prove the classic result that any orientation preserving homeomorphism can be written as a composition of Dehn twists [8]. Lickorish-Wallace theorem is proved by using the aforementioned theorem.

CONTENTS

1. BASIC SURFACE TOPOLOGY

Surfaces are one of the most interesting objects in topology. In this chapter, we skim through some standard results of surface topology. We also define Euler characteristic and genus of a surface.

1.1 Surfaces and Triangulations

Definition 1.1.1. An *n-dimensional manifold* is a hausdorff topological space such that every point has a neighbourhood homeomorphic to an n-dimensional open disc. A 2-manifold is called a *surface*.

Definition 1.1.2. An *n-dimensional manifold with boundary* is a hausdorff topological space such that every point has a neighbourhood homeomorphic to an n-dimensional open disc or an half disc. The points with half-disc neighbourhoods are called boundary points.

Definition 1.1.3. A surface S is *triangulable* if there is a two complex structure K such that $S = |K|$ and satisfies following conditions:

(i). K has only triangular cells.

(ii). Any two triangles are identified along a single edges or a single vertex or are disjoint.

This triangulated complex K is called a *triangulation* on S.

It is a very classic result about surfaces that every surface permits a triangulation [11]. Moreover, every compact surface has a triangulation with finite triangles. [7, Theorem 4.12]

1.2 Euler characteristic and genus

Definition 1.2.1. Let V, E, F be the number of vertices, edges and faces in a triangulation of a compact surface S respectively. The Euler characteristic, denoted by $\chi(S)$ is given by

$$\chi(S) = V - E + F.$$

The following theorem [7, Theorem 5.13] about Euler characteristic is a well-known result and gives us the well definedness of the property.

Theorem 1.2.2. *Euler characteristic does not depend on triangulation.*

Remark 1.2.3. Let S be a surface with boundary and S^* be the surface (without boundary) obtained by sewing disks onto the k boundary circles. Then, euler characteristic of S is given by

$$\chi(S) = \chi(S^*) - k.$$

Definition 1.2.4. Let S be a compact surface. The *genus* of S, denoted by $g(S)$ is given by

$$g(S) = \begin{cases} \frac{1}{2}(2-\chi), & \text{if } S \text{ is orientable, and} \\ 2-\chi, & \text{if } S \text{ is non-orientable.} \end{cases}$$

The genus of a surface S with boundary is the genus of S^*.

Definition 1.2.5. Let S_1 and S_2 be two surfaces. Remove a small disk from each S_1 and S_2, then glue the boundary cirlces of these discs together to form a new surface called *connected sum* of S_1 and S_2, denoted by $S_1 \# S_2$.

Theorem 1.2.6. $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$

Proof. Let T_i be triangulation of S_i. Let v_i, e_i, f_i be the number of vertices, edges and faces respectively of triangles in the triangulation T_i. Removing a disc from each of the surface is equivalent of removing a triangle from the triangulation. Therefore, the connected sum is equivalent to removing a triangle each from the triangulations and gluing them along the boundaries of these triangles. Therefore, in the new triangulated complex, number of vertices, edges and faces are $v_1 + v_2 - 3$, $e_1 + e_2 - 3$, $f_1 + f_2 - 2$, respectively.

$$\begin{aligned}\chi(S_1 \# S_2) &= (v_1 + v_2 - v_3) - (e_1 + e_2 - 3) + (f_1 + f_2 - 2) \\ &= (v_1 - e_1 + f_1) + (v_2 - e_2 + f_2) - 2 \\ &= \chi(S_1) + \chi(S_2) - 2.\end{aligned}$$

□

The following is a direct consequence of Theorem 1.2.6.

Corollary 1.2.7. $g(S_1 \# S_2) = g(S_1) + g(S_2)$.

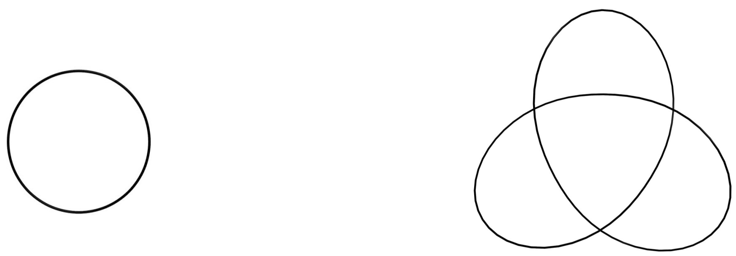
Corollary 1.2.8. If S' is the surface obtained by attaching the endpoints of a strip to a surface S. Then, $\chi(S') = \chi(S) - 1$.

2. BASIC KNOT THEORY

In this chapter, after introducing some basic concepts in knot theory, we go on to study Seifert surfaces and give the definition of genus of a knot. Finally, we prove the additivity of genus of a knot, which gives us the prime factorisation.

2.1 Knots and Links

Definition 2.1.1. A *knot* is an embedding of a circle S^1 into S^3. The disjoint union of m knots is called *link of m components*.



(a) Unknot. **(b)** Trefoil Knot.

Fig. 2.1: Knots.

Definition 2.1.2. A *knot diagram* is a two dimensional projection of a knot with transverse intersections at crossings, without triple points (see Figure 2.2) and equipped with crossing information.

Fig. 2.2: These are not allowed.

Figure 2.3 shows the knot diagrams for Trefoil and Figure-Eight knots.

(a) Trefoil Knot. **(b)** Figure-Eight knot.

Fig. 2.3: Knot Diagrams.

Definition 2.1.3. Two links L_1 and L_2 are said to be *equivalent* if there exists a continuous family of homeomorphisms $h_t : S^3 \to S^3,\ t \in [0, 1]$ such that h_0 is identity and $h_1(L_1) = L_2$. The homeomorphisms h_0 and h_1 are said to be *ambient isotopic*.

Definition 2.1.4. A link for which each component has been given an orientation is called an *oriented link* (see Figure 2.4).

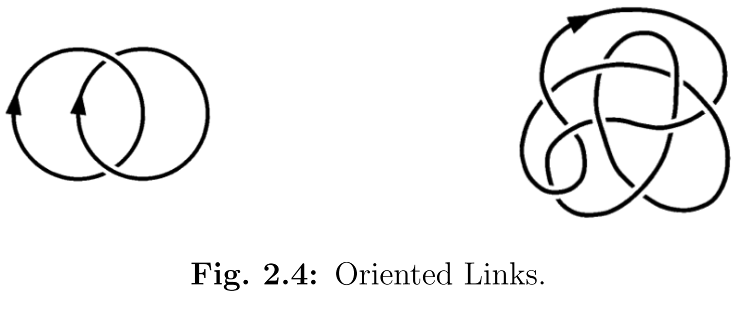


Fig. 2.4: Oriented Links.

Definition 2.1.5. Let K_1 and K_2 be two oriented knots such that they are embedded in distinct copies of S^3. Remove a small ball from each copy of S^3 that meets the knot in an unknotted spanning arc and then identify together the resulting boundary spheres and their intersection with the knots so that all orientations match up. The resulting knot is called *connected sum* of K_1 and K_2, denoted by $K_1 \# K_2$ (see Figure 2.5).

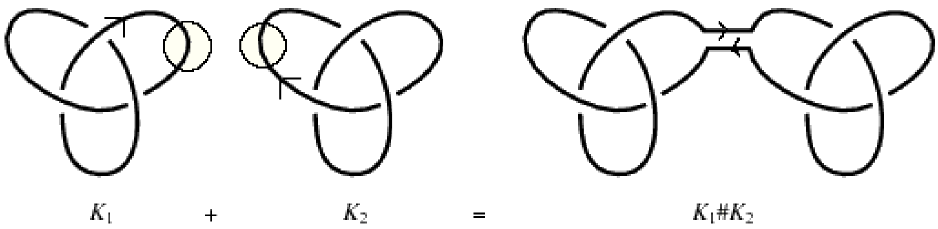


Fig. 2.5: Connected sum of knots.

The connected sum of knots has following properties:

1. It is commutative (see Figure 2.6).

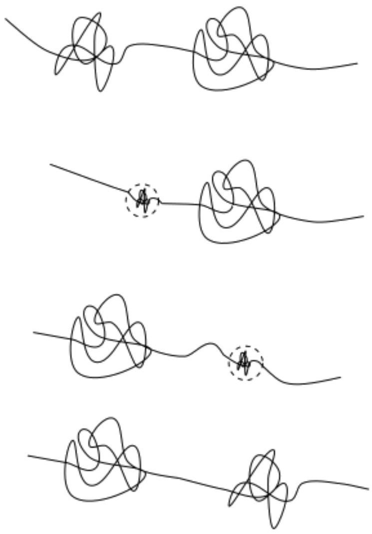


Fig. 2.6: Commutativity of connected sum.

2. It is associative.

3. It has an identity, that is unknot.

Definition 2.1.6. A knot K is a prime knot, if $K = K_1 \# K_2$ then either K_1 or K_2 is the unknot.

2.2 Seifert surfaces and Knot genus

Definition 2.2.1. A *Seifert surface* for an oriented link L in S^3 is a connected compact oriented surface contained in S^3 that has L as its oriented boundary.

Theorem 2.2.2. (Seifert's algorithm)*: Any oriented link in S^3 has a Seifert surface.*

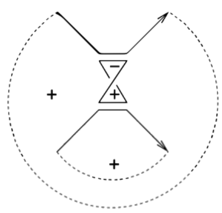
Proof. For a link L, we give an algorithm to construct a Seifert surface.

1. Give the link an orientation.

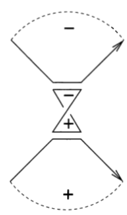
2. Manipulate all the crossings in the way described in diagrams below. We will end up with circles, which are called *Seifert circuits*. Seifert circuits bound discs in S^3.

3. Add rectangular strips with a half twist at the crossings connecting the disks.

4. Orient the surface in the following way.



(a) If disks are on top of each other.



(b) If the disks are adjacent.

This process yields a surface which could be disconnected. We connect the disconnected parts by a cylinder. Thus, following this algorithm we get a connected compact oriented surface with the link L as its boundary. □

Definition 2.2.3. The genus $g(K)$ of a knot K is defined by

$$g(K) = \min\{g(F) : F \text{ is a Seifert surface for } K\}.$$

Example 2.2.4. The unknot has disk as a Seifert surface, which implies $g(\text{unknot}) \leq 0$. Therefore, unknot is a 0-genus knot.

Theorem 2.2.5. *Let K be a knot and F be the Seifert surface obtained by the Seifert's algorithm on a diagram of K, say D. Suppose D has n crossings and s Seifert circuits Then:*

(i). $\chi(F) = s - n$, *and*

(ii). $g(K) \leq \frac{1}{2}(n - s + 1)$.

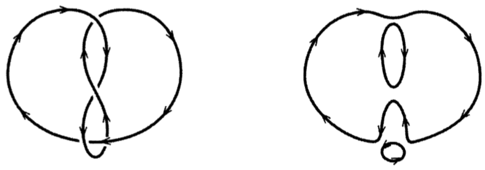
Proof. We know that $\chi(\text{disk}) = 1$. Here, the knot diagram D has n crossings and s Seifert circuits. This means that we have to attach the endpoints n strips to s disks to obtain the Seifert surafce F. Then, (i) follows by Corollary 1.2.8.

As F is a surface with one boundary component, it follows by definiton that

$$\begin{aligned} g(F) &= \frac{1}{2}(n - s + 2) - 1 \\ &= \frac{1}{2}(n - s + 1). \end{aligned}$$

Now, because $g(K)$ is minimum over all Seifert surfaces (ii) follows. □

Example 2.2.6. The diagram of figure-eight knot below has 3 Seifert circuits and 4 crossings. Therefore, $g(\text{trefoil}) \leq \frac{1}{2}(4 - 3 + 1) = 1$, which implies that figure-eight is a 1-genus knot. This follows because figure-eight is not equivalent to unknot.



Theorem 2.2.7. *For any two knots K_1 and K_2,*

$$g(K_1 \# K_2) = g(K_1) + g(K_2).$$

Proof. Let F_i be the minimal genus surface for K_i. Let $S \subset S^3$ be a 2-sphere which separates K_1 from K_2. Then $F_1 \cap S = F_2 \cap S$ is an arc on S. Hence, the surface, $F_1 \cup F_2$ has the knot K as its boundary and forms a Seifert surface for K. The genus of $F_1 \cup F_2$ is $g(F_1) + g(F_2)$. Therefore,

$$g(K_1 \# K_2) \leq g(K_1) + g(K_2).$$

The idea of the proof of reverse inequality is to construct Seifert surfaces for K_1 and K_2, say P and Q, from the minimal Seifert surface of K, say F, such that the genus of ther union is equal to genus of K. If such a construction is possible, the result follows easily. Suppose F_1 and F_2 to be the minimal Seifert surfaces for K_1 and K_2 respectively, then we have

$$g(K) = g(F) = g(P) + g(Q) \geq g(F_1) + g(F_2) = g(K_1) + g(K_2).$$

Let S be the separating sphere of K_1 and K_2. Let S divide K into two arcs α_1 and α_2, and let β be any curve joining two points of intersection of S with K. Then, $\alpha_1 \cup \beta = K_1$ and $\alpha_2 \cup \beta = K_2$. By general position argument,we can assume that F and S intersect transversally, that is $F \cap S$ is a 1-manifold. In particular, it is a collection of simple closed curves and the arc β.

Let C be the innermost simple closed curve, that is the curve which bounds a disc D on S and $D \cap F = \phi$. Now, cut F along C and attach two parallel disks on the either side of D resulting in a new surface F'. This surface also has K as its boundary.

The above surgery has an effect of removing a handle from a surface (if it is still connected after surgery), which decreases the genus of a surface by 1. So, if F' is connected, it contradicts the minimality of genus of F. Hence, F' is disconnected.

Repeating this surgery till all the intersections are removed will result in a Seifert surface, P for K, with the same genus as F and intersecting S only in β. Thus, S separates P into Seifert surfaces of K_1 and K_2 and the result follows. □

Corollary 2.2.8. No non-trivial knot has an additive inverse.

Corollary 2.2.9. There are infinitely many distinct knots.

Proof. Suppose K is a nontrivial knot, and nK denotes connected sum $\overset{n}{\underset{i=1}{\#}}\mathrm{K}$. Since $mK = nK$ if and only if $m = n$, there are infinitely many distinct knots. □

Corollary 2.2.10. A knot of genus 1 is prime.

Corollary 2.2.11. A knot can be expressed as a finite sum of prime knots.

Proof. Suppose a knot is non-prime, then it can be expressed as a sum of knots of smaller genus. The assertion follows by inducting on the genus. □

This proves that every knot can be factorised into prime knots. We will prove uniqueness (up to order) of this factorisation in next chapter.

3. UNIQUE PRIME FACTORIZATION THEOREM FOR KNOTS

We have proved that the knots can be factorised into prime knots. This chapter is dedicated entirely to the proof of uniqueness of this prime factorisation.

3.1 Preliminaries

Definition 3.1.1. A topological embedding $i : M \to N$ of a k-dimensional manifold M into an n-dimensional manifold N is *locally flat* at $x \in M$ if there exists a neighbourhood U of $i(x)$ in N such that $(U, U \cap i(M)) \cong (\mathbb{R}^n, \mathbb{R}^k)$. An embedding is said to be *locally flat* if it is locally flat at each point x of its domain.

The following theorem is a generalisation of Jordan curve theorem in 3-dimensions [2].

Theorem 3.1.2. (Schönflies theorem) *Let $e : S^2 \to S^3$ be any local flat embedding. Then $S^3 - eS^2$ has two components, the closure of each of which is a ball.*

Remark 3.1.3. The hypothesis of local flatness is required because there happen to be weird embeddings of S^2 into S^3. One example is the Alexander's Horned Sphere (see Figure 3.1).

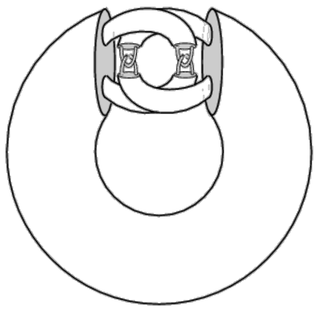


Fig. 3.1: Alexander's Horned Sphere.

3.2 Unique Prime Factorisation theorem

Definition 3.2.1. Let $S_j, 1 \leq j \leq m$, be a system of disjoint 2-spheres embedded in S^3, bounding $2m$ balls $B_i, 1 \leq i \leq 2m$, in S^3, and denote by $B_j, B_{c(j)}$ the two balls bounded by S_j. If B_i contains the s balls $B_{l(1)}, \ldots, B_{l(s)}$ as proper subsets, $R_i = B_i \setminus \left(\bigcup_{q=1}^{s} \operatorname{int}(B_l(q))\right)$ is called the *domain* R_i. The spheres S_j are said to be *decomposing* with respect to a knot $K \subset S^3$ if the following conditions are fulfilled:

1. Each sphere S_j meets K in two points.
2. The arc $\alpha_i = K \cap R_i$, oriented as K, and completed by simple arcs on the boundary of R_i to represent a knot $K_i \subset R_i \subset B_i$, is prime. K_i is called the factor of K determined by B_i.

The set $S = \{S_j, 1 \leq j \leq m\}$ is called the *decomposing sphere system* (DSS) for K (see Figure 3.2).

Lemma 3.2.2. Let S be a DSS for K with m spheres. Then the factoring has $m+1$ prime knots: $K = \mathop{\#}_{i=1}^{m+1} K_i$.

Proof. The case $m = 0$ is vacuously true. The case $m = 1$ implies that there is a separating sphere between connected sum of two knots, which is true. Assume that the lemma is true for any value less than m. Let B_p be an innermost ball, that is, there is no ball B such that $B \subset B_x$. Replace the

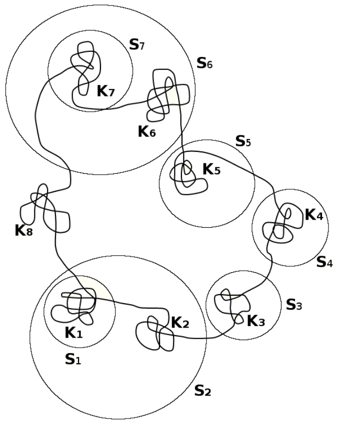


Fig. 3.2: Decomposing Sphere system.

arc $K'_p = K \cap B_p$ with any arc joining two points of $K \cap S_p$ on S_p. This forms a new knot $\hat{K}$. By the definition of factorising sphere, it is clear that $K = \hat{K} \# K_p$. Let $\hat{S} = S \setminus \{S_p\}$, then $\hat{S}$ is a DSS for $\hat{K}$ and it has $m-1$ spheres. By our induction hypothesis, this implies that $\hat{K}$ is decomposed into m factors that is $\overset{m}{\underset{i=1}{\#}} \hat{K}_i$. Therefore, S gives the factorization $K = \overset{m}{\underset{i=1}{\#}} \hat{K}_i \# \hat{K}_p$, which has $m+1$ terms, as required. □

Definition 3.2.3. Let S and S' be DSS's for the knot K. Then $S \sim S'$ if they determine the same factorizations of K.

Lemma 3.2.4. Let $S = \{S_1, \ldots, S_m\}$ be a DSS for K. Let B_k be an outermost ball within B_i. Then $B_{c(k)}$ and B_i determine the same knot.

Proof. As B_k is the outermost ball in B_i, $B_{c(k)}$ contains every ball outside B_k except B_i. Hence, the knot determined by B_i is same as the knot determined by $B_{c(k)}$. □

Lemma 3.2.5. Let $S = \{S_1, \ldots, S_m\}$ be a DSS for K. Let $\hat{S}_j$ be another 2−sphere in S^3, disjoint from each $S_i \in S$, that bounds $\hat{B}_j$. Let $\hat{S} = (S \setminus \{S_j\}) \cup \{\hat{S}_j\}$. Suppose B_j is outermost in $\hat{B}_j$ and that $\hat{B}_j$ determines the same knot K_j (relative to $\hat{S}$) as B_j does (relative to S). Then $\hat{S} \sim S$.

Proof. To prove the equivalence we have to prove that balls in both the DSSs determine the same knots. The proof involves determining the knot factors of each and every ball of both the DSS.

By hypothesis B_j and $\hat{B}_j$ determine the same factor K_j. Hence, the region $M = \bigcup_{B_i \subset \hat{B}_j} \text{int}(B_i)$, where $S_i \in S$, determines the unknot. This implies that $\hat{B}_{c(j)}$ and $B_{c(j)}$ determine the same factor $K_{c(j)}$.

Now, consider balls lying inside $\hat{B}_j$. Let $i \neq j$ and assume $B_i \subset \hat{B}_j$. Suppose B_i is not the outermost ball in $\hat{B}_j$. This means that there is a k such that $B_i \subset B_k \subset \hat{B}_j$. Clearly, the knot determined by B_i in $\hat{S}$ is same as the one in S. As it is not the outermost, $B_{c(i)}$ determines the same knot $K_{c(i)}$ in both the DSS.

If B_i is the outermost ball, then B_i determines the factor K_i. $B_{c(i)}$ contains every ball contained in $\hat{B}_{c(j)}$ and $\hat{B}_j$ (except the ones inside B_i). Hence, it determines the same knot as B_j, that is, K_j with respect to $\hat{S}$. The knot $K_{c(i)}$ is determined by $\hat{B}_{c(j)}$, since B_i is an outermost ball in $B_{c(j)}$.

The same can be done for balls lying outside $\hat{B}_j$ and the lemma follows.

□

Theorem 3.2.6 (Unique Prime factorisation of knots)**.** *Let S and S' be DSSs for K. Then $S \sim \mathbf{S}'$.*

Proof. Let $S = \{S_1, \ldots, S_m\}$ and $S' = \{S'_1, \ldots, S'_n\}$ be two DSSs for K. We prove this by induction on $m+n$. For $m+n=0$, it is vacuously true. The spheres are assumed to be in general position.

Suppose there exists B_i such that $B_i \cap S' = \emptyset$, and does not contain any other B_p or B'_p. Then, there exists S'_j, S_i is outermost within some S'_j with respect to S', otherwise S' would not be a DSS for K. Now, replace S'_j by S_i, by Lemma 3.2.5, $S' \sim S''$, where $S'' = (S' \setminus \{S'_j\}) \cup S_i$.

Let $\hat{K} = (K \setminus B_i) \cup K_i$. Then, $K = \hat{K} \# K_i$. Let $\hat{S} = S \setminus S_i$, and $\hat{S''} = S'' \setminus S_i$. $\hat{S}$ and $\hat{S''}$ are DSSs for $\hat{K}$. By induction hypothesis, $\hat{S} \sim \hat{S''}$ and thus they give the same factors $\mathop{\#}\limits_{j=1}^{m} \hat{K}_j$. Then by definition of factoring, S and S'' give $\mathop{\#}\limits_{j=1}^{m} \hat{K}_j \# K_i$. Hence, $S' \sim S'' \sim S$.

Suppose there is no such ball. Let B'_j be an innermost ball with respect to S and S'. Let λ be an innermost curve of $S'_j \cap S$. Thus there exists a disk $D \subset S'_j$ with $\partial D = \lambda$ and $\text{int}(D) \cap S = \emptyset$.

Now there exists i such that $\lambda \subset S_i$ and $D \subset B_i$ and $\text{int}(D) \subset \text{int}(B_i)$.

So, D divides B_i into two balls, B_{i_1} and B_{i_2}. One of these determines the factor K_i and the other determines an unknot.

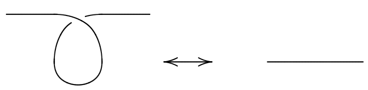
Without loss of generality, let B_{i_1} determine K_i. By surgery, reduce the number of intersections and apply the previous case. □

4. JONES POLYNOMIAL

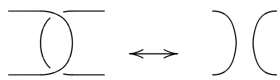
This chapter is dedicated to understanding one of the most important invariants of links called Jones Polynomial, named after V.F.R Jones. We introduce Reidemeister moves, which are one of the most important ways of modifying knots without actually changing them. We define Jones Polynomial using Kauffmann Bracket.

4.1 Reidemeister moves and Kauffmann Bracket

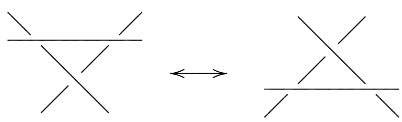
Definition 4.1.1. (*Reidemeister moves*) Two links L_1 and L_2 are equivalent if they are related by a sequence of *Reidemeister moves*. The three types of Reidemeister moves are shown below.



Move-I



Move-II



Move-III

Definition 4.1.2. The *Kauffmann Bracket* of a link diagram in S^2 is a

Laurent polynomial with integer coefficients in an indeterminate A. It is defined using these three rules:

(i) $\langle \bigcirc \rangle = 1.$

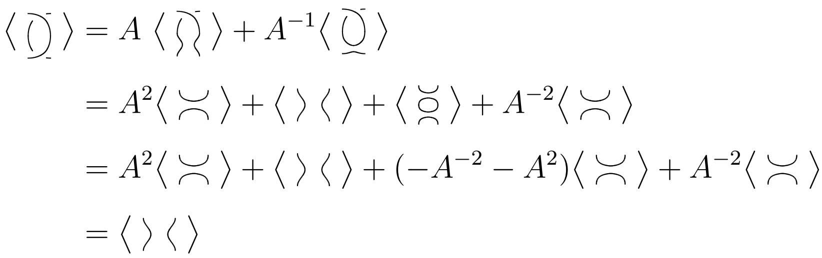
(ii) $\langle D \cup \bigcirc \rangle = (-A^{-2} - A^{2})\langle D \rangle.$

(iii) $\langle \times \rangle = A\langle\,)\,(\, \rangle + A^{-1}\langle \asymp \rangle$

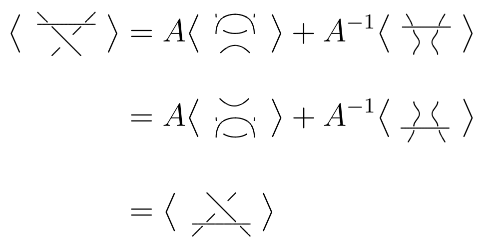
It is clear from the definition that any ambient isotopy on S^2 is not going to change the bracket polynomial of a link diagram.

Lemma 4.1.3. Kauffmann Bracket of a Link L is invariant over *Type-II* and *Type-III* Reidemeister moves.

Proof. We have

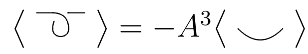


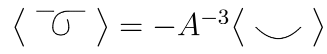
Hence, its invariant under *Type-II* Reidemeister move.
Now, we have



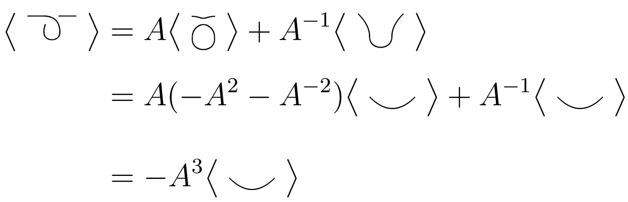
Hence, its invariant under *Type-III* Reidemeister move. □

Lemma 4.1.4. *Type-I* Reidemeister moves changes the bracket polynomial in the following way:

(i) 

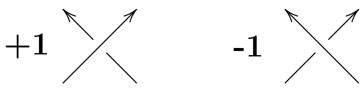
(ii) 

Proof. We have



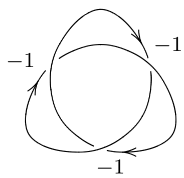
Hence, (i) follows. Similarly, (ii) can be proved. □

Each crossing in an oriented link can be given signs in the following fashion:



Definition 4.1.5. The *writhe* $w(L)$ of a diagram L of an oriented link is defined as the sum of the signs of crossings of L.

Example 4.1.6. Consider the following diagram K of an oriented trefoil knot.



Here, $w(K) = -1 - 1 - 1 = -3$.

It is easy to check that *Type-II* and *Type-III* Reidemeister moves would not change the writhe. *Type-I* Reidemeister moves changes the writhe by ±1.

4.2 Jones Polynomial

Definition 4.2.1. The Jones Polynomial $V(D)$ of a diagram D of an oriented link L is the Laurent polynomial with integer coefficients, defined by

$$V(D) = (-A)^{-3w(D)}\langle D\rangle \text{ with } A^{-2} = t^{1/2}$$

Theorem 4.2.2. *The Jones Polynomial is a oriented link invariant.*

Proof. It is enough to prove that any Reidemeister move will not change the Jones Polynomial. Clearly, *Type-II* and *Type-III* Reidemeister moves will not change the Jones polynomial, as $\langle D\rangle$ and $w(D)$ are unchanged by them.

It remains to check the invariance in the case of *Type-I* Reidemeister move. From Lemma 4.1.4, the bracket polynomial of a diagram D is changed by a factor of $-A^3$ and $w(D)$ changes by $+1$. So

$$(-A)^{-3(w(D)+1)}(-A^3\langle D\rangle\ ) = (-A)^{-3w(D)}\langle D\rangle$$

A similar argument works for the other *Type-I* Reidemeister move. Therefore, the expression is unchanged by *Type-I* Reidemeister move and the assertion follows. □

It follows from the theorem that, any diagram D of an oriented link L will give us the same polynomial.

Definition 4.2.3. Let L be an oriented link with two components K_1 and K_2. The *linking number* $lk(K_1, K_2)$ is the half of the sum of signs of crossings involving a strand each from K_1 and K_2.

Proposition 4.2.4. Linking number is an oriented link invariant [12, Theorem 3.8.2].

Remark 4.2.5. Let c denote the crossings of a diagram D of an oriented link L. Define a function ε such that $\varepsilon(c)$ gives the sign of the crossing c. Therefore,

$$w(D) = \sum_{c\in D}\varepsilon(c), \text{ and } lk(K_1, K_2) = \tfrac{1}{2}\sum_{c\in K_1\cap K_2}\varepsilon(c).$$

Proposition 4.2.6. Let L be an oriented link and L^* be the oriented link obtained after reversing the orientation of one of the components, say K, of L. Then,

$$V(L^*) = t^{-3lk(K,L-K)}V(L).$$

Proof. Let D be a diagram of L. We know that $w(D) = \sum_{c\in D} \varepsilon(c)$ and $lk(K, L-K) = \frac{1}{2} \sum_{c\in K\cap L-K} \varepsilon(c)$. Let D^* be the diagram of $L*$. Then

$$\begin{aligned} w(D^*) &= \sum_{c\in D} \varepsilon(c) \\ &= w(D) - 2 \sum_{c\in K\cap L-K} \varepsilon(c) \\ &= w(D) - 4lk(K, L-K). \end{aligned}$$

Note that $\langle D^*\rangle = \langle D\rangle$ as bracket polynomial is independent of orientation. Now,

$$\begin{aligned} V(L^*) &= A^{-3w(D^*)}\langle D^*\rangle \\ &= A^{-3(w(D)-4lk(K,L-K))}\langle D^*\rangle \\ &= A^{12lk(K,L-K)}A^{-3(w(D)}\langle D\rangle \\ &= t^{-3lk(K,L-K)}. \qquad\qquad \because A^2 = t^{-1/2}. \end{aligned}$$

Hence, the proposition follows. ☐

Proposition 4.2.7. The Jones Polynomial invariant is a function

$$V : \{\text{Oriented links in } S^3\} \longrightarrow \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

such that

(i) $V(\bigcirc) = 1$.

(ii) Let L_+, L_- and L_0 are the same link, except for at one crossing as shown in the Figure 4.1, then

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0.$$

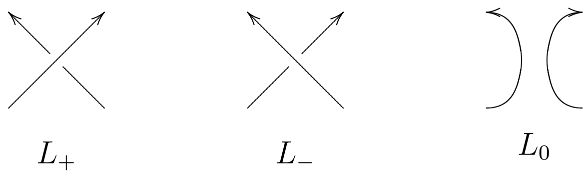


Fig. 4.1

Proof. We know that, $w(\bigcirc) = 0$ and $\langle \bigcirc \rangle = 1$. Hence, $V(\bigcirc) = 1$ and (i) follows.
For (ii), we know that ,

$$\langle \diagup\!\!\!\!\!\diagdown \rangle = A\langle\ )\,(\ \rangle + A^{-1}\langle \asymp \rangle. \tag{4.1}$$

$$\langle \diagdown\!\!\!\!\!\diagup \rangle = A^{-1}\langle\ )\,(\ \rangle + A\langle \asymp \rangle. \tag{4.2}$$

Multiplying (4.1) with A, (4.2) with A^{-1} and subtracting gives

$$A\langle \diagup\!\!\!\!\!\diagdown \rangle - A^{-1}\langle \diagdown\!\!\!\!\!\diagup \rangle = (A^2 - A^{-2})\ \langle\ )\,(\ \rangle. \tag{4.3}$$

It is clear from the diagrams above that

$$w(L_+) - 1 = w(L_0) = w(L_-) - 1.$$

Multiply $(-A)^{-3w(L_0)}$ on both sides of (4.3)

$$-A^4V(L_+) + A^{-4}V(L_-) = (A^2 - A^{-2})V(L_0).$$

After substituting $t^{-1/2} = A^2$, we get

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0.$$

□

Remark 4.2.8. We can easily observe that,

$$V(K_1 \# K_2) = V(K_1) \cdot V(K_2). \tag{4.4}$$

This is because we can calculate the bracket polynomial of the connected sum of knots by operating first on only first summand. This is true also for links, but the connected sum is not well defined in links.

Example 4.2.9. *(A limitation of Jones Polynomial)* Let L be a link which is disjoint union of a trefoil knot and an unknot and K be another trefoil. Now, $L \#_1 K$ be the link obtained by performing connected sum operation between K and unknot of L and $L \#_2 K$ be the link obtained by performing connected sum operation between K and trefoil knot of L. Clearly, $L \#_1 K$ and $L \#_2 K$ are not equivalent, but they have the same Jones polynomial by (4.4).

5. DEHN-LICKORISH THEOREM

The main objects of study in this chapter are surface homeomorphisms. We define Mapping class groups and also a special class of surface homeomorphisms called Dehn twists. Our aim is to prove the classic result about surface homeomorphisms which was independently proved by Lickorish and Dehn (see Theorem 5.2.11).

5.1 Mapping Class Groups

Definition 5.1.1. Let S be a surface; f_0 and f_1 be self-homeomorphisms of S. f_0 and f_1 are said to be isotopic if there exists a homeomorphism $H : S \times [0,1] \to S$ such that:

1. $H(x,t) = h_t(x)$ where each h_t is a self-homeomorphism of S.

2. $h_0 = f_0, h_1 = f_1$.

The following result by Munkres [10] will help us to understand the idea of orientation preserving homeomorphisms using diffeomorphisms.

Theorem 5.1.2. *Every homeomorphism of a compact surface S is isotopic to a diffeomorphism.*

Definition 5.1.3. Let h be a homeomorphism of a surface S. Then h is called an orientation preserving homeomorphism, if any diffeomorphism isotopic to h is orientation preserving.

Let $\mathrm{Homeo}^+(S, \partial S)$ be the set of all orientation preserving self-homeomorphisms of S that is identity on ∂X. Let $\mathrm{Homeo}_0(S, \partial S)$ be the set of all elements in $\mathrm{Homeo}^+(S, \partial S)$ which are isotopic to identity and these isotopies are also identity on boundary.

Proposition 5.1.4. $\text{Homeo}^+(S, \partial S)$ forms a group.

Proof. We will show that this set satisfies the group axioms with composition as group operation:

1. Let $f, g \in \text{Homeo}^+(S, \partial S)$. It is clear that $f \circ g$ is also a homeomorphism. It is orientation preserving because the composition of orientation preserving diffeomorphisms is orientation preserving.

2. Associativity is property of the composition.

3. For every $f \in \text{Homeo}^+(S, \partial S)$, $f^{-1} \in \text{Homeo}^+(S, \partial S)$, since inverse of a homeomorphism is a homeomorphism and orientation preserving follows from its corresponding diffeomorphism.

4. The identity map of S, denoted as Id_S, is an orientation preserving homeomorphism and $f \circ \text{Id}_S = f = \text{Id}_S \circ f, \forall f \in \text{Homeo}^+(S, \partial S)$.

Thus, $\text{Homeo}^+(S, \partial S)$ forms a group. □

Proposition 5.1.5. $\text{Homeo}_0(S)$ is a normal subgroup of $\text{Homeo}^+(S)$.

Proof. Let $n_1, n_2 \in \text{Homeo}_0(S, \partial S)$ and N_i be isotopies of n_i for $i = 1, 2$.

1. Define $F : S \times [0, 1] \to S$ by $F(x, t) = N_1(N_2(x, t), t)$. This gives an isotopy from $n_1 \circ n_2$ to Id_S.

2. Define $G : S \times [0, 1] \to S$ by $G(x, t) = N_1^{-1}(x, t)$. This gives an isotopy from n_1^{-1} to Id_S.

3. Define $H : S \times [0, 1] \to S$ by $H(x, t) = f \circ N_1(f^{-1}(x), t)$, where $f \in \text{Homeo}^+(S, \partial S)$.

Thus, $\text{Homeo}_0(S, \partial S)$ is a normal subgroup of $\text{Homeo}^+(S, \partial S)$. Note that all isotopies are fixing the boundary pointwise as required.

□

Definition 5.1.6. The mapping class group of a surface S is defined as the quotient group

$$\text{MCG}(S) = \text{Homeo}^+(S, \partial S)/\text{Homeo}_0(S, \partial S).$$

Remark 5.1.7. Let $f, g \in \mathrm{MCG}(S)$. Then $f = g$ iff $fg^{-1} = n \implies f = ng$, where $n \in \mathrm{Homeo}_0(S, \partial S)$. Suppose $f = n_1 g_1 n_2 g_2$ i.e., $f \in N_S g_1 N_S g_2$, then $f \in N_S g_1 g_2$, where $N_S = \mathrm{Homeo}_0(S, \partial S)$. This implies that there exists $n_3 \in N$ such that $f = n_3 g_1 g_2$, in other words, f is isotopic to $g_1 g_2$. From now on, we will denote $\mathrm{Homeo}_0(S, \partial S)$ by N_S.

5.2 Dehn Lickorish Theorem

Definition 5.2.1. Let C be a simple closed curve on S. Consider an annular neighbourhood of C, i.e, $C \times [0,1] \cong S^1 \times [0,1]$, say A. Cut S along C, twist one of the free ends of cylinder through an angle of 2π and glue it together again. This is called a *Dehn twist* about the curve C (see Figure 5.1).

In other words, Dehn twist is a homeomorphism $\tau : S \to S$ such that $\tau|_{S-A}$ is the identity and $\tau|_A$ by $\tau(e^{i\theta}, t) = (e^{i(\theta - 2\pi t)}, t)$.

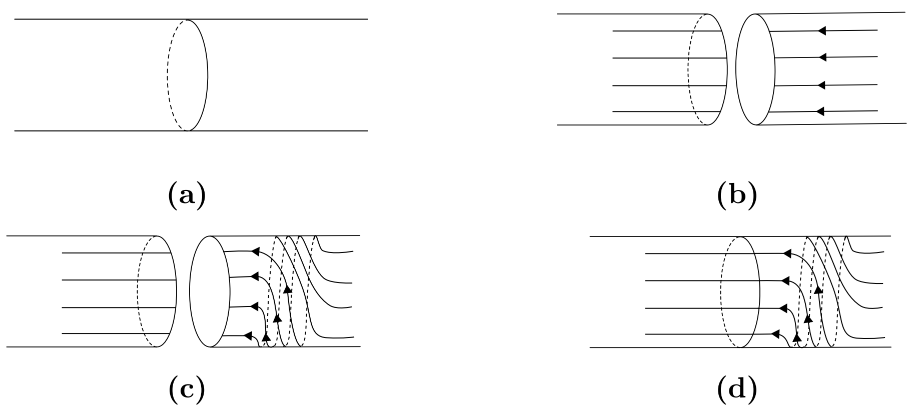


Fig. 5.1: Dehn Twist.

Similarly, if we twist in the other direction, we get the inverse of τ which is the homeomorphism $\tau^{-1} : S \to S$ such that $\tau^{-1}|_{S-A}$ is the identity and $\tau|_A$ by $\tau(e^{i\theta}, t) = (e^{i(\theta + 2\pi t)}, t)$.

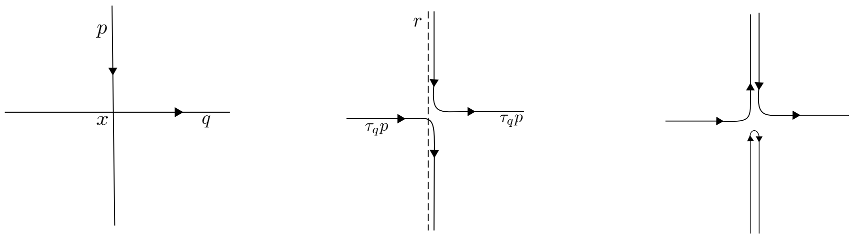
Definition 5.2.2. Let p and q be two simple closed curves in a surface S. The curve p is said to be twist equivalent to q, denoted as $p \sim_\tau q$, if there exists a sequence of Dehn twists $h_1, \ldots h_m$ and n, a homeomorphism isotopic to identity such that $n h_1 \cdots h_m p = q$.

Proposition 5.2.3. $\sim_\tau$ is an equivalence relation.

Proof. Reflexivity of $\sim_\tau$ is straightforward. Let $p \sim_\tau q$. This implies that there exists a sequence of Dehn twists $h_1, \dots h_m$ and $n \in N_S$ such that $nh_1 \cdots h_m p = q \implies p = h_m^{-1} \cdots h_1^{-1} n^{-1} q \implies p = n' h_m^{-1} \cdots h_1^{-1} q$ for some n'. Hence, $q \sim_\tau p$. Let $p \sim_\tau q$ and $q \sim_\tau r$. This implies that there exists $h_1, \cdots h_m, g_1, \cdots g_k$ and $n_1, n_2 \in N_S$ such that $p = n_1 h_1 \cdots h_m q$ and $q = n_2 g_1 \cdots g_n r$. Then, $p = n_1 h_1 \cdots h_m n_2 g_1 \cdots g_n r \implies p = n_3 h_1 \cdots h_m g_1 \cdots g_n r$, for some $n_3 \in N_S$. Hence, $p \sim_\tau r$. Thus, $\sim_\tau$ is an equivalence relation. □

Lemma 5.2.4. If p and q are two simple closed curves in S intersecting at only one point, then $p \sim_\tau q$.

Proof. Let p and q be as shown in Figure 5.2a. Let the intersection point of p and q be x and τ_q be the Dehn twist about the curve q. We see that $\tau_q p$ is a copy p, which is cut at x and traverses q to return to x and follows the



(a) **(b)** **(c)**

Fig. 5.2

remaining (see Figure 5.2b) . Now, perform a Dehn twist along the curve r, as in the second picture, then $\tau_r \tau_q p$ is shown in Figure 5.2c. Then, there exists an isotopy of S sending $\tau_r \tau_q p$ to q. Hence, $\exists\ n \in N_S$ and Dehn twists τ_r, τ_q such that $n \tau_r \tau_q p = q$; i.e., $p \sim_\tau q$. □

Figure 5.3 illustrates the proof of previous lemma with the example of meridian and longitude of the torus.

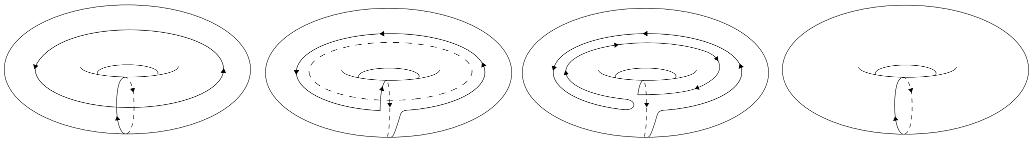


Fig. 5.3

Corollary 5.2.5. Let $p_1, \ldots, p_n$ be simple closed curves on S such that p_i and p_{i+1} intersect at one point for $1 \leq i \leq n-1$. Then, $p_1 \sim_\tau p_n$.

Proof. As p_1 and p_2 intersect at only one point, from Lemma 5.2.4, it follows that $p_1 \sim_\tau p_2$. Similarly, $p_2 \sim_\tau p_3$. Hence, by transitivity of $\sim_\tau$, $p_1 \sim_\tau p_3$. Repeating this, we get $p_1 \sim_\tau p_n$. □

Lemma 5.2.6. Let p and q be two simple closed curves in X and A be a neighbourhood of q in S. Then, there exists a path p_* such that

(i) $p \sim_\tau p_*$,

(ii) $p_* \cap (S \setminus A) \subset p \cap (S \setminus A)$, and

(iii) either p_* does not intersect q or intersects it twice with zero algebraic intersection.

Proof. Let m be the number of points of intersection of p and q.
Case 1: If $m = 0$, then the lemma is trivial.
Case 2: If $m = 1$, then by Lemma 5.2.4, $p \sim_\tau q$. We have an $n \in N_S$ such that $nq = q'$ (see Figure 5.4), such that $q' \cap (S \setminus A) = \emptyset$. As $q \sim_\tau q'$, by transitivity, $p \sim_\tau q'$ and q does not intersect q'. Hence, $q' = p_*$.

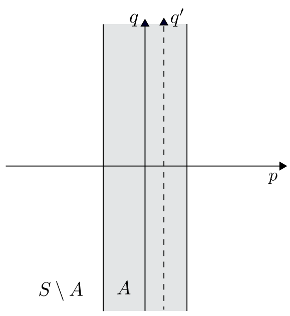


Fig. 5.4

Case 3: If $m = 2$, and p has opposite orientations at the points of intersection, then take $p = p_*$ (see Figure 5.4).

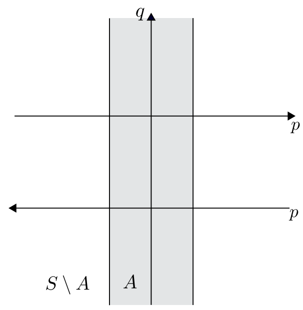


Fig. 5.5

Now, assume that the lemma is true for all p and q intersecting at less than k points, i.e., $m < k$. Let p and q be two curves with $m = k$.

Case 4: Suppose there are two adjacent points of intersection on q, say α and β, such that p goes in the same direction at these points. Let P and Q be two points in A, in a neighbourhood of α and β respectively. Join the points P and Q by a line segment cutting p and q once in A. From P traverse along p till the point Q to form a simple closed curve, say p_1 (see Figure 5.6). As p_1 cuts p at one point, $p_1 \sim_\tau p$. There is an $n \in N_S$ such that $np_1 \cap (S \setminus A) \subset p \cap (S \setminus A)$. Also, observe that np_1 cuts q at less than k points. Hence, by induction hypothesis, there exists an l such that the lemma is true for np_1 and q. It is clear that $l \cap (S \setminus A) \subset p \cap (S \setminus A)$ and

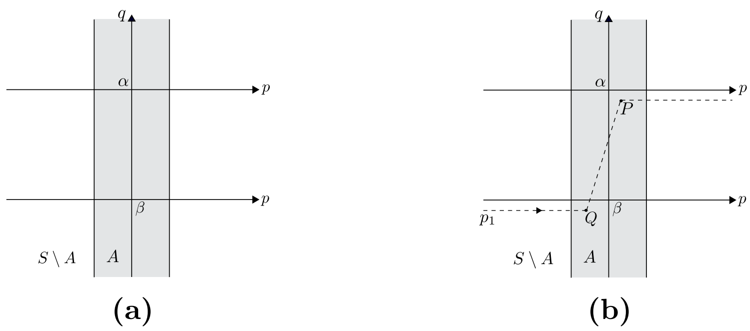
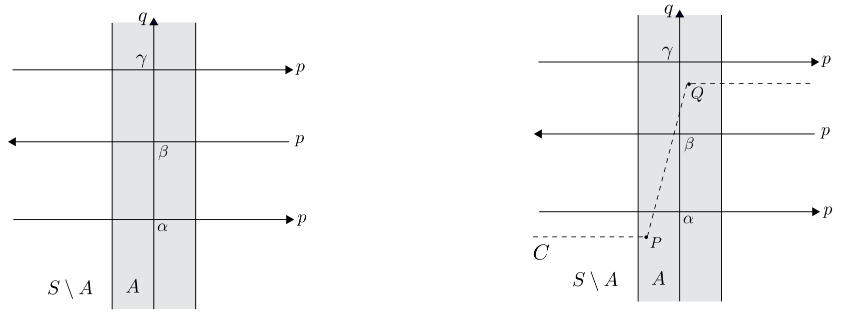


Fig. 5.6

by transitivity $p \sim_\tau l$. Hence, l is our required p_*.

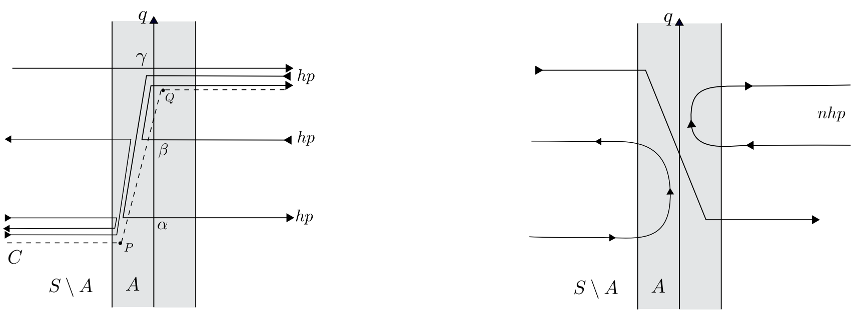
Case 5: Suppose there are three adjacent points α, β and γ on q such that the direction of p alternates accordingly at these points (see Figure 5.7a). β will lie on either the segment $\overrightarrow{\gamma\alpha}$ or $\overrightarrow{\alpha\gamma}$. Assume that it lies on $\overrightarrow{\gamma\alpha}$. Let P and Q be points in A such that they lie in the neighbourhood of α and γ respectively. Join P and Q by a line segment in A. From Q, traverse the path along p to P to get simple closed curve, say C (see Figure 5.7b).



(a) **(b)**

Fig. 5.7

Now, perform a Dehn twist h along C on S, taking p to hp as shown in Figure 5.8a. From the figure, it is clear that there exists an $n \in N_S$ such that nhp is as shown in Figure 5.8b and $nhp \cap (S \setminus A) \subset p \cap (S \setminus A)$. Now



(a) **(b)**

Fig. 5.8

nhp intersects q at atmost $k-2$ points, as atleast two points of intersection were removed and none added by our modifications. Hence, there exists an l

such that the lemma is true for nhp and l. It is clear that $l \cap (S \setminus A) \subset p \cap (S \setminus A)$ and by transitivity $p \sim_\tau l$. Hence, l is our required p_*. □

Corollary 5.2.7. Let $p, q_1, \cdots, q_n$ be simple closed curves on S such that $q_i's$ are pairwise disjoint. Then, there exists p_* such that :

(i) $p \sim_\tau p_*$, and

(ii) either p_* does not intersect q_i or intersects it twice with zero algebraic intersection, for every i.

Proof. Let A_i be the neighbourhood of q_i such that $A_i \cap A_j = \emptyset$. Suppose p intersects q_1 at only one point, then $p \sim_\tau q_1$ and there exists $n \in N_S$ such that nq_1 is disjoint from q_1. Then, nq_1 is the required p_*.

Suppose p intersects q_1 at more than one point, then by applying Lemma 5.2.6, we get $p_1 \sim_\tau p$ such that p_1 intersects q_1 atmost twice with zero algebraic intersection and $p_1 \cap (S \setminus A_1) \subset p \cap (S \setminus A_1)$. Now, if p_1 intersects q_2 at only one point then there exists $n' \in N_S$ such that $n'q_2$ is our required p_*. If p_1 intersects q_2 at more than one point, then again apply Lemma 5.2.6 to get $p_2 \sim_\tau p_1$ such that p_2 intersects q_2 atmost twice with zero algebraic intersection and $p_2 \cap (S \setminus A_2) \subset p_1 \cap (S \setminus A_2)$. Now, as $q_1 \subset X \setminus A_2$, p_2's intersection with q_1 is same as p_1. Repeating this way, finally we get p_k satisfying the properties of p_*. □

From the classification theorem of surfaces, we know that every closed, connected, orientable surfaces is connected sum of tori. In other words, any

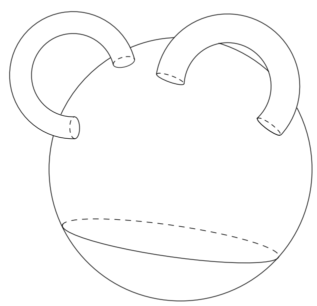


Fig. 5.9: A typical surface.

closed, connected, orientable surface can be thought of as sphere with handles attached. A typical such surface is shown in Figure 5.9. From now on, we think of a closed, connected oriented surface of genus g as a sphere with g handles attached.

Definition 5.2.8. Let $c_\alpha, c_\beta, c_\gamma$ be as shown in Figure 5.10. A curve is said to *meet the handle* if it intersects c_β (see Figure 5.11a), otherwise it *does not meet the handle* (see Figure 5.11b). A curve is said to *go through the handle* if it does not meet the handle and intersects c_γ odd number of times (see Figure 5.11c).

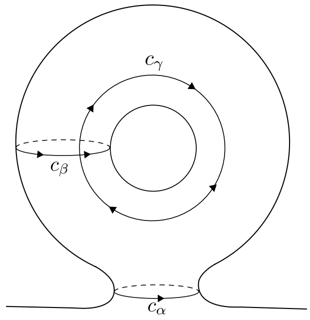
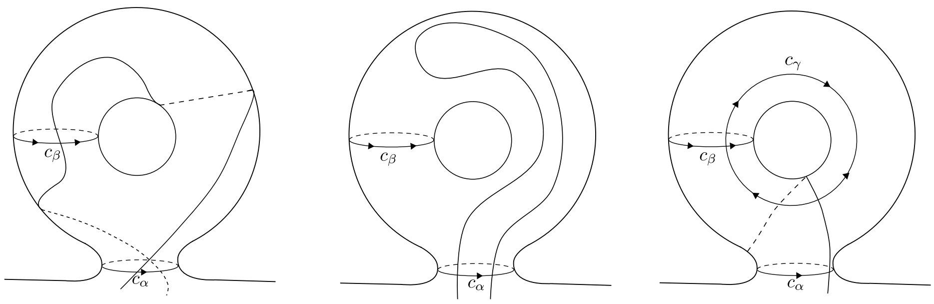


Fig. 5.10: A handle with $c_\alpha, c_\beta, c_\gamma$.



(a) **(b)** **(c)**

Fig. 5.11

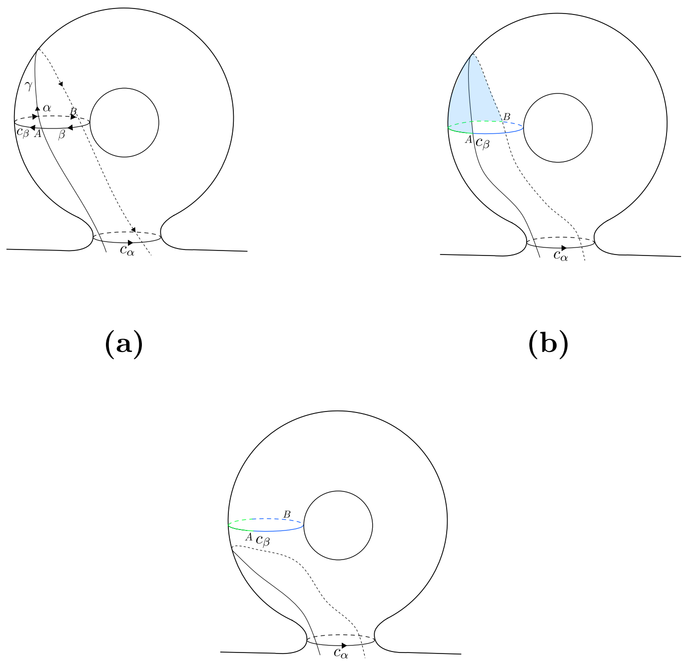
Lemma 5.2.9. Let p be a simple closed curve in S. Then there exists a p_* such that:

(i) $p \sim_\tau p_*$, and

(ii) p_* does not meet any of the handles of S.

Proof. Let $q_1, \ldots q_k$ be the collection of all $c'_\alpha s$ and $c'_\beta s$ on S. Thus, this is a collection of pairwise disjoint curves on S. Applying Corollary 5.2.7, we get a simple closed curve l which either does not intersect q_i or intersects it twice with zero algebraic intersection and $p \sim_\tau l$. Now, we will reduce this intersection by performing an isotopy on l.

At any handle, l enters the handle cutting c_α, then cuts c_β twice with zero algebraic intersection and returns cutting c_α in the opposite direction. Let A and B be the points of intersection of l with some q_i, which is a c_β of some handle. Assume A to be the first point we encounter when we traverse along l, according to its orientation. Let α be the path $\overrightarrow{AB}$ on q_i, β be the path $\overrightarrow{BA}$ on q_i, γ be the part of l from A to B in the direction of the curve (see Figure 5.12a). Then, $[\alpha * \beta] = [q_i]$. The simple closed curves $\gamma * \overline{\alpha}$ and $\gamma * \beta$ can be seen as loops based at A. Any part of l cannot follow c_γ of the



(a) **(b)**

(c)

Fig. 5.12

handle as it would result in non-zero algebraic intersection. The only possibilities left for $\gamma * \overline{\alpha}$ and $\gamma * \beta$ is that they are homotopic to q_i. Suppose $[\gamma * \beta] = [\alpha * \beta]$, then $[\gamma] * [\beta] = [\alpha] * [\beta] \implies [\gamma] = [\alpha] \implies [\gamma] * [\overline{\alpha}] = \mathrm{Id}$.

Thus, $\gamma * \overline{\alpha}$ bounds a disk (see Figure 5.12b). We can perform an isotopy to move the curve through this disk and remove the intersection (see Figure 5.12c). □

Lemma 5.2.10. Let f be homeomorphism of disk which is identity on boundary. Then f is isotopic to identity. In other words, $MCG(D^2)$ is trivial.

Proof. Consider for $0 \leq t < 1$

$$F(x,t) = \begin{cases} (1-t)f(\frac{x}{1-t}), & \text{if } 0 \leq |x| \leq 1-t, \text{ and} \\ x, & \text{if } 1-t \leq |x| \leq 1. \end{cases}$$

Clearly, $F(x,0) = f(x)$ and $F(x,1) = x = \mathrm{Id}(x)$, and this defines and isotopy from f to the identity. □

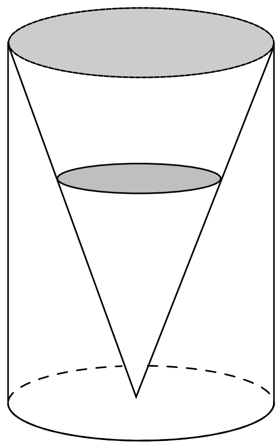


Fig. 5.13: Pictorial representation of the isotopy.

This isotopy can be seen as increasing the portion of Identity on the disk and ultimaley covering the entire disc (see Figure 5.13).

Theorem 5.2.11. *Any orientation preserving homeomorphism of a closed, connected, orientable surface S is isotopic to the product of a sequence of Dehn-twists.*

Proof. The theorem will be proved in two steps. The first step is to prove that given a homeomorphism, composing it with Dehn twists and a homeomorphism isotopic to identity, we get another homeomorphism which is identity on all c_β's of S. Next, step will be to prove that given any homeomorphism of a disk with k holes which is identity on boundary, composing it with Dehn twists and a homeomorphism isotopic to identity, we get another homeomorphism which is identity.

Step-1: Let $p_1, \ldots, p_k$ be the $c'_\beta s$ of the handles of the surface S and $h \in \text{Homeo}^+(S)$. We will prove that there exists $n \in N_S$ and s, a product of Dehn twists such that $nshp_i = p_i$, for all $1 \leq i \leq k$ i.e., nsh is identity on p_i $\forall i$. Assume that this is true for all $i \leq t$, we will prove that it is true for $t + 1$.

By induction hypothesis, there exists $n \in N_S$ and s, a product of Dehn twists such that $nshp_i = p_i$, forall $1 \leq i \leq t$. Let $nshp_{t+1}$ be denoted by q. By Lemma 5.2.9, there exists a p_* such that p_* does not meet any handles and $p_* \sim_\tau q$. As $p_* \sim_\tau q$, there exists $n_1 \in N_S$ and a product of Dehn twists s_1 such that $n_1 s_1 q = p_*$. We can choose this n_1, s_1 in such a way that $n_1 s_1 nshp_i = p_i$ for all $i \leq t$. This is because q does not intersect any of the $p_i, i \leq t$, since nsh fixes $p_i, i \leq t$. Thus, by the proof of Lemma 5.2.9, the construction of p_* can be done without affecting p_i for $i \leq t$.

Observe that p_* is not trivial, as it is a homemorphic copy of a non-trivial curve p_{t+1}. Moreover, p_* lies completely on the sphere component of S, as it does not meet any handle of S. By Jordan Curve theorem, p_* must divide sphere into two components. We know that, p_* is non-separating, since p_{t+1} is non-separating. So, there exists a handle which connects the two components of sphere. Hence, p_* goes through some handles of S.

Since p_{t+1} is not homologous to any of the p_i, we have $nshp_{t+1} = q$ is not homologous to any of the $nshp_i$, as homeomorphism induces an isomorphism of homologies. As $nshp_i = p_i$, for $i \leq t$, q is not homologous to p_i, for $i \leq t$. Hence, p_* cannot be a linear combination of p_i which means that p_* has to pass through some handles which do not contain p_i as c_β.

Let H be such a handle. Take curves l and m such as shown in the Figure 5.14. Then, by Lemma 5.2.4 and the transitivity of twist equivalence, $p_* \sim_\tau q$. Hence, $\exists\ n_2 \in N_S$ and a product of Dehn twists, s_2 such that $n_2 s_2 p_* = p_{t+1}$. By the proof of Lemma 5.2.4, n_2, s_2 can be chosen such that $n_2 s_2 n_1 s_1 nshp_i = p_i$,for $i \leq t$. As $q = nshp_{t+1} = p_*$ and $n_2 s_2 p_* = p_{t+1}$, we get that $n_2 s_2 n_1 s_1 nshp_{t+1} = p_{t+1}$.

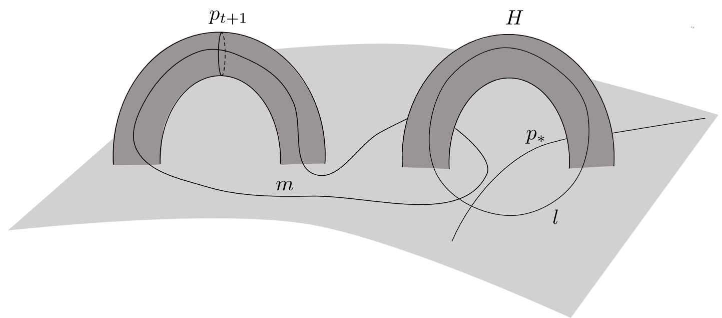
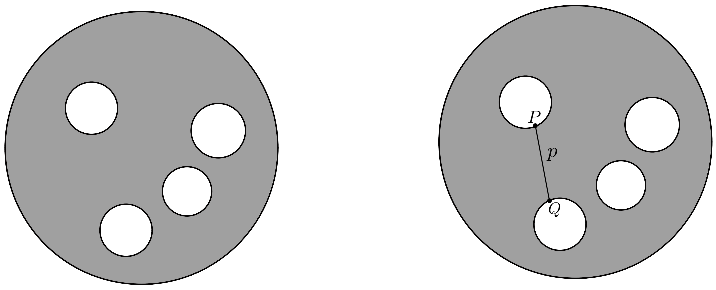


Fig. 5.14

We know that, there exists $n_3 \in N_S$, such that $n_3 s_2 s_1 s = n_2 s_2 n_1 s_1 ns$. Thus, we have $n' = n_3$ and $s' = s_2 s_1 s$ such that $n's'hp_i = p_i, i \leq t+1$.

If we cut our surface along all $p_i's$, we get a disc with k holes and $n's'h$ would an orientation preserving homeomorphism of a disc with k holes which is identity on the boundary.

Step-2: We will prove that, if D_u be the disc with u holes and f, a homeomorphism which is identity on the boundary, then $\exists n \in \text{Homeo}_0(X)$ and s, a product of Dehn twists such that $nsf = \text{Id}$. The case $u = 1$ is done by Lemma 5.2.10. Now, assume it is true for D_k, we have to prove it for D_{k+1}. Let f be a homeomorphism of D_{k+1} such that it is identity on the boundary.



(a) **(b)**

Fig. 5.15

Let p be a path from a point P on one boundary circle to Q on another boundary circle. We can assume that f is identity on small intervals near the boundary circle. Let $P_1, \ldots P_r$ be the points of the intersection of fp and p, in the order of their appearance on p, such that f is identity on PP_1.

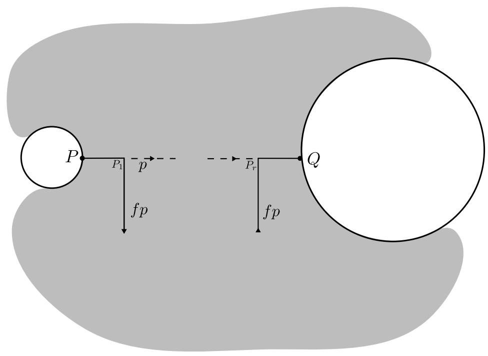


Fig. 5.16

Case-I: Suppose fp is in the same direction at P_1 and P_2 (see Figure 5.17a). Consider the curve C, which starts in a neighbourhood of P_1 and goes till P_2 then f_p to reach back to P_1 (see Figure 5.17b). Perform a Dehn twist along this curve to get sfp as in Figure 5.17c . There exists $n \in N_S$ such that $nsfp$ is identity on PP_2.

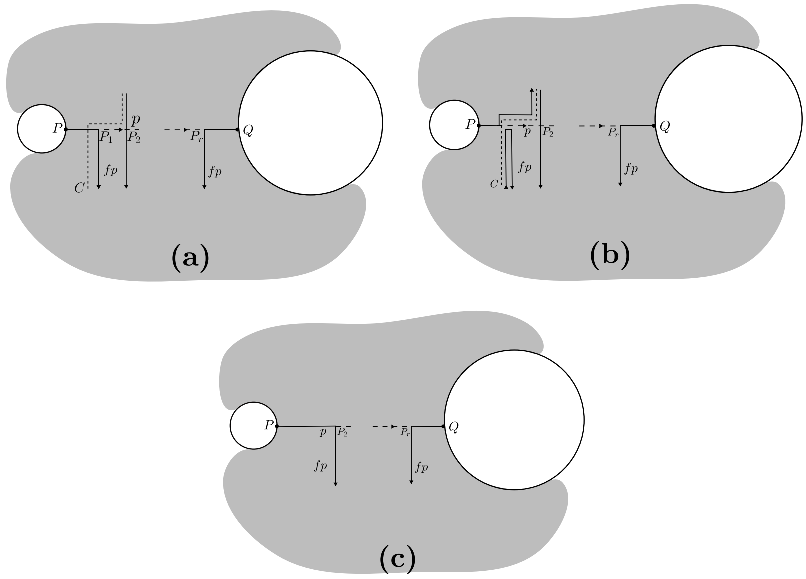


Fig. 5.17

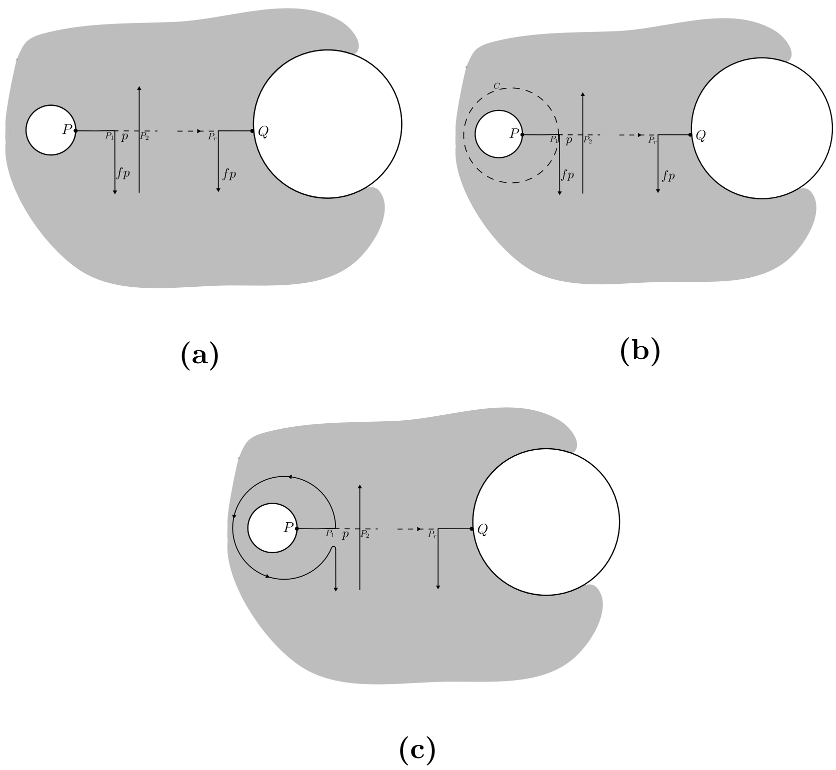


Fig. 5.18

Case-II: Suppose fp is in different directions at P_1 and P_2 (see Figure 5.18a). Consider the curve C in Figure 5.18b . Perform a Dehn twist along C to get a curve which has same direction at these points which reduces to Case-I (see Figure 5.18c). Repeating this we get $n'' \in N_S$ and s'', a product of Dehn twists such that $n''s''f$ is identity on p. Now, cut D_{k+1} along p, this results in a disk with k holes and $n''s''f$ is a homeomorphism which is identity on the boundary. Hence, $n'''s'''f = \mathrm{Id}_{D_{k+1}}$.

From Step 1, take $f = n's'h$, then $n'''s'''n's'h = \mathrm{Id}$. This implies that $h = (s')^{-1}(n')^{-1}(s''')^{-1}(n''')^{-1}$. Hence, $h = \eta\sigma$ for some $\eta \in N_S$ and a product of Dehn twists, σ. □

6. 3-MANIFOLDS AND LICKORISH-WALLACE THEOREM

As the title suggests, the objective of this chapter is to prove Lickorish-Wallace Theorem. This is a neat application of Dehn-Lickorish theorem proved in the previous chapter.

6.1 Preliminaries

Definition 6.1.1. Let M be a 3-manifold and e be the embedding of two 2-dimensional discs into the boundary ∂M. Then, $M \cup_e (D^2 \times I)$ is called M with an 1-handle added.

Definition 6.1.2. A handlebody of genus g is an orientable 3-manifold that is 3-ball with g 1-handles added.

Definition 6.1.3. A Heegaard splitting of a closed, connected, orientable 3-manifold M is a pair of handlebodies X and Y contained in M such that $X \bigcup Y \cong M$ and $X \cap Y = \partial X = \partial Y$.

The following result is an important theorem in 3-manifolds [9, Lemma 12.12]:

Theorem 6.1.4. *Any closed connected orientable 3-manifold has a Heegaard splitting.*

6.2 Lickorish-Wallace Theorem

Theorem 6.2.1. *Any closed connected orientable 3-manifold is homeomorphic to S^3 from which have been removed a finite set of disjoint solid tori and are sewn back in a different way.*

Proof. Let M be any 3-manifold and V_1, V_2 form a heegard splitting of M i.e., $M \cong V_1 \bigcup_f V_2$ where $f : \partial V_1 \to \partial V_2$. We know that, there exists an $i : \partial V_1 \to \partial V_2$ such that $S^3 \cong V_1 \bigcup_i V_2$. Without loss of generality, assume that $f^{-1}i : \partial V_1 \to \partial V_1$ is orientation preserving, then by Theorem 5.2.11, $f^{-1}i = ns$, where $n \in N_S$ and s is a product of Dehn twists. Suppose that $f^{-1}i$ is just a single Dehn twist λ.

Let C be the curve on ∂V_1 along which you perform λ and A be an annulus. Imbed $A \times [0,1]$ in V_1 such that $A \times \{0\}$ is A and $A \times (0,1]$ lies in the interior of V_1. Let $T = A \times [1/2, 1]$. Define $j : V_1 \setminus T \to V_1 \setminus T$ such that

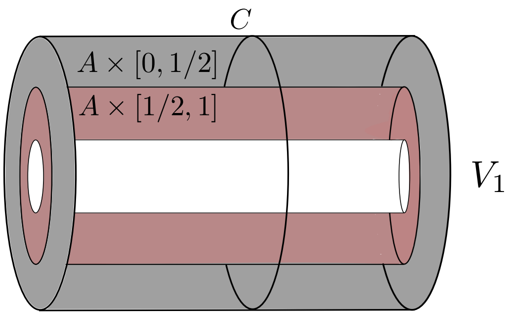


Fig. 6.1

$j|_{A\times[0,1/2]}(x,t) = (\lambda(x), t)$ and elsewhere it is identity. Clearly, j is a homeomorphism.

$$\begin{array}{ccc} V_1 \setminus T & \xrightarrow{\lambda} & V_1 \setminus T \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ V_2 & \xrightarrow{\text{Id}} & V_2 \end{array}$$

The maps indicated in the diagram above are the maps of the boundary used to glue the handlebodies. We know that, $V_1 \setminus T \bigcup_i V_2 \cong S^3 \setminus T$ and $V_1 \setminus T \bigcup_f V_2 \cong M \setminus T$. Define $h : S^3 \setminus T \to M \setminus T$ as

$$h(x) = \begin{cases} j(x), & \text{if } x \in V_1 \setminus T, \text{and} \\ x, & \text{if } x \in V_2. \end{cases}$$

We have to prove that h is well defined. It suffices to prove that $h(x) = h(i(x))$, when $x \in \partial V_1$. Let $x \in \partial V_1$, then

$$\begin{aligned} h(x) &= f(h(x)) \\ &= f(j(x)) \quad \because h|_{V_1} = j. \\ &= f(\lambda(x)) \quad \because j|_{\partial V_1} = \lambda. \\ &= i(x) \quad \because f^{-1}i = \lambda. \\ &= h(i(x)) \quad \because h|_{V_2} = \text{Id}. \end{aligned}$$

Thus, $M \setminus T \cong S^3 \setminus T$. Now to reattach the removed torus, we will have perform the Dehn twist and attach accordingly. Thus, if we remove the torus T and reattach it with a twist, we get M.

Now, if we have $f^{-1}i$ to be a composition of Dehn twists, we can select annuli and their regular neighbourhoods such that the tori being removed are disjoint. Thus, removing disjoint solid tori from S^3 and gluing them back with a twist will give us any 3-manifold. □

We can think of the solid torii being removed as a neighbourhood of a link in S^3. Thus, we can obtain any 3-manifold by removing neighbourhoods of links and attaching them back with a twist.

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