Background

Algebraic graph theory is a branch of Mathematics in which algebraic methods, particularly those employed in group theory and linear algebra, are used to solve graph-theoretic problems. An important subbranch of algebraic graph theory is spectral graph theory, which involves the study of the spectra of matrices associated with the graph such as its adjacency matrix, and its relation to the properties of the graph.

The spectral gap of a graph is the difference in magnitude of the two largest eigenvalues of its adjacency matrix. Graphs which have large spectral gaps are of great interest from the viewpoint of communication networks, as they exhibit strong connectivity properties.

Definition. Let \( X = (V,E) \) be an undirected \( k \)-regular of order \( n \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \), and let
\[
\lambda(X) = \max\{|\lambda_i| \mid |\lambda_i| \neq k, 1 \leq i \leq n\}.
\]
Then \( X \) is said to be a Ramanujan graph if \( \lambda(X) \leq 2\sqrt{k-1} \).

As Ramanujan Graphs are known to maximize the spectral gap, they have been widely studied from an application perspective. However, Ramanujan graphs have also fascinated pure mathematicians alike, as they lie in the interface of Number Theory, Representation Theory, and Algebraic Geometry [5]. In 1988, a beautiful paper by A. Lubotzky, R. Phillips, and P. Sarnak [4], described the explicit construction of a Ramanujan Graph for every pair of distinct primes congruent to 1 modulo 4, thereby establishing the existence of an infinite family of such graphs. The primary goal of this project to study some basic concepts in algebraic graph theory, with a view towards understanding Ramanujan graphs and some of their properties.

Synopsis

We will begin by studying some fundamental topics in algebraic graph theory such as graph automorphisms, vertex-transitivity, group actions, and Cayley graphs. We will mostly follow [3], and we will use [2, 5] and [1] as additional references. Cayley graphs, which are graphs that encode the abstract structures of groups, will be one of our central objects of study in this project. We will try to understand the following important property that characterizes Cayley graphs:
**Theorem.** Every Cayley graph is vertex transitive. Furthermore, every connected vertex-transitive graph is a retract of a Cayley graph.

After attaining a basic understanding of Cayley graphs, we will begin studying the spectral properties of regular graphs. The first key result that we will encounter in this direction is:

**Theorem.** Let $X$ be a $k$-regular graph. Then

(i) $k$ is an eigenvalue of $X$,

(ii) the multiplicity of $k$ is equal to number of connected components of $X$, and

(iii) for any eigenvalue $\lambda$ of $X$, we have $|\lambda| \leq k$.

We shall then study the spectra of Cayley graphs of cyclic groups, also known as circulant graphs, which will lead us to the following theorem.

**Theorem.** Suppose that $[0,a_2,a_3,...,a_n]$ is the first row of adjacency matrix of circulant graph $X$. Then the eigenvalues of $X$ are given by

$$\lambda_r = \sum_{j=2}^{n} a_j \omega^{(j-1)r}, \text{ where } 0 \leq r \leq n-1 \text{ and } \omega = e^{2\pi i/n}.$$ 

As a direct application of this theorem, we shall establish that for any $n$, the $n$-cycle graph $C_n$ is a Ramanujan graph. Using these results, we shall derive conditions on $n, m$ under which the graph Cartesian product $C_n \square C_m$ will be a Ramanujan graph. Finally, if time permits, we shall study the spectral properties of the Cayley graphs of other finite groups such as $S_n, Q_n$ and $Z_m \ltimes_k Z_n$.

**References**


