Self-dual representations with vectors fixed under an Iwahori subgroup

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Abstract

Let G be the group of F-points of a split connected reductive F-group over a non-Archimedean local field F of characteristic 0. Let π be an irreducible smooth self-dual representation of G. The space W of π carries a non-degenerate G-invariant bilinear form (,) which is unique up to scaling. The form is easily seen to be symmetric or skew-symmetric and we set $\varepsilon(\pi) = \pm 1$ accordingly. In this article, we show that $\varepsilon(\pi) = 1$ when π is a generic representation of G with non-zero vectors fixed under an Iwahori subgroup I.

Introduction

Let G be a group and (π, V) be an irreducible complex representation of G such that $\pi \simeq \pi^\vee$ ($\pi^\vee$ denotes the dual or contragredient of π). Assuming Schur's Lemma applies, this isomorphism gives rise to a non-degenerate G-invariant bilinear form on V which is unique up to scalars. It follows that the form is either symmetric or skew-symmetric. Accordingly, we set

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if the form is symmetric,} \\ -1 & \text{if the form is skew-symmetric,} \end{cases}$$

which we call the sign of π.

For G finite, Frobenius and Schur showed over a century ago that the irreducible self-dual representations π with $\varepsilon(\pi) = 1$ are exactly the real irreducible representations of G. We recall that an irreducible representation τ of G is real if there is a real G-invariant subspace $\mathcal{W}$ of the space $\mathcal{V}$ of τ such that $dim\mathcal{V} = dim\mathcal{W}$, or equivalently $\mathcal{V} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{W}$. Frobenius and Schur also obtained a striking expression for the sign $\varepsilon(\pi)$ in terms of the character χ_π of π:

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$$\varepsilon(\pi) = \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g^2).$$

If the irreducible representation π is not self-dual, then the right side is 0.

The sign $\varepsilon(\pi)$ has been extensively studied for connected compact Lie groups and certain classes of finite groups of Lie type. For connected compact Lie groups the sign can be computed using the dominant weight attached to the representation π (see [2] pg. 261-264). In [6], Gow showed that for q a power of an odd prime and F_q the finite field with q elements, irreducible self-dual complex representations of $\mathrm{SO}(n, F_q)$ are always realizable over $\mathbb{R}$. He also showed that the same is true for any non-faithful representation of $\mathrm{Sp}(n, F_q)$. The proofs involve a detailed analysis of the conjugacy classes of these groups. In [8], Prasad introduced an elegant idea to compute the sign for a certain class of representations of finite groups of Lie type. These representations are called generic. He used this idea to determine the sign for many finite groups of Lie type, avoiding extensive conjugacy class computations. It is natural to look at the question of determining the sign in the setting of reductive p-adic groups. These groups share many of the features of finite groups of Lie type but have a more complicated representation theory. Prasad has initiated such a study. In fact, in [9] he extends the results of [8] to reductive p-adic groups and computes the sign for generic representations of certain classical groups in some cases. In [13], Vinroot shows that the sign is one for an irreducible self-dual representation of $\mathrm{GL}(n, F)$ where F is a p-adic field. More recently in [10], Prasad and Ramakrishnan have looked at signs of irreducible self-dual discrete series representations of $\mathrm{GL}_n(D)$, for D a finite dimensional p-adic division algebra, and have proved a remarkable formula that relates the signs of these representations and the signs of their Langlands parameters.

In this paper, we determine the sign $\varepsilon(\pi)$ when π is a certain type of representation of an arbitrary connected reductive p-adic group G which is split over the underlying p-adic field F. Suppose K_0 is a special maximal compact open subgroup of G and π has non-zero vectors fixed under K_0. In this situation, the space of K_0-fixed vectors in π is one-dimensional and it follows easily that $\varepsilon(\pi) = 1$. A natural question is what is $\varepsilon(\pi)$ if π has non-zero Iwahori fixed vectors. There is strong evidence that the sign is one in the Iwahori fixed case. We address a particular case of this problem for generic representations. To be more precise, we prove the following theorem.

Theorem 0.1 (Main Theorem)**.** *Let (π, W) be an irreducible smooth self-dual representation of G with non-zero vectors fixed under an Iwahori subgroup in G. Suppose that π is also generic. Then $\varepsilon(\pi) = 1$.*

The paper is organized as follows. In section 1, we introduce the sign associated to a self-dual representation. In section 2, we motivate the problem by discussing representations of some classical groups. In section 3, we recall some important results we need to prove the main theorem. In section 4, we prove the main theorem.

1. Self-dual representations and signs

In this section, we briefly discuss the notion of signs associated to self-dual representations.

Let F be a non-Archimedean local field and G be the group of F-points of a connected reductive algebraic group. Let (π, W) be a smooth irreducible representation of G. We write $(\pi^\vee, W^\vee)$ for the smooth dual or contragredient of (π, W) and $\langle\ ,\ \rangle$ for the canonical non-degenerate G-invariant pairing on $W \times W^\vee$ (given by evaluation). Let $s : (\pi, W) \to (\pi^\vee, W^\vee)$ be an isomorphism. The map s can be used to define a bilinear form on W as follows:

$$(w_1, w_2) = \langle w_1, s(w_2)\rangle, \quad \forall w_1, w_2 \in W.$$

It is easy to see that $(\ ,\ )$ is a non-degenerate G-invariant form on W, i.e., it satisfies,

$$(\pi(g)w_1, \pi(g)w_2) = (w_1, w_2), \quad \forall w_1, w_2 \in W.$$

Let $(\ ,\ )_*$ be a new bilinear form on W defined by

$$(w_1, w_2)_* = (w_2, w_1)$$

Clearly, this form is again non-degenerate and G-invariant. It follows from Schur's Lemma that

$$(w_1, w_2)_* = c(w_1, w_2)$$

for some non-zero scalar c. A simple computation shows that $c \in \{\pm 1\}$. Indeed,

$$(w_1, w_2) = (w_2, w_1)_* = c(w_2, w_1) = c(w_1, w_2)_* = c^2(w_1, w_2).$$

We set $c = \varepsilon(\pi)$. It clearly depends only on the equivalence class of π. In sum, the form $(\ ,\ )$ is symmetric or skew-symmetric and the sign $\varepsilon(\pi)$ determines its type.

2. Representations of some classical groups

We use a theorem of Waldspurger (Chapter 4.II.1 in [7]) to show that many representations of classical groups are self-dual. Throughout this section, we let F be a non-Archimedean local field of characteristic $\neq 2$ and W be a finite dimensional vector space over F. We write $\mathfrak{O}$ for the ring of integers in F, $\mathfrak{p}$ for the unique maximal ideal of $\mathfrak{O}$ and k for the residue field. We let $\langle\ ,\ \rangle$ be a non-degenerate symmetric or skew-symmetric form on W. We take

$$G = \{g \in \mathrm{GL}(W) \mid \langle gw, gw' \rangle = \langle w, w' \rangle\}.$$

For $x \in \mathrm{GL}(W)$ such that $xGx^{-1} = G$ and (π, V) a representation of G, we let π^x denote the representation of G defined by conjugation (i.e., $\pi^x(g) = \pi(xgx^{-1})$).

We recall the statement of Waldspurger's theorem below.

Theorem 2.1 (Waldspurger)**.** *Let π be an irreducible admissible representation of G and $\pi^\vee$ be the smooth-dual (contragredient) of π. Let $x \in \mathrm{GL}(W)$ be such that $\langle xw, xw' \rangle = \langle w', w \rangle$, for all $w, w' \in W$. Then $\pi^x \simeq \pi^\vee$.*

Orthogonal groups. Suppose the form $\langle\, , \rangle$ is symmetric so that G is the orthogonal group $\mathrm{O}(W)$. Let π be any irreducible admissible representation of G. Then $x = 1 \in G$ satisfies $\langle xw, xw' \rangle = \langle w', w \rangle$ for all $w, w' \in W$. By Theorem 2.1, it follows that $\pi \simeq \pi^\vee$. So in the case of orthogonal groups every irreducible representation π is self-dual.

Special orthogonal groups. Suppose the dimension of W is odd. Take $G = \mathrm{SO}(W) = \mathrm{O}(W) \cap \mathrm{SL}(W)$ and π to be an irreducible admissible representation of G. Since $\mathrm{O}(W) \simeq \mathrm{SO}(W) \times \{\pm 1_W\}$, it follows that there exists an irreducible representation $\tilde{\pi}$ of $\mathrm{O}(W)$ such that $\tilde{\pi} \simeq \pi \otimes \chi$ (where χ is a character of $\{\pm 1_W\}$). Since $\chi = \chi^{-1}$ and $\tilde{\pi} \simeq \tilde{\pi}^\vee$, it follows that $\pi \simeq \pi^\vee$.

Symplectic groups. Let W be a $2n$-dimensional vector space over F with basis $\{e_1, \ldots, e_n, f_n, \ldots, f_1\}$. We take $\langle \cdot, \cdot \rangle$ to be the skew-symmetric bilinear form on W defined as

$$\langle x, y \rangle = x_1 y_{2n} + \cdots + x_n y_{n+1} - x_{n+1} y_n - \cdots - x_{2n} y_1.$$

and G is the symplectic group $Sp(W)$ corresponding to the above form. We show that a certain class of representations of G is always self-dual. To be more precise, we prove,

Theorem 2.2. *Let (π, V) be an irreducible admissible representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Then $\pi \simeq \pi^\vee$.*

Consider $x = \left[\begin{smallmatrix} -I_n & 0 \\ 0 & I_n \end{smallmatrix}\right] \in \mathrm{GL}(W)$ (where I_n is the $n \times n$ identity matrix). It is easy to see that $\langle xw, xw' \rangle = \langle w', w \rangle$. By Theorem 2.1, $\pi^x \simeq \pi^\vee$. To prove $\pi \simeq \pi^\vee$, it suffices to show that $\pi \simeq \pi^x$. Observe that $xIx^{-1} = I$ and $\pi^I = (\pi^x)^I$. Since $\pi^I = (\pi^x)^I \neq 0$, they can be realized as simple modules over $\mathcal{H}(G, I)$ (where $\mathcal{H}(G, I)$ is the Iwahori-Hecke algebra). Let $f \bullet v$ and $f \star v$ denote the action of $\mathcal{H}(G, I)$ on π^I and $(\pi^x)^I$, respectively. It will follow that $\pi \simeq \pi^x$ if π^I and $(\pi^x)^I$ are equivalent as $\mathcal{H}(G, I)$-modules. We establish this equivalence by showing that the map $\phi = 1_V$ (identity map on V) defines an intertwining map between π^I and $(\pi^x)^I$.

Before we continue, we fix a collection of coset representatives for the affine Weyl group $\widetilde{\mathcal{W}}$ and record two lemmas we need.

Let w_n be the matrix of the linear transformation such that $w_n(e_n) = -f_n$, $w_n(f_n) = e_n$ and $w_n(e_k) = e_k$ for $k \neq n$. For $i = 1, 2, \dots, n-1$, we let w_i be the matrix of the linear transformation such that $w_i(e_i) = e_{i+1}, w_i(e_{i+1}) = e_i, w_i(f_i) = f_{i+1}, w_i(f_{i+1}) = f_i$ and fixes all the other basis vectors. For $\ell = (l_1, \dots, l_n) \in \mathbb{Z}^n$, we let w_ℓ to be the matrix of the linear transformation such that $w_\ell(e_i) = \varpi^{l_i} e_i$ and $w_\ell(f_i) = \varpi^{-l_i} f_i$.

The group $\left\langle w_\ell, w_n, w_i \;\middle|\; \ell \in \mathbb{Z}^n, i = 1, 2, \dots, n-1 \right\rangle$ contains a collection of coset representatives for the affine Weyl group $\widetilde{\mathcal{W}}$.

Lemma 2.3. *For $f \in \mathcal{H}(G, I)$, let $f^x \in \mathcal{H}(G, I)$ be the function $f^x(g) = f(x^{-1}gx)$. The following statements are true.*

(i) $f \star v = f^x \bullet v$.

(ii) *For $g = i_1 w i_2 \in G$ (Bruhat Decomposition), $f^x(g) = f(xwx^{-1})$ and $f(g) = f(w)$.*

Proof. Clearly $f \star v = f^x \bullet v$. Indeed,

$$f \star v = \int_G f(g)\pi^x(g)v dg = \int_G f(x^{-1}gx)\pi(g)v dg = f^x \bullet v.$$

(ii), Clearly follows from the fact that x normalizes I and f is I bi-invariant. $\square$

Lemma 2.4. *For $w \in \widetilde{W}$, $xwx^{-1} \in IwI$.*

Proof. If $w \in \widetilde{W}$, we can write $w = u_1 u_2 \dots u_n$ for $u_k \in \{w_n, w_i, w_\ell\}$. We assume that this expression for w is minimal and we say that w has height n and denote it as $h(w) = n$. We will use induction on the height $h(w)$ to show that $xwx^{-1} \in T_\circ w = wT_\circ \subset IwI$. Suppose that $h(w) = 1$. A simple computation shows that conjugation by the element x fixes the elements w_i, w_ℓ and fixes w_n up to multiplication by elements in $T_\circ \subset I$, i.e., $xw_ix^{-1} = w_i$ for $i = 1, \dots, n-1$, $xw_\ell x^{-1} = w_\ell$ for $\ell \in \mathbb{Z}^n$ and $xw_nx^{-1} = tw_n = w_nt^*$ for some $t, t^* \in T_\circ \subset I$. Suppose $h(w) = 2$, i.e., $w = u_1u_2$ for $u_1, u_2 \in \{w_n, w_i, w_\ell\}$. Using the above conjugation formulae, it is clear that we have

$$
\begin{aligned}
xwx^{-1} &= xu_1x^{-1}xu_2x^{-1} \\
&= t_1u_1t_1^{-1}t_2u_2t_2^{-1} \\
&= t_1u_1u_2\underbrace{u_2^{-1}t_1^{-1}t_2u_2}_{\in T_\circ}t_2^{-1} \\
&= t_1u_1u_2t_1' \\
&= t_1\underbrace{u_1u_2t_1'u_2^{-1}u_1^{-1}}_{\in T_\circ}u_1u_2 \\
&= tu_1u_2
\end{aligned}
$$

where $t \in T_\circ$.

Assume that the result is true for all words w such that $h(w) \leq n-1$. Suppose $w = u_1u_2 \dots u_n$. Now

$$
\begin{aligned}
xwx^{-1} &= xu_1u_2 \dots u_{n-1}x^{-1}xu_nx^{-1} \\
&= t_1u_1u_2 \dots u_{n-1}t_2u_n \\
&= t_1u_1u_2 \dots u_{n-1}u_n\underbrace{u_n^{-1}t_2u_n}_{\in T_\circ} \\
&= t_1wt_1' \\
&= t_1\underbrace{wt_1'w^{-1}}_{\in T_\circ}w \\
&= tw
\end{aligned}
$$

where $t \in T_\circ$. $\square$

We now prove Theorem 2.2. To prove $\phi = 1_V$ is an intertwining map, we need to show that $f \bullet v = f \star v = f^x \bullet v$. By Lemma 2.3, it suffices to show that $f(xwx^{-1}) = f(w)$, for all $w \in \widetilde{\mathcal{W}}$. By Lemma 2.4, it follows that conjugation by the element x fixes every element in $\widetilde{W}$ (up to multiplication by elements in I). The result now follows.

3. Results used in proof of main theorem

In this section, we recall the important results used in the proof of main theorem.

3.1. Restriction of representations to subgroups

We recall some results about restricting an irreducible representation to a subgroup. These results hold when G is a locally compact totally disconnected group and H is an open normal subgroup of G such that G/H is finite abelian. For a more detailed account, we refer the reader to [5] (Lemmas 2.1 and 2.3).

Theorem 3.1 (Gelbart-Knapp)**.** *Let π be an irreducible admissible representation of G. Suppose that G/H is finite abelian. Then*

(i) $\pi|_H$ is a finite direct sum of irreducible admissible representations of H.
(ii) When the irreducible constituents of $\pi|_H$ are grouped according to their equivalence classes as

$$\pi|_H = \bigoplus_{i=1}^{M} m_i \pi_i$$

with the π_i irreducible and inequivalent, the integers m_i are equal.

Theorem 3.2 (Gelbart-Knapp)**.** *Let G be a locally compact totally disconnected group and H be an open normal subgroup of G such that G/H is finite abelian, and let π be an irreducible admissible representation of H. Then*

(i) There exists an irreducible admissible representation $\tilde{\pi}$ of G such that $\tilde{\pi}|_H$ contains π as a constituent.
(ii) Suppose $\tilde{\pi}$ and $\tilde{\pi}'$ are irreducible admissible representations of G whose restrictions to H are multiplicity free and contain π. Then $\tilde{\pi}|_H$ and $\tilde{\pi}'|_H$ are equivalent and $\tilde{\pi}$ is equivalent with $\tilde{\pi}' \otimes \chi$ for some character χ of G that is trivial on H.

3.2. Unramified principal series and representations with Iwahori fixed vectors

We state an important characterization of representations with non-zero vectors fixed under an Iwahori subgroup due to Borel and Casselman. We refer to ([1], [4]) for a proof.

Throughout this section, we let G be the group of F-points of a connected reductive algebraic group defined and split over F. We write T for a maximal F-split torus in G. We also fix a Borel subgroup B defined over F such that $B \supset T$ and write U for the unipotent radical of B. Given a smooth representation (ρ, W) of T, we write $\operatorname{Ind}_B^G \rho$ for the resulting parabolically induced representation.

Theorem 3.3 (Borel-Casselman)**.** *Let (π, W) be any irreducible admissible representation of G. Then the following assertions are equivalent.*

(i) There are non-zero vectors in W invariant under I.
(ii) There exists some unramified character μ of T such that π imbeds as a sub-representation of $\operatorname{Ind}_B^G \mu$.

3.3. Prasad's idea for computing the sign

Prasad gives a criterion to compute the sign for an irreducible smooth self-dual generic representation of a p-adic group G. Before we continue, we recall the definition of a generic representation and record an important fact. A representation π is called generic if there exists a non-degenerate character ψ of U such that $\mathrm{Hom}_U(\pi,\psi) \neq 0$. Also if π is generic then $dim_{\mathbb{C}}\,\mathrm{Hom}_U(\pi,\psi) = 1$. In [9], he shows that for generic representations the sign is determined by the value of the central character ω_π at a special central element. We recall his result below.

Theorem 3.4 (Prasad). *Let π be an irreducible smooth self-dual representation of G and let K be a compact open subgroup. Let s be an element of G which normalizes K and whose square belongs to the center of G. Let $\psi_K : K \to \mathbb{C}^\times$ be a one dimensional representation of K which is taken to its inverse by the inner conjugation action of s on K. Suppose that the character ψ_K of K appears with multiplicity 1 in the restriction of π to K. Then $\varepsilon(\pi)$ is 1 if and only if the element s^2 belonging to the center of G operates by 1 on π.*

3.4. Compact approximation of Whittaker models

For $m \in \mathbb{Z}$, Rodier constructs a sequence (K_m, ψ_m) of compact open subgroups K_m and characters ψ_m of K_m such that the following are satisfied.

(i) K_m converges to U and
(ii) $\psi_m|_{K_m \cap U} = \psi|_{K_m \cap U}$

We refer the reader to ([11], section III, pg. 155) for the construction of (K_m, ψ_m) and a more detailed account of his results.

We fix an integer l large enough and call the pair (K_l, ψ_l) as the compact approximation of (U, ψ). To simplify notation, we write (K, ψ_K) for the compact approximation (K_l, ψ_l). We now recall an important result of Rodier which we need in the proof of the main theorem.

Theorem 3.5 (Rodier). *Let π be an irreducible smooth representation of G and ψ be a non-degenerate character of U. There then exists a compact open subgroup K of G and a character ψ_K of K such that*

$$dim_{\mathbb{C}}\,\mathrm{Hom}_K(\pi,\psi_K) = dim_{\mathbb{C}}\,\mathrm{Hom}_U(\pi,\psi).$$

Therefore, if π is generic, $dim_{\mathbb{C}}\,\mathrm{Hom}_K(\pi,\psi_K) = 1$.

4. Main Theorem

In this section, we prove the main theorem. We recall the statement below.

Theorem 4.1 (Main Theorem). *Let (π, W) be an irreducible smooth self-dual representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Suppose that π is also generic. Then $\varepsilon(\pi) = 1$.*

Throughout this section, we let G be the group of F-points of a connected reductive algebraic group defined and split over F. We write T for a maximal F-split torus in G. We also fix a Borel subgroup B defined over F such that $B \supset T$. We write U for the unipotent radical of B (respectively, $\bar{U}$ for the T-opposite of U) and fix a non-degenerate character ψ of U such that $\mathrm{Hom}_U(\pi, \psi) \neq 0$ (ψ exists since π is generic). We let X and $X^\vee$ be the character and cocharacter groups of T. We write Φ and $\Phi^\vee$ for the set of roots and coroots and Δ for the set of simple roots of T. Since T is F-split, we have unique subgroups $T_\circ$ and T_1 of T such that $T = T_\circ \times T_1$. To be more precise, the isomorphism $F^\times \simeq \mathfrak{O}^\times \times \mathbb{Z}$ given by $x\varpi^n \mapsto (x, n)$ induces the following isomorphism

$$T \simeq X^\vee \otimes F^\times \simeq X^\vee \otimes \mathfrak{O}^\times \oplus X^\vee \otimes \mathbb{Z}$$

and we take $T_\circ$ and T_1 to be the subgroups of T such that $T_\circ \simeq X^\vee \otimes \mathfrak{O}^\times$ ($\alpha^\vee \otimes y \to \alpha^\vee(y)$) and $T_1 \simeq X^\vee \otimes \mathbb{Z}$ ($\alpha^\vee \otimes n \to \alpha^\vee(\varpi^n)$). In what follows, we let $Z_\circ = Z \cap T_\circ$ and $Z_1 = Z \cap T_1$. We write ω_π for the central character.

4.1. Center of G is connected

In this section, we prove the result when G has connected center. In this case, we use Rodier's result (Theorem 3.5) to get a compact open subgroup K and a character ψ_K of K which appears with multiplicity one in $\pi|_K$. We show that there exists an element $s \in T_\circ$ satisfying the hypotheses of Prasad's Theorem (Theorem 3.4). Finally we use the fact that π has non-zero Iwahori fixed vectors to show that $\varepsilon(\pi) = 1$.

Lemma 4.2. *Suppose $s \in T$ is such that $\alpha(s) = -1$ for all simple roots $\alpha \in \Delta$. The following are true.*

(i) For $u \in U$, we have $\psi(sus^{-1}) = \psi^{-1}(u)$.
(ii) The element s^2 belongs to the center of G.
(iii) The element s normalizes the compact open subgroup K and the inner conjugation by s takes the character ψ_K to its inverse.

Proof. Since U is generated by U_α, $\alpha \in \Phi$, it is enough to show that $\psi(sus^{-1}) = \psi^{-1}(u)$ for $u \in U_\alpha$. For $u = x_\alpha(\lambda) \in U_\alpha$ we have,

$$\psi(sus^{-1}) = \psi(x_\alpha(\alpha(s)\lambda)) = \psi(x_\alpha(-\lambda)) = \psi^{-1}(x_\alpha(\lambda)) = \psi^{-1}(u).$$

For (ii), since $\alpha(s) = -1$ it is clear that $\alpha(s^2) = 1$ for all simple roots $\alpha \in \Delta$. Since $Z(G) = \bigcap_{\alpha \in \Delta} \mathrm{Ker}(\alpha)$, the result follows.

For (iii), Follows trivially from the construction of K and the character ψ_K. $\square$

Theorem 4.3. *Let (π, W) be an irreducible smooth self-dual generic representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Suppose there exists an element $s \in T_\circ$ such that $\alpha(s) = -1$ for all simple roots α. Then $\varepsilon(\pi) = 1$.*

Proof. By Theorem 3.4, it is enough to show that $\omega_\pi(s^2) = 1$. Let $v \neq 0 \in \pi^I$. We have $v = \pi(s^2)v = \omega_\pi(s^2)v$. From this it follows that $\omega_\pi(s^2) = 1$. $\square$

Theorem 4.4. *There exists $s \in T_\circ$ such that $\alpha(s) = -1$ for all the simple roots α.*

Proof. We know that $X^\vee \otimes F^\times \simeq T$, via $y \otimes \lambda \mapsto y(\lambda)$. Since $F^\times \simeq \mathfrak{O}^\times \rtimes \mathbb{Z}$, we see that $T \simeq X^\vee \otimes \mathfrak{O}^\times \oplus X^\vee \otimes \mathbb{Z}$. Now define $f : T \longrightarrow \prod_{\alpha_i \in \Delta} F^\times$ by

$$f(y \otimes \lambda) = (\lambda^{\langle \alpha_1, y \rangle}, \ldots, \lambda^{\langle \alpha_k, y \rangle}).$$

We now show that there exists $y \in X^\vee$ such that $\langle \alpha_i, y \rangle$ is an odd integer for every simple root $\alpha_i, i = 1, \ldots, k$. Since Z is connected $X/\mathbb{Z}\Phi$ is torsion free. Since Δ spans Φ we see that $\mathbb{Z}\Phi = \mathbb{Z}\Delta$. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}\Delta \longrightarrow X \longrightarrow X/\mathbb{Z}\Delta \longrightarrow 0. \tag{4.1}$$

Since (4.1) is an exact sequence of finitely generated free abelian groups, it is split, i.e., $X = \mathbb{Z}\Delta \oplus L$, where $L \simeq X/\mathbb{Z}\Delta$. Let $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z})$. Clearly, g extends to an element of $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ (say trivial on L). Since $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \simeq X^\vee$, there exists $y \in X^\vee$ such that $g = \langle -, y \rangle$. We now choose $h \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z})$ such that $h(\alpha_i)$ is odd for every $\alpha_i, i = 1, 2, \ldots, k$. Then $h(\alpha_i) = \langle \alpha_i, y \rangle$ is an odd integer. Now consider the element $y \otimes -1 \in X^\vee \otimes \mathfrak{O}^\times$. Let $s = y(-1)$. Then $s \in T_\circ$ clearly acts by -1 on all the simple root subgroups U_α of U, i.e.,

$$\begin{aligned} s x_{\alpha_i}(\mu) s^{-1} &= x_{\alpha_i}(\alpha_i(s)\mu) \\ &= x_{\alpha_i}(\alpha_i(y(-1))\mu) \\ &= x_{\alpha_i}((-1)^{\langle \alpha_i, y \rangle}\mu) \\ &= x_{\alpha_i}(-\mu). \end{aligned}$$

$\square$

4.2. Center of G is not connected

In this section, we prove the result when the center of G is not connected. We construct a split connected reductive F-group $\tilde{G}$ with a maximal F-split torus $\tilde{T}$. The group $\tilde{G}$ is constructed in such a way that the center $\tilde{Z}$ is connected and contains G as a subgroup. We also have a decomposition for $\tilde{T}$ similar to T (i.e., $\tilde{T} = \tilde{T}_\circ \times \tilde{T}_1$). Throughout we write $\tilde{Z}_\circ = \tilde{Z} \cap \tilde{T}_\circ$, $\tilde{Z}_1 = \tilde{Z} \cap \tilde{T}_1$ and $\omega_\circ = \omega_\pi|_{Z_\circ}$, $\omega_1 = \omega_\pi|_{Z_1}$.

4.2.1. Construction of $(\tilde{G}, \tilde{T})$

Let $q : X \to X/\mathbb{Z}\Phi$ be the canonical quotient map. Choose a free abelian group L of finite rank such that there exists a surjective map $p : L \to X/\mathbb{Z}\Phi$. Let p_1 and p_2 be the projection maps from $X \times L$ onto X and L respectively. Let

$$\tilde{X} = \{(x, l) \in X \times L \mid q(x) = p(l)\}.$$

Clearly $\tilde{X}$ is a free abelian group of finite rank. Let $\tilde{\Phi} = \{(\alpha, 0) \mid \alpha \in \Phi\}$. The map $\alpha \mapsto (\alpha, 0)$ induces an injection $\mathbb{Z}\Phi \hookrightarrow \tilde{X}$ and we identify its image under the map with $\mathbb{Z}\tilde{\Phi}$. Let $\tilde{X}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\tilde{X}, \mathbb{Z})$. Given $\tilde{\alpha} \in \tilde{\Phi}$ we want to describe $\tilde{\alpha}^\vee \in \tilde{\Phi}^\vee \subset \tilde{X}^\vee$. Now $\tilde{\alpha} = (\alpha, 0)$ for some $\alpha \in \Phi$. For this α, there exists $\alpha^\vee \in \Phi^\vee \subset X^\vee \simeq \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$. Let $\tilde{x} = (x, 0) \in \tilde{X}$. Define $\tilde{\alpha}^\vee(\tilde{x}) = \tilde{\alpha}^\vee((x, 0)) = \alpha^\vee(p_1(\tilde{x}))$. Clearly $\tilde{\alpha}^\vee \in \mathrm{Hom}_{\mathbb{Z}}(\tilde{X}, \mathbb{Z})$. It is easy to see that $(\tilde{X}, \tilde{\Phi}, \tilde{X}^\vee, \tilde{\Phi}^\vee)$ is a root datum. By the classification theorem for split groups, the existence of $(\tilde{G}, \tilde{T})$ follows. Since $\tilde{X}/\mathbb{Z}\tilde{\Phi} \hookrightarrow L$, it follows that it is torsion free and the center $\tilde{Z}$ of $\tilde{G}$ is connected. As an example, it is easy to see that when $G = \mathrm{SL}(n, F)$, we get $\tilde{G} = \mathrm{GL}(n, F)$. Indeed, for $G = \mathrm{SL}(n, F)$, $X/\mathbb{Z}\Phi$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. We get $\tilde{G} = \mathrm{GL}(n, F)$, by taking $L = \mathbb{Z}$ and p to be the canonical projection from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$.

4.2.2. Extension of the central character

In this section, we show that there exists a character ν of $\tilde{Z}$ which extends the central character ω_π and satisfies $\nu^2 = 1$. Before we proceed, we note that $\omega_\pi^2 = 1$. Indeed, $\pi \simeq \pi^\vee$, implies that $\omega_\pi = \omega_{\pi^\vee} = \omega_\pi^{-1}$.

Lemma 4.5. *There exists an unramified character $\mu : T \to \mathbb{C}^\times$ such that $\mu|_Z = \omega_\pi$.*

Proof. Since π has Iwahori fixed vectors, there exists an unramified character μ of T such that $\pi \hookrightarrow \mathrm{Ind}_B^G \mu$. Let (ρ, E) be an irreducible sub-representation of $\mathrm{Ind}_B^G \mu$ that is isomorphic to π. Let $x \in Z, f \in E, g \in G$. Clearly,

$$(\rho(x)f)(g) = f(gx) = f(xg) = \mu(x)f(g) \tag{4.2}$$

On the other hand,

$$(\rho(x)f)(g) = \omega_\rho(x)f(g) \tag{4.3}$$

From (4.2) and (4.3) it follows that $\omega_\rho(x) = \mu(x) = \omega_\pi(x)$. □

Since μ is unramified it follows that $\mu|_Z = \omega_1$ (since $\mu|_{Z_\circ} = 1$ and $\omega_\pi = \omega_\circ\omega_1$). If we can extend ω_1 to a self-dual character $\tilde{\omega}_1$ of $\tilde{Z}_1$ then we get a self-dual character ν of $\tilde{Z}$ extending the central character ω_π. We record the result in a lemma below.

Lemma 4.6. *Suppose that $\tilde{\omega}_1$ is an extension of ω_1 to $\tilde{Z}_1$. Then there exists $\nu : \tilde{Z} \to \{\pm 1\}$ such that ν extends ω_π.*

Proof. For $\tilde{z} = \tilde{z}_0\tilde{z}_1 \in \tilde{Z}$, $\tilde{z}_0 \in \tilde{Z}_\circ, \tilde{z}_1 \in \tilde{Z}_1$ define $\nu(\tilde{z}) = \tilde{\omega}_1(\tilde{z}_1)$. Clearly ν is a well-defined character of $\tilde{Z}$ and $\nu|_Z = \tilde{\omega}_1|_Z = \omega_1 = \mu|_Z = \omega_\pi$. □

From Lemma 4.6, it follows that we can extend the central character ω_π to a self-dual character ν of $\tilde{Z}$ if there exists an extension $\tilde{\omega}_1$ of ω_1. Consider the map $\omega_1' : Z_1/Z_1^2 \to \{\pm 1\}$ defined by $\omega_1'(aZ_1^2) = \omega_1(a)$. Since Z_1/Z_1^2 is an elementary abelian 2-group, ω_1' can be thought of as a $\mathbb{Z}_2$-linear map. If the natural map from Z_1/Z_1^2 to $\tilde{Z}_1/\tilde{Z}_1^2$ is an embedding, then we can extend the $\mathbb{Z}_2$-linear map ω_1' to a $\mathbb{Z}_2$-linear map $\tilde{\omega}_1'$ of $\tilde{Z}_1/\tilde{Z}_1^2$. Now defining $\tilde{\omega}_1(a) = \tilde{\omega}_1'(a\tilde{Z}_2)$ gives us an extension of ω_1. The natural map is an embedding precisely when $\tilde{Z}_1^2 \cap Z_1 \subset Z_1^2$. We record the result in the following lemma.

Lemma 4.7. *The natural map Z_1/Z_1^2 to $\tilde{Z}_1/\tilde{Z}_1^2$ is an embedding and ω_1 extends to a character $\tilde{\omega}_1$ of $\tilde{Z}_1$.*

Proof. It is enough to show that $\tilde{Z}_1^2 \cap Z_1 \subset Z_1^2$. Consider $z_1 \in \tilde{Z}_1^2 \cap Z_1$. Clearly $z_1 = \tilde{z}_1^2$ for some $\tilde{z}_1 \in \tilde{Z}_1$. It is enough to show that $\tilde{z}_1 \in T$ (since $\tilde{z}_1 \in T$ implies $\tilde{z}_1 \in \tilde{Z}_1 \cap T = Z_1$). Since $T \hookrightarrow \tilde{T}$ there exists a sub-torus S such that $\tilde{T} = \tilde{T}_\circ \times \tilde{T}_1 = T \times S$. Clearly, $\tilde{T}_1 = T_1 \times S_1$. Indeed,

$$\tilde{T} = \tilde{T}_\circ \times \tilde{T}_1 = T_\circ \times T_1 \times S_\circ \times S_1 = T_\circ \times S_\circ \times T_1 \times S_1.$$

Now $\tilde{z}_1 \in \tilde{Z}_1 \subset \tilde{T}_1 = T_1 \times S_1$. Therefore $\tilde{z}_1 = t_1 s_1$ for $t_1 \in T_1$, $s_1 \in S_1$. Also $z_1 = \tilde{z}_1^2 = t_1^2 s_1^2 \in Z_1 \subset T_1$. We see that $s_1^2 = 1$. Since $\tilde{T}_1$ is torsion free it follows that $s_1 = 1$ and $\tilde{z}_1 \in T$. $\square$

4.2.3. The irreducible representation $\tilde{\pi}$ of $\tilde{G}$

In this section, we show that there exists an irreducible representation $\tilde{\pi}$ of $\tilde{G}$ which contains π with multiplicity one on restriction to G.

The main idea behind the proof is Theorem 3.2. We first extend the representation π to an irreducible representation $\pi\nu$ of $\tilde{Z}G$ and show that the group $\tilde{G}/\tilde{Z}G$ is finite abelian. Before we continue, we recall a result of Serre which we use in proving the finiteness of $\tilde{G}/\tilde{Z}G$. We take $\bar{F}$ to be the algebraic closure of F and $\Gamma = \mathrm{Gal}(\bar{F}/F)$.

Proposition 4.8 (Serre)**.** *If A is a finite Γ module , $H^n(\Gamma, A)$ is finite for every n.*

Proof. See Proposition 14, Sec. 5.1 in [12]. $\square$

Let (π, W) be an irreducible representation of G and ν be a self-dual character of $\tilde{Z}$ extending the central character ω_π. Clearly, $\pi\nu$ is a well-defined irreducible representation of $\tilde{Z}G$.

We now prove the finiteness of $\tilde{G}/\tilde{Z}G$.

Theorem 4.9. *$\tilde{G}/\tilde{Z}G$ is a finite abelian group.*

Proof. Clearly, $\tilde{G} = \tilde{T}G$. Now $\tilde{G}/\tilde{Z}G = \tilde{T}G/\tilde{Z}G = (\tilde{T}\tilde{Z}G)/\tilde{Z}G \simeq \tilde{T}/(\tilde{T} \cap \tilde{Z}G) = \tilde{T}/\tilde{Z}T$. It follows that $\tilde{G}/\tilde{Z}G$ is abelian. Let $m : T(\bar{F}) \times \tilde{Z}(\bar{F}) \to \tilde{T}(\bar{F})$

be the multiplication map. By dimension considerations, the map m is surjective and it is easy to see that $\mathrm{Ker}(m) = \{(z, z^{-1}) \mid z \in Z(\bar{F})\}$. Considering $Z(\bar{F})$ embedded diagonally in $T(\bar{F}) \times \tilde{Z}(\bar{F})$, we get the following exact sequence of abelian groups

$$1 \longrightarrow Z(\bar{F}) \longrightarrow T(\bar{F}) \times \tilde{Z}(\bar{F}) \xrightarrow{m} \tilde{T}(\bar{F}) \longrightarrow 1.$$

Γ clearly acts on these groups and applying Galois cohomology, we get a long exact sequence of cohomology groups

$$\begin{aligned} 1 \longrightarrow Z(\bar{F})^\Gamma \longrightarrow T(\bar{F})^\Gamma \times \tilde{Z}(\bar{F})^\Gamma \longrightarrow \tilde{T}(\bar{F})^\Gamma \longrightarrow H^1(\Gamma, Z(\bar{F})) \\ \longrightarrow H^1(\Gamma, T(\bar{F}) \times \tilde{Z}(\bar{F})) \longrightarrow H^1(\Gamma, \tilde{Z}(\bar{F})) \longrightarrow \cdots \end{aligned}$$

We note that $H^1(\Gamma, T(\bar{F}) \times \tilde{Z}(\bar{F})) = 1$ (Hilbert 90) and we get the short exact sequence

$$1 \longrightarrow Z \longrightarrow T \times \tilde{Z} \xrightarrow{m} \tilde{T} \xrightarrow{\varphi} H^1(\Gamma, Z(\bar{F})) \longrightarrow 1. \tag{4.4}$$

From (4.4), it follows that φ is surjective, $\mathrm{Im}(m) = T\tilde{Z} = \mathrm{Ker}(\varphi)$, and $\tilde{T}/\tilde{Z}T \simeq H^1(\Gamma, Z(\bar{F}))$. It is enough to show that $H^1(\Gamma, Z(\bar{F}))$ is finite. Let Z° be the identity component of the algebraic group Z. Consider the short exact sequence

$$1 \longrightarrow Z^\circ(\bar{F}) \longrightarrow Z(\bar{F}) \longrightarrow Z(\bar{F})/Z^\circ(\bar{F}) \longrightarrow 1. \tag{4.5}$$

Applying Galois cohomology again to (4.5), we get the sequence

$$\begin{aligned} 1 \longrightarrow Z^\circ \longrightarrow Z \longrightarrow Z/Z^\circ \longrightarrow H^1(\Gamma, Z^\circ(\bar{F})) \longrightarrow H^1(\Gamma, Z(\bar{F})) \\ \longrightarrow H^1(\Gamma, Z(\bar{F})/Z^\circ(\bar{F})) \longrightarrow \cdots \end{aligned}$$

Since $Z^\circ(\bar{F})$ is connected, we have $H^1(\Gamma, Z^\circ(\bar{F})) = 1$ and it follows that $H^1(\Gamma, Z(\bar{F})) \hookrightarrow H^1(\Gamma, Z(\bar{F})/Z^\circ(\bar{F}))$. Since F is a local field of char 0 and $Z(\bar{F})/Z^\circ(\bar{F})$ is a finite abelian group, $H^1(\Gamma, Z(\bar{F})/Z^\circ(\bar{F}))$ is finite (Proposition 4.8). Hence the result follows. ☐

By Theorem 3.1 and Theorem 3.2, we get an irreducible representation $(\tilde{\pi}, V)$ of $\tilde{G}$ which breaks up as a finite direct sum of distinct irreducible representations $(\pi_1, W_1), \ldots, (\pi_k, W_k)$ each occurring with the same multiplicity m on restriction to $\tilde{Z}G$ and containing $\pi\nu$ as a constituent. Without loss of generality, we assume that $\pi_1 \simeq \pi\nu$ (i.e., $W_1 \simeq W$ as representations of $\tilde{Z}G$). To simplify notation, we again denote the restriction of π_i's to G by π_i so that $\tilde{\pi}|_G = m\pi_1 \oplus m\pi_2 \oplus \ldots \oplus m\pi_k$ and $\pi \simeq \pi_1$. We now show that each π_i occurs with multiplicity one in $\tilde{\pi}|_G$.

Lemma 4.10. *The representation $(\tilde{\pi}, V)$ of $\tilde{G}$ is generic and each irreducible representation π_i occurs with multiplicity one.*

Proof. $\mathrm{Hom}_U(\tilde{\pi}, \psi)$ contains $\mathrm{Hom}_U(\pi, \psi)$. So the first space is non-zero. That is, $\tilde{\pi}$ is generic and so the space is one-dimensional, and hence $m = 1$. ☐

4.2.4. Choosing $\tilde{\pi}$ with non-zero $\tilde{I}$ fixed vectors

In this section, we show that the representation $\tilde{\pi}$ can be modified in such a way that it has non-zero vectors fixed under an Iwahori subgroup $\tilde{I}$ in $\tilde{G}$.

Lemma 4.11. *Suppose that τ_1 is a character of $\tilde{I}$ which is trivial on I. Then τ_1 extends to a character $\tilde{\tau}$ of $\tilde{G}$ which is trivial on G.*

Proof. Let $I^- = I \cap \bar{U}$ and $I^+ = I \cap U$. We know that $I = I^- T_\circ I^+$. Since $U = \tilde{U}$ we have $\tilde{I} = I^- \tilde{T}_\circ I^+$. Now

$$\tilde{I}/I = \tilde{T}_\circ I/I = \tilde{T}_\circ/\tilde{T}_\circ \cap I = \tilde{T}_\circ/T_\circ.$$

It follows that we can consider τ_1 as a character of $\tilde{T}_\circ$ which is trivial on $T_\circ$. We first extend τ_1 to a character $\tilde{\tau}_1$ of $\tilde{T}$ by making it trivial on $\tilde{T}_1$, i.e., $\tilde{\tau}_1(\tilde{t}_\circ \tilde{t}_1) = \tau_1(\tilde{t}_\circ)$. Now define an extension $\tilde{\tau}$ of $\tilde{\tau}_1$ to $\tilde{G}$ as $\tilde{\tau}(\tilde{t}g) = \tilde{\tau}_1(\tilde{t}), \tilde{t} \in \tilde{T}, g \in G$ (this is possible since $\tilde{G} = \tilde{T}G$). Using

$$\tilde{G}/G = \tilde{T}G/T = \tilde{T}/T = \tilde{T}_\circ/T_\circ \times \tilde{T}_1/T_1$$

it follows that $\tilde{\tau}$ is well-defined and a character of $\tilde{G}$. □

Theorem 4.12. *The representation $(\tilde{\pi}\tau^{-1}, V)$ of $\tilde{G}$ has non-zero $\tilde{I}$ fixed vectors.*

Proof. Let $v \neq 0 \in V$ be such that $\tilde{\pi}(i)v = v$, for all $i \in I$ (v exists since $\tilde{\pi}|_G \supset \pi$ and π has non-zero vectors fixed under I). Let $V_0 = \mathrm{Span}_{\mathbb{C}}\{\tilde{\pi}(k)v \mid k \in \tilde{I}\}$. Clearly, V_0 is an invariant subspace for $\tilde{I}$ and thus we get a representation (ρ, V_0) of $\tilde{I}$. Suppose $\rho = \tau_1 \oplus \cdots \oplus \tau_k$, where each τ_i is an irreducible representation of $\tilde{I}$. We know that $1_I \subset \rho$. Pick an irreducible component, say τ_1, that contains 1_I. By Clifford theory, $I \leq \mathrm{Ker}(\tau_1)$. Since $\tilde{I}/I$ is a compact abelian group, it follows that τ_1 is a character of $\tilde{I}$ which is trivial on I. By Lemma 4.11, τ_1 extends to a character $\tilde{\tau}$ of $\tilde{G}$ trivial on G. Consider the irreducible representation $\tilde{\pi}\tau^{-1}$. Clearly it has an $\tilde{I}$ fixed vector. Indeed for w in the space of τ_1 and $k \in \tilde{I}$, we have

$$(\tilde{\pi}\tau^{-1})(k)w = \tau^{-1}(k)\tilde{\pi}(k)w = \tau^{-1}(k)\tau_1(k)w = w.$$

□

It is easy to see that $\tilde{\pi}\tau^{-1}|_G$ contains the representation π with multiplicity one, in addition to having non-zero $\tilde{I}$ fixed vectors. To simplify notation, we will denote the representation $\tilde{\pi}\tau^{-1}$ as $\tilde{\pi}$.

4.2.5. Sign of $\tilde{\pi}$

In this section, we attach a sign $\varepsilon(\tilde{\pi})$ to the representation $\tilde{\pi}$. We also give a formula to compute $\varepsilon(\tilde{\pi})$ and show that $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$. Finally we show that $\varepsilon(\tilde{\pi}) = 1$ to complete the proof of the main theorem.

Consider the representation $\tilde{\pi}^\vee$. This is again an irreducible representation of $\tilde{G}$ which on restriction to $\tilde{Z}G$ contains the representation $\pi\nu$ with multiplicity 1 (since $\nu = \nu^{-1}$ and $\pi \simeq \pi^\vee$). By Theorem 3.2, there is a character χ of $\tilde{G}$

trivial on $\tilde{Z}G$ such that $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$. We use this isomorphism to define a non-degenerate bilinear form $[\,,\,]$ on V.

Lemma 4.13. *There exists a non-degenerate form* $[\,,\,] : V \times V \to \mathbb{C}$ *satisfying* $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2]$.

Proof. Since $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$, there exists a non-zero map $\phi : V \to V^\vee$ such that $\tilde{\pi}^\vee(g)(\phi(v)) = \phi((\tilde{\pi} \otimes \chi)(g)v)$. Let $\langle\,,\,\rangle$ be the canonical $\tilde{G}$ invariant pairing. We define $[\,,\,] : V \times V \to \mathbb{C}$ as $[v_1, v_2] = \langle v_1, \phi(v_2)\rangle$. Clearly this form is non-degenerate and satisfies $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2]$. $\square$

The form $[\,,\,]$ is unique up to scalars and is easily seen to be symmetric or skew-symmetric as before, i.e.,

$$[v_1, v_2] = \varepsilon(\tilde{\pi})[v_2, v_1].$$

where $\varepsilon(\tilde{\pi}) \in \{\pm 1\}$. We call $\varepsilon(\tilde{\pi})$ the sign of $\tilde{\pi}$.

Let $[\,,\,] : V \times V \longrightarrow \mathbb{C}$ be the non-degenerate bilinear form on V(obtained above). Suppose that $[\,,\,]\big|_{W_1 \times W_j} = 0$, for all $j = 2, 3, \cdots, k$. Then it is easy to see that $[\,,\,]\big|_{W_1 \times W_1}$ is non-degenerate. We now show that $[\,,\,]\big|_{W_1 \times W_j} = 0$ for $j = 2, 3, \cdots, k$.

Lemma 4.14. $[\,,\,]\big|_{W_1 \times W_j} = 0$, *for all* $j = 2, 3, \cdots, k$.

Proof. Suppose $[\,,\,]\big|_{W_1 \times W_j} \neq 0$. Let $v \in W_1$ and $u \in W_j$ be such that $[v, u] \neq 0$. Let $\phi(w) = \phi_w$ be defined as $\phi_w(v) = [v, w]$. Then clearly ϕ is a non-zero intertwining map between $\pi_1^\vee$ and π_j. Since $\pi_1^\vee \simeq \pi_1$ and the representations π_i are distinct (up to isomorphism), the lemma follows. $\square$

We know that $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2]$, for all $g \in \tilde{G}, v_1, v_2 \in V$. Now if $g \in G$ then $\chi(g) = 1$ and we have $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = [v_1, v_2]$. In particular if $v_1 \in W_1$ and $v_j \in W_j$, then $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_j] = [v_1, v_j]$. Since $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k, \tilde{\pi}(g)v_1 = \pi_1(g)v_1, \tilde{\pi}(g)v_j = \pi_j(g)v_j$.

Lemma 4.15. *With notation as above,* $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$.

Proof. Since $\chi(g) = 1$ for $g \in G$, we have

$$\begin{aligned} [\tilde{\pi}(g)w_1, \tilde{\pi}(g)\dot{w}_1] &= [\pi_1(g)w_1, \pi_1(g)\dot{w}_1] \\ &= [w_1, \dot{w}_1]. \end{aligned}$$

i.e., $[\,,\,]\big|_{W_1 \times W_1}$ is G-invariant. Since $\pi_1 \simeq \pi$, we have $[w_1, \dot{w}_1] = \varepsilon(\pi)(w_1, \dot{w}_1)$. But we also know that $[w_1, \dot{w}_1] = \varepsilon(\tilde{\pi})[\dot{w}_1, w_1]$. The result follows. $\square$

By Lemma 4.15, it follows that the sign $\varepsilon(\pi)$ is completely determined by the sign $\varepsilon(\tilde{\pi})$. Since $\tilde{Z}$ is connected, applying Theorem 4.4 we get an element $s \in \tilde{T}_\circ$ such that $\alpha(s) = -1$ for all simple roots α of $\tilde{T}$. Also $\tilde{\pi}$ is generic, implies that there exists a compact approximation $(\tilde{K}, \psi_{\tilde{K}})$ of $(\tilde{U}, \psi)$ (follows from Theorem 3.5).

We now show that the sign $\varepsilon(\tilde{\pi})$ is controlled by the central character $\omega_{\tilde{\pi}}$ and the character χ. Before we proceed, we record a lemma we need.

Lemma 4.16. *Let V_0, V_1 be irreducible $\tilde{K}$-invariant subspaces of V. Let $\rho_1 = \tilde{\pi}|_{V_0}$ and $\rho_2 = \tilde{\pi}|_{V_1}$. Let $b : V_0 \to V_1^\vee$ be the map $v_0 \mapsto [-, v_0]$. If $b \neq 0$, then $\rho_2 \simeq \rho_1^\vee \chi^{-1}$.*

Proof. It is easy to see that b defines an intertwining map between ρ_2 and $\rho_1^\vee \chi^{-1}$. The result now follows from Schur's Lemma. □

Let V_0 be the space of $\psi_{\tilde{K}}$ and $v_0 \in V_0$. Since $[\,,\,]$ is non-degenerate, $b(v_0)(v_1) = [v_1, v_0] \neq 0$ for some $v_1 \in V_1$, where V_1 is an irreducible $\tilde{K}$-invariant subspace of V. We write ρ for the restriction of $\tilde{\pi}|_{V_1}$. By Lemma 4.16, it follows that $\rho \simeq \psi_{\tilde{K}}^{-1} \chi^{-1}$. Since χ is smooth, we can in fact choose $\tilde{K}$ such that χ is trivial on $\tilde{K}$. It follows that any vector $v_0 \in V_0$ has to pair non-trivially with some vector in the space of $\psi_{\tilde{K}}^{-1}$, i.e., $[\tilde{\pi}(s)v_0, v_0] \neq 0$. We use this in the following theorem.

Theorem 4.17. *Let $(\tilde{\pi}, V)$ be the irreducible representation of $\tilde{G}$ obtained above. Then $\varepsilon(\tilde{\pi}) = \omega_{\tilde{\pi}}(s^2)\chi(s)$.*

Proof. Clearly $s^2 \in \tilde{Z}$. Since $\tilde{\pi}$ is generic, it follows by Theorem 3.5 that there exists a compact open subgroup $\tilde{K}$ and a character $\psi_{\tilde{K}}$ of $\tilde{K}$ such that $\psi_{\tilde{K}}$ occurs with multiplicity one in $\tilde{\pi}|_{\tilde{K}}$. Let V_0 be the space of $\psi_{\tilde{K}}$ and $0 \neq v_0 \in V_0$. Now

$$\begin{aligned}
[\tilde{\pi}(s)v_0, \tilde{\pi}(s^2)v_0] &= \omega_{\tilde{\pi}}(s^2)[\tilde{\pi}(s)v_0, v_0] \\
&= \chi^{-1}(s)[v_0, \tilde{\pi}(s)v_0]
\end{aligned}$$

It follows that $\varepsilon(\tilde{\pi}) = \omega_{\tilde{\pi}}(s^2)\chi(s)$. □

Using $\tilde{\pi}$ has Iwahori fixed vectors and $s^2 \in \tilde{T}_\circ$, we see that $\omega_{\tilde{\pi}}(s^2) = 1$. It will follow that $\varepsilon(\pi) = 1$ once we show that $\chi(s) = 1$. We do this by showing that χ is an unramified character. Before we continue, we recall a result about intertwining maps which we need in the proof.

For H a compact open subgroup of $\tilde{G}$ and ρ an irreducible representation of H, we let $\hat{H}$ denote the set of equivalence classes of irreducible smooth representations of H, $H^g = g^{-1}Hg$, $g \in \tilde{G}$ and ρ^g the irreducible representation of H^g defined as $x \to \rho(gxg^{-1})$.

Remark 4.18. For $i = 1, 2$, let H_i be a compact open subgroup of $\tilde{G}$ and let $\rho_i \in \hat{H}_i$. Let (Π, V) be an irreducible representation of $\tilde{G}$ which contains both ρ_1 and ρ_2. There then exists $g \in \tilde{G}$ such that $\mathrm{Hom}_{H_1^g \cap H_2}(\rho_1^g, \rho_2) \neq 0$.

We refer the reader to [3] (Chapter 3, Section 11, Proposition 1) for a proof of the above remark.

Theorem 4.19. *The character χ is unramified. In particular $\chi(s) = 1$.*

Proof. We know that $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$. Since $\tilde{\pi}$ has non-trivial $\tilde{I}$ fixed vectors it follows that $\tilde{\pi}^\vee$ and hence $\tilde{\pi} \otimes \chi$ has non-trivial $\tilde{I}$ fixed vectors. Therefore $(\tilde{\pi} \otimes \chi)\big|_{\tilde{I}} \supset 1$ and $(\tilde{\pi} \otimes \chi)\big|_{\tilde{I}} \supset \chi$. By the previous remark (Remark 4.18) there exists $g \in \tilde{G}$ such that $\mathrm{Hom}_{\tilde{I}^g \cap \tilde{I}}(1^g, \chi) \neq 0$. Since $\tilde{G} = \coprod_{w \in \widetilde{W}} \tilde{I} w \tilde{I}$ we see that $\mathrm{Hom}_{\tilde{I}^w \cap \tilde{I}}(1^w, \chi) \neq 0$ when $g \in \tilde{I} w \tilde{I}$. From this it follows that $\chi(h) = 1, \forall h \in \tilde{I}^w \cap \tilde{I}$. Since $\tilde{T}_\circ \subset \tilde{I}^w \cap \tilde{I}$ it follows that χ is unramified. $\square$

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[1] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35** (1976), 233–259. MR 0444849 (56 #3196)

[2] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1995, Translated from the German manuscript, Corrected reprint of the 1985 translation. MR 1410059 (97i:22005)

[3] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for GL(2)*, Grundlehren der Mathematischen Wissenschaften, 335 Springer - Verlag, Berlin, 2006.

[4] W. Casselman, *The unramified principal series of $\mathfrak{p}$-adic groups. I. The spherical function*, Compositio Math. **40** (1980), no. 3, 387–406. MR 571057 (83a:22018)

[5] S. S. Gelbart and A. W. Knapp, *L-indistinguishability and R groups for the special linear group*, Adv. in Math. **43** (1982), no. 2, 101–121. MR 644669 (83j:22009)

[6] R. Gow, *Real representations of the finite orthogonal and symplectic groups of odd characteristic*, J. Algebra **96** (1985), no. 1, 249–274. MR 808851 (87b:20015)

[7] C. Mœglin, M.F. Vignéras, and J.L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin, 1987. MR 1041060 (91f:11040)

[8] D. Prasad, *On the self-dual representations of finite groups of Lie type*, J. Algebra **210** (1998), no. 1, 298–310. MR 1656426 (2000a:20025)

[9] ______, *On the self-dual representations of a p-adic group*, Internat. Math. Res. Notices (1999), no. 8, 443–452. MR 1687319 (2000d:22019)

[10] D. Prasad and D. Ramakrishnan, *Self-dual representations of division algebras and Weil groups: a contrast*, Amer. J. Math. **134** (2012), no. 3, 749–767. MR 2931222

[11] F. Rodier, *Modèle de Whittaker et caractères de représentations*, Noncommutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1974), Springer, Berlin, 1975, pp. 151–171. Lecture Notes in Math., Vol. 466. MR 0393355 (52 #14165)

[12] J.P. Serre, *Galois cohomology*, English ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, Translated from the French by Patrick Ion and revised by the author. MR 1867431 (2002i:12004)

[13] C. R. Vinroot, *Involutions acting on representations*, J. Algebra **297** (2006), no. 1, 50–61. MR 2206367 (2006j:20071)