SELF-DUAL REPRESENTATIONS OF $\mathrm{SL}(n, F)$

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Abstract. Let F be a non-Archimedean local field of characteristic 0 and $G = \mathrm{SL}(n, F)$. Let (π, W) be an irreducible smooth self-dual representation G. The space W of π carries a non-degenerate G-invariant bilinear form $(\ ,\ )$ which is unique up to scaling. The form $(\ ,\ )$ is easily seen to be symmetric or skew-symmetric and we set $\varepsilon(\pi) = \pm 1$ accordingly. In this article, we show that $\varepsilon(\pi) = 1$ when π is an Iwahori spherical representation of G.

1. Introduction

Let G be a group and (π, V) be an irreducible complex representation of G. Suppose that $\pi \simeq \pi^\vee$ ($\pi^\vee$ is the dual or contragredient representation). In the presence of Schur's lemma, it is easy to see that there exists a non-degenerate G-invariant bilinear form on V which is unique up to scalars, and consequently is either symmetric or skew-symmetric. Accordingly, we set

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if the form is symmetric,} \\ -1 & \text{if the form is skew-symmetric,} \end{cases}$$

which we call the sign of π. In this paper, we study this sign for a special class of representations of $\mathrm{SL}(n, F)$.

The sign $\varepsilon(\pi)$ has been well studied for connected compact Lie groups and certain classes of finite groups of Lie type. If G is a connected compact Lie group, it is known that the sign can be computed using the dominant weight attached to the representation π (see [3] pg. 261-264). For finite groups of Lie type, computing the sign involves tedious conjugacy class computations. We refer to the following paper of Gow ([5]) where the sign is studied for such groups. In [8], Prasad introduced an elegant idea to compute the sign for a certain class of representations of finite groups of Lie type. He used this idea to determine the sign for many classical groups of Lie type, avoiding difficult computations. In recent times, there has been a lot of interest in studying these signs in the setting of reductive p-adic groups. In [9], Prasad extended the results of [8] to the case of reductive p-adic groups and computed the sign of certain classical groups. The disadvantage of his method is that it works only for representations admitting a Whittaker model. In [11], Roche and Spallone discuss the relation between twisted sign (see section 1 in [11]) and the ordinary sign and describe a way of studying the ordinary sign using the twisted sign. More recently in [10], Prasad and Ramakrishnan have looked at signs of irreducible self-dual discrete series representations of $\mathrm{GL}_n(D)$, for D a finite dimensional p-adic division algebra, and have proved a remarkable formula that relates the signs of these representations and the signs of their Langlands parameters.

In this paper, we compute $\varepsilon(\pi)$ for any irreducible smooth self-dual representation of $\mathrm{SL}(n, F)$ with non-trivial vectors fixed under an Iwahori subgroup I. To be more precise, we prove the following

Theorem 1.1 (Main Theorem)**.** *Let $G = \mathrm{SL}(n, F)$ and (π, W) be an irreducible smooth self-dual representation of G with non-trivial vectors fixed under an Iwahori subgroup I. Then $\varepsilon(\pi) = 1$.*

Using the main result (Theorem 4.1) of [1], it follows that $\varepsilon(\pi) = 1$, if we assume that the representation π generic. The key idea in this paper is to use the results of Roche and Spallone ([11]) and reduce the problem to computing the twisted sign (explained later) of a certain generic representation of a Levi subgroup of G.

The paper is organized as follows. In section 2, we introduce the notion of twisted and ordinary signs attached to the representation π. In section 3, we recall the results which we need in the proof of the main theorem. In the final section (section 4), we prove the main theorem.

2. Some preliminaries on signs

In this section, we briefly discuss the notion of twisted and ordinary signs associated to representations.

Let F be a non-Archimedean local field and G be the group of F-points of a connected reductive algebraic group. Let (π, W) be a smooth irreducible complex representation of G. We write $(\pi^\vee, W^\vee)$ for the smooth dual or contragredient of (π, W) and $\langle\, ,\, \rangle$ for the canonical non-degenerate G-invariant pairing on $W \times W^\vee$ (given by evaluation). Let θ be a continuous automorphism of G of order at most two. Let (π^θ, W) be the θ-twist of π defined by

$$\pi^\theta(g)w = \pi(\theta(g))w.$$

Suppose that $\pi^\theta \simeq \pi^\vee$. Let $s : (\pi^\theta, W) \to (\pi^\vee, W^\vee)$ be an isomorphism. The map s can be used to define a bilinear form on W as follows:

$$(w_1, w_2) = \langle w_1, s(w_2)\rangle, \quad \forall w_1, w_2 \in W.$$

It is easy to see that $(\, ,\,)$ is a non-degenerate form on W that satisfies the following invariance property

$$(\pi(g)w_1, \pi^\theta(g)w_2) = (w_1, w_2), \quad \forall w_1, w_2 \in W. \tag{2.1}$$

Let $(\, ,\,)_*$ be a new bilinear form on W defined by

$$(w_1, w_2)_* = (w_2, w_1)$$

Clearly, this form is again non-degenerate and G-invariant in the sense of (2.1). It follows from Schur's Lemma that

$$(w_1, w_2)_* = c(w_1, w_2)$$

for some non-zero scalar c. A simple computation shows that $c \in \{\pm 1\}$. Indeed,

$$(w_1, w_2) = (w_2, w_1)_* = c(w_2, w_1) = c(w_1, w_2)_* = c^2(w_1, w_2).$$

We set $c = \varepsilon_\theta(\pi)$ and call it the twisted sign of π. It clearly depends only on the equivalence class of π. If θ is the trivial automorphism of G, we simply write $\varepsilon(\pi)$

instead of $\varepsilon_1(\pi)$ and call it the ordinary sign. In sum, the form (,) is symmetric or skew-symmetric and the sign $\varepsilon_\theta(\pi)$ determines its type.

2.1. Let θ be an automorphism of G of order at most 2 and suppose that $\pi^\theta \simeq \pi^\vee$. Consider the automorphism θ' of G defined by

$$\theta' = \mathrm{Int}(h) \circ \theta$$

for some $h \in G$, where $\mathrm{Int}(h)$ denotes the inner automorphism $g \to hgh^{-1}$ of G. In this situation, it is clear that $\pi^{\theta'} \simeq \pi^\vee$. A simple computation shows that

$$\varepsilon_{\theta'}(\pi) = \varepsilon_\theta(\pi)\omega_\pi({}^\theta hh) \tag{2.2}$$

where ω_π is the central character. For specific details of this computation, we refer the reader to 1.2.1 in [11].

3. Some results we need

In this section, we recall some results used in the proof of the main result of this paper.

3.1. Restriction of representations to subgroups. We recall some results about restricting an irreducible representation of $\mathrm{GL}(n, F)$ to $\mathrm{SL}(n, F)$. For a more comprehensive treatment of these results, we refer the reader to [13].

Theorem 3.1 (Tadić)**.** *Let $\tilde{G} = \mathrm{GL}(n, F)$ and $G = \mathrm{SL}(n, F)$. The following are satisfied:*

a) *Let $\tilde{\tau}$ be an irreducible smooth representation of $\tilde{G}$. Then $\tilde{\tau}|_G$ is a finite direct sum of irreducible smooth representations of G, each occurring with multiplicity one. (Theorem 1.2, Lemma 2.1 in* [13]*)*

b) *Given an irreducible smooth representation τ of G, there exists an irreducible smooth representation $\tilde{\tau}$ of $\tilde{G}$ such that τ is isomorphic to a sub representation of $\tilde{\tau}|_G$. (Proposition 2.2 in* [13]*)*

c) *If τ_1 and τ_2 are two irreducible smooth representations of $\tilde{G}$ such that they share an irreducible component τ on restriction to G, then there exists a character χ of $\tilde{G}$ trivial on G such that $\tau_1 \otimes \chi \simeq \tau_2$. (Corollary 2.5 in* [13]*)*

Remark 3.2. Let $\mathcal{P} = (n_1, \ldots, n_k)$ be a partition of n. Take $\tilde{M}$ to be the block diagonal subgroup of $\tilde{G}$ corresponding to the partition $\mathcal{P}$, and M to be the corresponding subgroup in G. The above results also apply in this situation.

3.2. Unramified principal series and representations with Iwahori fixed vectors. We state an important characterization of representations with non-zero vectors fixed under an Iwahori subgroup due to Borel and Casselman. We refer the reader to ([2], [4]) for a proof.

Throughout this section, we let G be the group of F-points of a connected reductive algebraic group defined and split over F. We write T for a maximal F-split torus in G. We also fix a Borel subgroup B defined over F such that $B \supset T$ and write U for the unipotent radical of B. Given a smooth representation (ρ, W) of

T, we write $\operatorname{Ind}_B^G \rho$ for the resulting parabolically induced representation.

Theorem 3.3 (Borel-Casselman)**.** *Let (π, W) be any irreducible smooth representation of G. Then the following assertions are equivalent.*

(i) *There are non-zero vectors in W invariant under I.*

(ii) *There exists some unramified character μ of T such that π imbeds as a sub representation of $\operatorname{Ind}_B^G \mu$.*

3.3. Compact approximation of Whittaker models. We continue with the same notaion as in section 3.2 above. We let ψ denote a non-degenerate character of U. For $\ell \in \mathbb{Z}$, Rodier constructs a sequence (K_ℓ, ψ_ℓ) of compact open subgroups K_ℓ and characters ψ_ℓ of K_ℓ such that the following are satisfied.

(i) K_ℓ converges to U and
(ii) $\psi_\ell|_{K_\ell \cap U} = \psi|_{K_\ell \cap U}$

We refer the reader to ([12], section III, pg. 155) for the construction of (K_ℓ, ψ_ℓ) and a more detailed account of his results.

We fix an integer m large enough and call the pair (K_m, ψ_m) as the compact approximation of (U, ψ). To simplify notation, we write (K, ψ_K) for the compact approximation (K_m, ψ_m). We state an important result of Rodier which we need in the proof of the main theorem.

Theorem 3.4 (Rodier)**.** *Let π be an irreducible smooth representation of G and ψ be a non-degenerate character of U. There then exists a compact open subgroup K of G and a character ψ_K of K such that*

$$dim_{\mathbb{C}} \operatorname{Hom}_K(\pi, \psi_K) = dim_{\mathbb{C}} \operatorname{Hom}_U(\pi, \psi).$$

Therefore, if π is generic, $dim_{\mathbb{C}} \operatorname{Hom}_K(\pi, \psi_K) = 1$.

3.4. Reduction to Tempered case. Throughout this section, we use the same notation and terminology as in [11]. In [11], Roche and Spallone reduce the problem of computing the θ-twisted sign to the case of tempered representations. We briefly recall their method below. For further details, we refer the reader to sections §3, §4 of [11].

Let θ be an involutory automorphism of G and suppose that $\pi^\theta \simeq \pi^\vee$. Let (P, τ, ν) be the triple associated to π via the Langlands' classification. Suppose that P has Levi decomposition $P = MN$. Under certain assumptions on the involution θ, they apply Casselman's pairing to show that $\varepsilon_\theta(\pi) = \varepsilon_\theta(\pi_N)$, where π_N is the Jacquet module of π. Using $\pi^\theta \simeq \pi^\vee$ and the fact that τ occurs with multiplicity one as a composition factor of π_N, they prove the following

Theorem 3.5 (Roche-Spallone)**.** *Let π be an irreducible smooth representation of G such that $\pi^\theta \simeq \pi^\vee$. Suppose the Langlands' classification attaches the triple (P, τ, ν) to π. Then $\tau^\theta \simeq \tau^\vee$ and $\varepsilon_\theta(\pi) = \varepsilon_\theta(\tau)$.*

4. Main Theorem

Throughout this section, we set $G = \mathrm{SL}(n, F)$ and $\tilde{G} = \mathrm{GL}(n, F)$. We write Z (respectively $\tilde{Z}$) for the center of G (respectively $\tilde{G}$), I (respectively $\tilde{I}$) for the Iwahori subgroup in G (respectively $\tilde{G}$) and $\mathfrak{O}$ for the ring of integers in F. We write T for a maximal F-split torus in G and $T(\mathfrak{O})$ for the $\mathfrak{O}$-points of T.

Let w_0 be the element with -1's and 1's alternating on the anti-diagonal and zeros elsewhere. It is clear that $w_0^2 \in Z$. Let 1_G be the trivial automorphism of G. Define an automorphism $\theta' : G \to G$ as $\theta'(g) = \mathrm{Int}(w_0) \circ 1_G$. Since π is self-dual, it is clear that $\pi^{\theta'} \simeq \pi^\vee$. We first observe that

$$\varepsilon_{\theta'}(\pi) = \varepsilon(\pi). \tag{4.1}$$

From (2.2), it is enough to show that $\omega_\pi(w_0^2) = 1$. We record the result in the following

Lemma 4.1. *Let w_0 be as above. Then $\omega_\pi(w_0^2) = 1$.*

Proof. It is clear that $\omega_\pi(w_0^2) = 1$ for n odd. We will prove that $\omega_\pi(w_0^2) = 1$ for even. Since π has non-trivial I fixed vectors, it follows from Theorem3.3 that there exists an unramified character μ of T such that $\pi \hookrightarrow \mathrm{Ind}_B^G \mu$. Let (ρ, E) be an irreducible subrepresentation of $\mathrm{Ind}_B^G \mu$ that is isomorphic to π. Let $x \in Z, f \in E, g \in G$. Clearly,

$$(\rho(x)f)(g) = f(gx) = f(xg) = \mu(x)f(g) \tag{4.2}$$

On the other hand,

$$(\rho(x)f)(g) = \omega_\rho(x)f(g) \tag{4.3}$$

From (4.2) and (4.3) it follows that $\omega_\rho(x) = \mu(x) = \omega_\pi(x)$. Since $w_0^2 = -1 \in T(\mathfrak{O})$ and μ is an unramified character, it follows that $\omega_\pi(w_0^2) = \mu(-1) = 1$.

□

Let (P, τ, ν) be the triple associated to π via the Langlands' classification. We let M and N denote the Levi component and the unipotent radical of the parabolic subgroup P. Before we proceed further, we observe that the automorphism θ' satisfies the hypotheses needed in order to apply *Casselman's pairing* as stated in §3 of [11]. To be more precise, we have

Lemma 4.2. *The involution θ' satisfies the following conditions*

- (i) *θ' is an automorphism of G as an algebraic group.*
- (ii) *θ' preserves T so that $\theta|_T$ is an involutory automorphism of the F-split torus T and*
- (iii) *θ' maps N to the opposite $\bar{N}$.*

Proof. (i) and (ii) are clearly satisfied. For (iii), Since $\pi \hookrightarrow \mathrm{ind}_P^G(\tau\nu)$, it follows (by taking duals) that $\pi^\vee$ is a quotient of $\mathrm{ind}_P^G(\tau^\vee\nu^{-1})$. In other words, we have

$$\pi^\vee \hookrightarrow \mathrm{ind}_{\bar{P}}^G(\tau^\vee\nu^{-1}). \tag{4.4}$$

Since $\pi \simeq \pi^{\theta'} \simeq \pi^\vee$, it follows that

$$\pi^\vee \hookrightarrow \mathrm{ind}_{\theta'(P)}^G(\tau^{\theta'}\nu^{\theta'}). \tag{4.5}$$

From (4.4) and (4.5), and the uniqueness of the Langlands' classification, it follows that $\bar{P} = \theta'(P)$ and $\tau^{\theta'} \simeq \tau^\vee$. In particular, we have $\theta'(N) = \bar{N}$. □

From Lemma 4.2 and Theorem 3.5, it follows that

$$\varepsilon_{\theta'}(\pi) = \varepsilon_{\theta'}(\tau). \tag{4.6}$$

We let $\mathcal{W}$ denote the space of τ. Throughout we write $I_M = I \cap M$. Before we continue, we observe that τ has nontrivial I_M fixed vectors. We record the result in the following lemma:

Lemma 4.3. *The representation τ has non-trivial I_M fixed vectors.*

Proof. Since $\pi \hookrightarrow \mathrm{ind}_P^G(\tau\nu)$ and $\pi^I \neq 0$, it follows that $(\pi_N)^{I_M} \neq 0$. Since $\tau\nu$ occurs as a composition factor of π_N, it follows that $(\tau\nu)^{I_M} \neq 0$ (refer Lemma 4.7 and Lemma 4.8 in [2]). Now using the fact that I_M is compact and ν is a (continuous) character of M taking positive real values, it is clear that $\nu|_{I_M} = 1$ and $\tau^{I_M} \neq 0$. □

Since $(\tau, \mathcal{W})$ is an irreducible tempered representation of M, it is also generic (see [6], [7]). Let $\tilde{M}$ be the corresponding subgroup in $\tilde{G}$ such that $\tilde{M} \cap G = M$. By Theorem 3.1, it follows that there exists an irreducible representation $(\tilde{\tau}, \mathcal{V})$ of $\tilde{M}$ such that $\tilde{\tau}|_M$ is a finite direct sum of irreducible representations (τ_i, W_i) of M, each occurring with multiplicity one and contains the representation τ. To be more precise, we have

$$\tilde{\tau}|_M = \bigoplus_{i=1}^{m} \tau_i$$

where τ_i are distinct irreducible representations of M and $\tau_1 \simeq \tau$. Consider the representation $(\tilde{\tau}^{\theta'})^\vee$. This is again an irreducible representation of $\tilde{M}$ which contains τ with multiplicity one on restriction to M. From Theorem 3.1, it follows that there exists a character χ of $\tilde{M}$ trivial on M such that

$$(\tilde{\tau}^{\theta'})^\vee \simeq \tilde{\tau} \otimes \chi. \tag{4.7}$$

Let $t \in \mathrm{Hom}_{\tilde{M}}((\tilde{\tau}^{\theta'})^\vee, \tilde{\tau} \otimes \chi)$ be an isomorphism. From (4.7), it is easy to see that there is a non-degenerate form $[\,,\,] : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ satisfying

$$[(\tilde{\tau} \otimes \chi)(g)v_1, \tilde{\tau}^{\theta'}(g)v_2] = [v_1, v_2] \tag{4.8}$$

where $[v_1, v_2] = \langle v_1, t(v_2) \rangle$. Indeed, for $g \in \tilde{M}, v_1, v_2 \in \mathcal{V}$, we have

$$\begin{aligned}
[(\tilde{\tau} \otimes \chi)(g)v_1, \tilde{\tau}^{\theta'}(g)v_2] &= \langle (\tilde{\tau} \otimes \chi)(g)v_1, t(\tilde{\tau}(\theta'(g))v_2) \rangle \\
&= t(\tilde{\tau}(\theta'(g))v_2)((\tilde{\tau} \otimes \chi)(g)v_1) \\
&= \chi^{-1}(\theta'(g))(\tilde{\tau}^{\theta'})^\vee(\theta'(g))t(v_2)((\tilde{\tau} \otimes \chi)(g)v_1) \\
&= \chi^{-1}(\theta'(g))t(v_2)(\tilde{\tau}^{\theta'}(\theta'(g^{-1})))((\tilde{\tau} \otimes \chi)(g)v_1) \\
&= \chi^{-1}(\theta'(g))\chi(g)t(v_2)(v_1) \\
&= [v_1, v_2].
\end{aligned}$$

The form $[\,,\,]$ is unique up to scalars and is easily seen to be symmetric or skew-symmetric as before, i.e.,

$$[v_1, v_2] = \varepsilon_{\theta'}(\tilde{\tau})[v_2, v_1]$$

where $\varepsilon_{\theta'}(\tilde{\tau}) \in \{\pm 1\}$.

Let $[\,,\,] : \mathcal{V}\times\mathcal{V} \longrightarrow \mathbb{C}$ be the non-degenerate bilinear form on $\mathcal{V}$(obtained above). Suppose that $[\,,\,]\big|_{W_1\times W_j} = 0$, for all $j = 2, 3, \cdots, k$. Then it is easy to see that $[\,,\,]\big|_{W_1\times W_1}$ is non-degenerate and satisfies the invariance condition

$$[\tilde{\tau}(g)w_1, \tilde{\tau}^{\theta'}(g)w_1'] = [w_1, w_1'] \tag{4.9}$$

where $w_1, w_1' \in W_1, g \in M$. From this it is easy to see that

$$\varepsilon_{\theta'}(\tilde{\tau}) = \varepsilon_{\theta'}(\tau). \tag{4.10}$$

Proof of (4.10) follows from a slight modification of Lemma 4.14, Lemma 4.15 in [1]. We also note that the representation $\tilde{\tau}$ is generic. This again follows from from the simple observation that $\mathrm{Hom}_U(\tilde{\tau}, \psi)$ contains $\mathrm{Hom}_U(\tau, \psi)(\neq 0)$ where ψ is a non-degenerate character of the unipotent radical U of M. Before we proceed further, we note that the representation $\tilde{\tau}$ can be chosen in such a way that it has non-trivial vectors fixed under $\tilde{I}_{\tilde{M}}$. This follows by replacing G, $\tilde{G}$ with M, $\tilde{M}$ in Lemma 4.11 and Theorem 4.15 in [1].

Suppose $\tilde{M} \simeq \mathrm{GL}(n_1, F)\times\cdots\times\mathrm{GL}(n_k, F)$. Write

$$w_0 = \begin{bmatrix} 0 & 0 & \dots & 0 & w_1 \\ 0 & 0 & \dots & w_2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ w_k & 0 & \dots & 0 & 0 \end{bmatrix},$$

where each w_i is an $n_i\times n_i$ matrix for $i = 1, 2, \dots, k$. Let $g = (g_1, \dots, g_k) \in \tilde{M}$. Since $\theta'(\tilde{M}) = \tilde{M}$, it follows that $\theta'(g) = h$ for some $h = (h_1, \dots, h_k) \in \tilde{M}$. In fact, for $1 \le i \le k$, we have $h_i = w_{k+1-i}(g_{k+1-i})w_{k+1-i}^{-1}$. Let η be an automorphism of G defined by $\eta(g) = xgx^{-1}$ where

$$x = \begin{bmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_k \end{bmatrix} \in \tilde{M}.$$

Let α be the automorphism of G given by $\alpha = \eta\circ\theta'$. The automorphism α has the property that it interchanges the blocks g_i and g_{k+1-i} of g for each $1 \le i \le k$. Since η is an inner automorphism and $\theta'(x)x = 1$, we have

$$\varepsilon_{\alpha}(\tilde{\tau}) = \varepsilon_{\theta'}(\tilde{\tau}). \tag{4.11}$$

It is enough to consider the following types of $\tilde{M}$:

$$\tilde{M} = \begin{cases} \mathrm{GL}(p, F)\times\mathrm{GL}(p, F) \\ \mathrm{GL}(p, F)\times\mathrm{GL}(m, F)\times\mathrm{GL}(p, F) \end{cases}$$

We first consider the case when $\tilde{M} \simeq \mathrm{GL}(p, F)\times\mathrm{GL}(p, F)$. Since $(\tilde{\tau}, \mathcal{V})$ is an irreducible generic representation of $\tilde{M}$, we have $\tilde{\tau} \simeq \rho_1\otimes\rho_2$, where $(\rho_1, \mathcal{V}_1)$ and $(\rho_2, \mathcal{V}_2)$ are irreducible generic representations of $\mathrm{GL}(p, F)$. Since $\tilde{\tau} \simeq \rho_1\otimes\rho_2$, we have an isomorphism $\phi : \mathcal{V}_1\otimes\mathcal{V}_2 \to \mathcal{V}$ satisfying

$$\phi(\rho_1(g_1)v_1\otimes\rho_2(g_2)v_2) = \tilde{\tau}(g)(\phi(v_1\otimes v_2)).$$

where $g = (g_1, g_2)$. We use the map ϕ to transfer the form on $\mathcal{V}$ to a form on $\mathcal{V}_1 \otimes \mathcal{V}_2$ in a natural way. To be more precise, we define $[\, ,\,]' : (\mathcal{V}_1 \otimes \mathcal{V}_2) \times (\mathcal{V}_1 \otimes \mathcal{V}_2) \to \mathbb{C}$ as

$$[v_1 \otimes v_2, w_1 \otimes w_2]' := [\phi(v_1 \otimes v_2), \phi(w_1 \otimes w_2)].$$

Replacing the automorphism θ' with α in (4.9), it is easy to see that it can be reformulated in terms of $\rho_1 \otimes \rho_2$ as

$$[v_1 \otimes v_2, w_1 \otimes w_2]' = \chi(g)[\rho_1(g_1)v_1 \otimes \rho_2(g_2)v_2, \rho_1(g_2)w_1 \otimes \rho_2(g_1)w_2]'. \tag{4.12}$$

Indeed, we have

$$\begin{aligned}
[v_1 \otimes v_2, w_1 \otimes w_2]' &= [\phi(v_1 \otimes v_2), \phi(w_1 \otimes w_2)] \\
&= [(\tilde{\tau} \otimes \chi)(g)(\phi(v_1 \otimes v_2)), \tilde{\tau}^{\alpha}(g)(\phi(w_1 \otimes w_2))] \\
&= \chi(g)[\tilde{\tau}(g)(\phi(v_1 \otimes v_2)), \tilde{\tau}^{\alpha}(g)(\phi(w_1 \otimes w_2))] \\
&= \chi(g)[\phi((\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2)), \phi((\rho_1 \otimes \rho_2)(\alpha(g))(w_1 \otimes w_2))] \\
&= \chi(g)[\rho_1(g_1)v_1 \otimes \rho_2(g_2)v_2, \rho_1(g_2)w_1 \otimes \rho_2(g_1)w_2]'
\end{aligned}$$

We know that ρ_1 and ρ_2 are irreducible representations of $\mathrm{GL}(p, F)$. Since the center of $\mathrm{GL}(p, F)$ is connected, it follows from Theorem 4.4 in [1], that there exists an element $s \in T(\mathfrak{O})$ (T is a maximal F-split torus in $\mathrm{GL}(p, F)$) such that $\alpha(s) = -1$ for each simple root α. Using Rodier's compact approximation (see Theorem 3.4) and genericity of the representations ρ_1 and ρ_2, we get pairs (K_m, ψ_m) and (K_n, ψ_n) such that $\rho_1|_{K_m} \supset \psi_m$ and $\rho_2|_{K_n} \supset \psi_n$ with multiplicity one. Choosing v_0, w_0 such that $v_0 \in$ space of ψ_m and $w_0 \in$ space of ψ_n and $s_0 = (s, s)$, it follows that

$$\varepsilon_{\alpha}(\tilde{\tau}) = \chi(s_0)\omega_{\tilde{\tau}}(s_0^2). \tag{4.13}$$

Indeed,

$$\begin{gathered}
\chi(s_0)[\rho_1(s)v_0 \otimes \rho_2(s)w_0, \rho_1(s^2)v_0 \otimes \rho_2(s^2)w_0]' = \\
\chi(s_0)\omega_{\rho_1}(s^2)\omega_{\rho_2}(s^2)[\rho_1(s)v_0 \otimes \rho_2(s)w_0, v_0 \otimes w_0]' = \\
[v_0 \otimes w_0, \rho_1(s)v_0 \otimes \rho_2(s)w_0]'.
\end{gathered}$$

Let $\tilde{I}_{\tilde{M}} = \tilde{I} \cap \tilde{M}$ be the Iwahori subgroup in $\tilde{M}$. Since $\alpha(\tilde{I}_{\tilde{M}}) = \tilde{I}_{\tilde{M}}$ and $\tilde{\tau}^{\tilde{I}_{\tilde{M}}} \neq 0$, it follows that $(\tilde{\tau}^{\alpha})^{\vee}$ has non-trivial $\tilde{I}_{\tilde{M}}$ fixed vectors. Now proceeding in a similar fashion as in Theorem 4.19 in [1], it follows that the character χ is unramified. In particular, $\chi(s_0) = 1$. Since $s_0 \in \tilde{T}(\mathfrak{O})$ and $\tilde{\tau}^{\tilde{I}_{\tilde{M}}} \neq 0$, it follows that $\omega_{\tilde{\tau}}(s_0^2) = 1$. The result follows.

If $\tilde{M} = \mathrm{GL}(p, F) \times \mathrm{GL}(m, F) \times \mathrm{GL}(p, F)$, then $\tilde{\tau} \simeq \rho_1 \otimes \rho_2 \otimes \rho_3$ where ρ_1, ρ_3 are irreducible representations of $\mathrm{GL}(p, F)$ and ρ_2 is an irreducible representation of $\mathrm{GL}(m, F)$. Since the automorphism α interchanges the blocks in a specific way, (in our case, α interchanges the $\mathrm{GL}(p, F)$ blocks and fixes the $\mathrm{GL}(m, F)$ block) choosing $s_0 = (s_1, s_2, s_1) \in \tilde{M}$ with $\alpha(s_1) = \alpha(s_2) = -1$ and proceeding as in the previous case, the result follows.

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