ON A TWISTED JACQUET MODULE OF GL(6) OVER A FINITE FIELD

KUMAR BALASUBRAMANIAN* AND HIMANSHI KHURANA

Abstract. Let F be a finite field and $G = \mathrm{GL}(6, F)$. In this paper, we explicitly describe a certain twisted Jacquet module of an irreducible cuspidal representation of G.

1. Introduction

Let F be a finite field and $G = \mathrm{GL}(n, F)$. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let π be any irreducible finite dimensional complex representation of G and ψ be an irreducible representation of N. Let $\pi_{N,\psi}$ be the sum of all irreducible representations of N inside π, on which π acts via the character ψ. It is easy to see that $\pi_{N,\psi}$ is a representation of the subgroup M_ψ of M, consisting of those elements in M which leave the isomorphism class of ψ invariant under the inner conjugation action of M on N. The space $\pi_{N,\psi}$ is called the *twisted Jacquet module* of the representation π. It is an interesting question to understand for which irreducible representations π, the twisted Jacquet module $\pi_{N,\psi}$ is non-zero and to understand its structure as a module for M_ψ.

In an earlier work of ours [1], inspired by the work of Prasad in [5], we studied the structure of a certain twisted Jacquet module of a cuspidal representation of $\mathrm{GL}(4, F)$. In this paper, we continue our study of the twisted Jacquet module for a cuspidal representation of $\mathrm{GL}(6, F)$. We refer the reader to Section 1 in [1] for a more elaborate introduction and the motivation to study the problem.

Before we state our result, we set up some notation. Let $G = \mathrm{GL}(6, F)$ and P be the maximal parabolic subgroup of G with Levi decomposition $P = MN$, where $M \simeq \mathrm{GL}(3, F) \times \mathrm{GL}(3, F)$ and $N \simeq \mathrm{M}(3, F)$. We write F_6 for the unique field extension of F of degree 6. Let ψ_0 be a fixed non-trivial additive character of F. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\psi_A : N \to \mathbb{C}^\times$ be the character of N given by

$$\psi_A\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) = \psi_0(\mathrm{Tr}(AX)). \tag{1.1}$$

Let $H_A = M_1 \times M_2$ where M_1 is the Mirabolic subgroup of $\mathrm{GL}(3, F)$ and $M_2 = w_0 M_1^\top w_0^{-1}$ where $w_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Let U be the subgroup of unipotent matrices

---

2020 *Mathematics Subject Classification.* Primary: 20G40.
Key words and phrases. Cuspidal representations, Twisted Jacquet module.
* indicates corresponding author.
Research of Kumar Balasubramanian is supported by the SERB grant: MTR/2019/000358.

in $\mathrm{GL}(6, F)$ and $U_A = U \cap H_A$. Clearly, we have $U_A \simeq U_1 \times U_2$ where U_1 and U_2 are the upper triangular unipotent subgroups of $\mathrm{GL}(3, F)$. For $k = 1, 2$, let $\mu_k : U_k \to \mathbb{C}^\times$ be the non-degenerate character of U_k given by

$$\mu_k \left(\begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi_0(x_{12} + x_{23}).$$

Let $\mu : U_A \to \mathbb{C}^\times$ be the character of U_A given by

$$\mu(u) = \mu_1(u_1)\mu_2(u_2)$$

where $u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$.

Theorem 1.1. *Let θ be a regular character of $F_6^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of G. Then*

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times} \otimes \mathrm{ind}_{U_A}^{H_A} \mu$$

as M_{ψ_A} modules.

We establish the above isomorphism by explicitly calculating the characters of π_{N,ψ_A} and $\theta|_{F^\times} \otimes \mathrm{ind}_{U_A}^{H_A}(\mu)$, and showing that they are equal at any arbitrary element of M_{ψ_A}.

The calculation of the twisted Jacquet module for $\mathrm{GL}(4, F)$, did not provide us with much insight to predict the structure of the twisted Jacquet module for $\mathrm{GL}(2n, F)$. This motivated us to study the problem for $\mathrm{GL}(6, F)$ to get a better understanding of its structure. Based on our computations in these particular cases, we formulate the following conjecture for $\mathrm{GL}(2n, F)$.

Let F_n be the unique field extension of F of degree n and let $A = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in$ $\mathrm{M}(n, F)$. We let $H_A = M_1 \times M_2$, $U_A = U_1 \times U_2$ where M_1, M_2, U_1, U_2 are appropriate subgroups of $\mathrm{GL}(n, F)$ as defined earlier.

Conjecture 1.2. *Let θ be a regular character of $F_n^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of G. Then*

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times} \otimes \mathrm{ind}_{U_A}^{H_A} \mu$$

as M_{ψ_A} modules.

The calculations involved in the $\mathrm{GL}(6, F)$ case are much more involved than in the case of $\mathrm{GL}(4, F)$ and we hope that some of these calculations may be useful in the general case. It is also an interesting problem to study the structure of π_{N,ψ_A} in the case when the finite field is replaced with a p-adic field. Our hope is that understanding the problem completely for the finite group case might help in understanding the problem in the p-adic case. We hope to study these problems in future.

2. Preliminaries

In this section, we mention some preliminary results that we need in our paper.

2.1. **Character of a Cuspidal Representation.** Let F be the finite field of order q and $G = \mathrm{GL}(m, F)$. Let F_m be the unique field extension of F of degree m. A character θ of $F_m^\times$ is called a "regular" character, if under the action of the Galois group of F_m over F, θ gives rise to m distinct characters of $F_m^\times$. It is a well known fact that the cuspidal representations of $\mathrm{GL}(m, F)$ are parametrized by the regular characters of $F_m^\times$. To avoid introducing more notation, we mention below only the relevant statements on computing the character values that we have used. We refer the reader to Section 6 in [3] for more precise statements on computing character values.

Theorem 2.1. *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(m, F)$ associated to θ. Let Θ_θ be its character. If $g \in \mathrm{GL}(m, F)$ is such that the characteristic polynomial of g is not a power of a polynomial irreducible over F. Then, we have*

$$\Theta_\theta(g) = 0.$$

Theorem 2.2. *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(m, F)$ associated to θ. Let Θ_θ be its character. Suppose that $g = s.u$ is the Jordan decomposition of an element g in $\mathrm{GL}(m, F)$. If $\Theta_\theta(g) \neq 0$, then the semisimple element s must come from $F_m^\times$. Suppose that s comes from $F_m^\times$. Let z be an eigenvalue of s in F_m and let t be the dimension of the kernel of $g - z$ over F_m. Then*

$$\Theta_\theta(g) = (-1)^{m-1}\Big[\sum_{\alpha=0}^{d-1} \theta(z^{q^\alpha})\Big](1 - q^d)(1 - (q^d)^2)\cdots(1 - (q^d)^{t-1}).$$

where q^d is the cardinality of the field generated by z over F, and the summation is over the distinct Galois conjugates of z.

See Theorem 2 in [5] for this version.

2.2. **Twisted Jacquet Module.** In this section, we recall the character and the dimension formula of the twisted Jacquet module of a representation π.

Let $G = \mathrm{GL}(k, F)$ and $P = MN$ be a parabolic subgroup of G. Let ψ be a character of N. For $m \in M$, let ψ^m be the character of N defined by $\psi^m(n) = \psi(mnm^{-1})$. Let

$$V(N, \psi) = \mathrm{Span}_{\mathbb{C}}\{\pi(n)v - \psi(n)v \mid n \in N, v \in V\}$$

and

$$M_\psi = \{m \in M \mid \psi^m(n) = \psi(n), \forall n \in N\}.$$

Clearly, M_ψ is a subgroup of M and it is easy to see that $V(N, \psi)$ is an M_ψ-invariant subspace of V. Hence, we get a representation $(\pi_{N,\psi}, V/V(N, \psi))$ of M_ψ. We call $(\pi_{N,\psi}, V/V(N, \psi))$ the twisted Jacquet module of π with respect to ψ. We write $\Theta_{N,\psi}$ for the character of $\pi_{N,\psi}$.

Proposition 2.3. *Let (π, V) be a representation of $\mathrm{GL}(k, F)$ and Θ_π be the character of π. We have*

$$\Theta_{N,\psi}(m) = \frac{1}{|N|}\sum_{n\in N} \Theta_\pi(mn)\overline{\psi(n)}.$$

We refer the reader to Proposition 2.3 in [1] for a proof.

Remark 2.4. Taking $m = 1$, we get the dimension of $\pi_{N,\psi}$. To be precise, we have

$$\dim_{\mathbb{C}}(\pi_{N,\psi}) = \frac{1}{|N|}\sum_{n\in N} \Theta_\pi(n)\overline{\psi(n)}.$$

2.3. **Character of the induced representation.** In this section, we recall the character formula for the induced representation of a group G. For a proof, we refer the reader to Chapter 3, Theorem 12 in [6].

Proposition 2.5. *Let G be a finite group and H be a subgroup of G. Let (π, V) be a representation of H and χ_π be the character of π. Then for each $s \in G$, the character of $\mathrm{ind}_H^G(\pi)$ is given by*

$$\chi_{\mathrm{ind}_H^G(\pi)}(s) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}st \in H}} \chi_\pi(t^{-1}st).$$

3. Dimension of the Twisted Jacquet Module

Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of G corresponding to the regular character θ of $F_6^\times$ and Θ_θ be its character. Throughout, we write $\mathrm{M}(n, m, r, q)$ for the set of $n \times m$ matrices of rank r over the finite field $F = F_q$. In this section, we calculate the dimension of π_{N,ψ_A}. Before we continue, we record some preliminary lemmas that we need.

Lemma 3.1. *Let $r \in \{0, 1, 2, 3\}$ and $X \in \mathrm{M}(3, 3, r, q)$. We have*

$$\Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) = \begin{cases} (q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1), & \text{if } r=0 \\ -(q-1)(q^2-1)(q^3-1)(q^4-1), & \text{if } r=1 \\ (q-1)(q^2-1)(q^3-1), & \text{if } r=2 \\ -(q-1)(q^2-1), & \text{if } r=3 \end{cases}$$

Proof. The result follows from Theorem 2.2 above. ∎

Let

$$X = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, AX = \begin{bmatrix} a & d & g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $\alpha \in F$ and $r \in \{0, 1, 2, 3\}$, consider the subset $Y_{3,r}^\alpha$ of $\mathrm{M}(3, F)$ given by

$$Y_{3,r}^\alpha = \{X \in \mathrm{M}(3, F) \mid \mathrm{Rank}(X) = r, \mathrm{Tr}(AX) = \alpha\}.$$

Lemma 3.2. *Let $r \in \{1, 2, 3\}$ and $\alpha, \beta \in F^\times$. Then we have*

$$\#Y_{3,r}^\alpha = \#Y_{3,r}^\beta.$$

Proof. Consider the map $\phi : Y_{3,r}^\alpha \to Y_{3,r}^\beta$ given by

$$\phi(X) = \alpha^{-1}\beta X.$$

Suppose that $\phi(X) = \phi(Y)$. Since $\alpha^{-1}\beta \neq 0$, it follows that ϕ is injective. For $Y \in Y_{3,r}^\beta$, let $X = \alpha\beta^{-1}Y$. Clearly, we have $\mathrm{Tr}(AX) = \alpha$ and $\mathrm{Rank}(X) = \mathrm{Rank}(Y) = r$. Thus ϕ is surjective and hence the result. ∎

Theorem 3.3. *Let θ be a regular character of $F_6^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(6, F)$. We have*

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = (q-1)^2(q^2-1)^2.$$

Proof. It is easy to see that the dimension of π_{N,ψ_A} is given by

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = \frac{1}{q^9} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(\mathrm{Tr}(AX))}. \tag{3.1}$$

We calculate the following sums

a) $S_1 = \sum_{X \in \mathrm{M}(3,3,0,q)} \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(\mathrm{Tr}(AX))}$

b) $S_2 = \sum_{\substack{X \in \mathrm{M}(3,3,1,q) \\ \mathrm{Tr}(AX)=0}} \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(\mathrm{Tr}(AX))} + \sum_{\substack{X \in \mathrm{M}(3,3,1,q) \\ \mathrm{Tr}(AX)=\alpha\neq 0}} \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(\mathrm{Tr}(AX))}$

c) $S_3 = \sum_{\substack{X \in \mathrm{M}(3,3,2,q) \\ \mathrm{Tr}(AX)=0}} \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(\mathrm{Tr}(AX))} + \sum_{\substack{X \in \mathrm{M}(3,3,2,q) \\ \mathrm{Tr}(AX)=\alpha\neq 0}} \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(\mathrm{Tr}(AX))}$

d) $S_4 = \sum_{\substack{X \in \mathrm{M}(3,3,3,q) \\ \mathrm{Tr}(AX)=0}} \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(\mathrm{Tr}(AX))} + \sum_{\substack{X \in \mathrm{M}(3,3,3,q) \\ \mathrm{Tr}(AX)=\alpha\neq 0}} \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(\mathrm{Tr}(AX))}$

separately to compute the dimension of π_{N,ψ_A}.

For a fixed $r \in \{0,1,2,3\}$ and $\alpha \in \{0,1\}$, we find a partition of $Y^{\alpha}_{3,r}$ into certain subsets, and compute the cardinality of each of these subsets to find the cardinality of $Y^{\alpha}_{3,r}$. We record the necessary information in the tables below.

For a), we clearly have

$$\begin{aligned} S_1 &= \Theta_\theta \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) \overline{\psi_0(0)} \\ &= (q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1). \end{aligned}$$

For b), we have

TABLE 1. Rank$(X) = 1$

Partition of $Y^0_{3,1}$	Cardinality	Partition of $Y^1_{3,1}$	Cardinality
$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ b & \lambda b & \beta b \\ c & \lambda c & \beta c \end{bmatrix} \right\}$	$(q^2-1)q^2$	$\left\{ \begin{bmatrix} 1 & \lambda & \beta \\ b & \lambda b & \beta b \\ c & \lambda c & \beta c \end{bmatrix} \right\}$	q^4
$\left\{ \begin{bmatrix} 0 & d & \lambda d \\ 0 & e & \lambda e \\ 0 & f & \lambda f \end{bmatrix} \right\}$	$(q^3-1)q$	-	-
$\left\{ \begin{bmatrix} 0 & 0 & g \\ 0 & 0 & h \\ 0 & 0 & k \end{bmatrix} \right\}$	q^3-1	-	-

Thus,

$$S_2 = \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)\left(\sum_{\substack{X\in \mathrm{M}(3,3,1,q)\\ \mathrm{Tr}(AX)=0}} \overline{\psi_0(0)} \quad + \sum_{\substack{X\in \mathrm{M}(3,3,1,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \overline{\psi_0(\alpha)}\right)$$

$$= \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)\left(\#Y^0_{3,1} - \#Y^1_{3,1}\right)$$

$$= -(q-1)(q^2-1)(q^3-1)(q^4-1)((q^2-1)q^2 + (q^3-1)q + (q^3-1) - q^4)$$

$$= -(q-1)^2(q^2-1)^2(q^8+2q^7+2q^6+q^5-2q^4-3q^3-4q^2-2q-1).$$

For d), we have

TABLE 2. $\mathrm{Rank}(X) = 3$

Partition of $Y^0_{3,3}$	Cardinality	Partition of $Y^1_{3,3}$	Cardinality
$\left\{\begin{bmatrix}0 & d & g\\ b & e & h\\ c & f & k\end{bmatrix}\right\}$	$(q^2-1)(q^3-q)(q^3-q^2)$	$\left\{\begin{bmatrix}1 & d & g\\ b & e & h\\ c & f & k\end{bmatrix}\right\}$	$q^2(q^3-q)(q^3-q^2)$

Thus,

$$S_4 = \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)\left(\sum_{\substack{X\in \mathrm{M}(3,3,3,q)\\ \mathrm{Tr}(AX)=0}} \overline{\psi_0(0)} \quad + \sum_{\substack{X\in \mathrm{M}(3,3,3,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \overline{\psi_0(\alpha)}\right)$$

$$= \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)\left(\#Y^0_{3,3} - \#Y^1_{3,3}\right)$$

$$= -(q-1)(q^2-1)((q^2-1)(q^3-q)(q^3-q^2) - q^2(q^3-q)(q^3-q^2))$$

$$= (q-1)^2(q^2-1)^2q^3.$$

For c), we let $X' = \begin{bmatrix}e & h\\ f & k\end{bmatrix}$. For $\alpha \in \{0,1\}$, we partition the set $Y^\alpha_{3,2}$ according to the rank of X' and count the cardinalities of each of these subsets. For $\mathrm{Rank}(X') \in \{0,1,2\}$ and $\alpha \in \{0,1\}$ and we record the cardinality of such subsets of $Y^\alpha_{3,2}$ in the following tables.

TABLE 3. $\mathrm{Rank}(X) = 2, \mathrm{Rank}(X') = 0$

Partition of $Y^0_{3,2}$	Cardinality	Partition of $Y^1_{3,2}$	Cardinality
$B_1 = \left\{ \begin{bmatrix} 0 & d & g \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \middle\vert (d,g) \neq (0,0) \right\}$	$(q^2-1)^2$	$B_2 = \left\{ \begin{bmatrix} 1 & d & g \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \middle\vert (d,g) \neq (0,0) \right\}$	$(q^2-1)^2$

TABLE 4. $\text{Rank}(X) = 2, \text{Rank}(X') = 2$

Partition of $Y^0_{3,2}$	Cardinality	Partition of $Y^1_{3,2}$	Cardinality
$C_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \lambda e + \beta h & e & h \\ \lambda f + \beta k & f & k \end{bmatrix} \right\}$	$(q^2-1)(q^2-q)(q^2)$	$C_4 = \left\{ \begin{bmatrix} 1 & d & 0 \\ d^{-1}e + \beta h & e & h \\ d^{-1}f + \beta k & f & k \end{bmatrix}, \begin{bmatrix} 1 & 0 & g \\ \lambda e + g^{-1}h & e & h \\ \lambda f + g^{-1}k & f & k \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)^2$
$C_2 = \left\{ \begin{bmatrix} 0 & d & 0 \\ \beta h & e & h \\ \beta k & f & k \end{bmatrix}, \begin{bmatrix} 0 & 0 & g \\ \beta e & e & h \\ \beta f & f & k \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)^2$	$C_5 = \left\{ \begin{bmatrix} 1 & d & g \\ d^{-1}(1-\beta g)e + \beta h & e & h \\ d^{-1}(1-\beta g)f + \beta k & f & k \end{bmatrix} \right\}$	$(q^2-1)(q^3-q^2)(q-1)^2$
$C_3 = \left\{ \begin{bmatrix} 0 & d & g \\ \beta(-gd^{-1}e + h) & e & h \\ \beta(-gd^{-1}f + k) & f & k \end{bmatrix} \right\}$	$(q^2-1)(q^2-q)^2(q-1)$	-	-

TABLE 5. $\text{Rank}(X) = 2, \text{Rank}(X') = 1$

Partition of $Y^0_{3,2}$	Cardinality	Partition of $Y^1_{3,2}$	Cardinality
$E_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ b & e & 0 \\ c & f & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & h \\ c & 0 & k \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)$	$F_1 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ b & 0 & h \\ c & 0 & k \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ b & e & 0 \\ c & f & 0 \end{bmatrix} \right\}$	$2(q^2-1)q^2$
$E_2 = \left\{ \begin{bmatrix} 0 & d & 0 \\ b & e & 0 \\ c & f & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & g \\ b & 0 & h \\ c & 0 & k \end{bmatrix} \right\}$	$2(q^2-1)^2(q-1)$	$F_2 = \left\{ \begin{bmatrix} 1 & d & 0 \\ \beta h & 0 & h \\ \beta k & 0 & k \end{bmatrix}, \begin{bmatrix} 1 & 0 & g \\ \beta e & e & 0 \\ \beta f & f & 0 \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)$
$E_3* = \left\{ \begin{bmatrix} 0 & 0 & g \\ \lambda e & e & 0 \\ \lambda f & f & 0 \end{bmatrix}, \begin{bmatrix} 0 & d & 0 \\ \lambda h & 0 & h \\ \lambda k & 0 & k \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)$	$F_3 = \left\{ \begin{bmatrix} 1 & 0 & g \\ b & 0 & h \\ c & 0 & k \end{bmatrix} \middle\vert (b,c) \neq (g^{-1}h, g^{-1}k) \right\} \cup \left\{ \begin{bmatrix} 1 & d & 0 \\ b & e & 0 \\ c & f & 0 \end{bmatrix} \middle\vert (b,c) \neq (d^{-1}e, d^{-1}f) \right\}$	$2(q^2-1)^2(q-1)$
$E_4 = \left\{ \begin{bmatrix} 0 & d & g \\ -d^{-1}\beta ge & e & 0 \\ -d^{-1}\beta gf & f & 0 \end{bmatrix}, \begin{bmatrix} 0 & d & g \\ -g^{-1}\beta dh & 0 & h \\ -g^{-1}\beta dk & 0 & k \end{bmatrix} \right\}$	$2(q^3-q)(q-1)^2$	$F_4 = \left\{ \begin{bmatrix} 1 & d & g \\ \beta h & 0 & h \\ \beta k & 0 & k \end{bmatrix}, \begin{bmatrix} 1 & d & g \\ \beta e & e & 0 \\ \beta f & f & 0 \end{bmatrix} \right\}$	$2q(q^2-1)(q-1)^2$
$E_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ b & \lambda h & h \\ c & \lambda k & k \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$(q^2-1)(q-1)(q^2-q)$	$F_5 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ b & e & \lambda e \\ c & f & \lambda f \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$(q^2-1)q^2(q-1)$
$E_6 = \left\{ \begin{bmatrix} 0 & d & 0 \\ \beta h & \lambda h & h \\ \beta k & \lambda k & k \end{bmatrix}, \begin{bmatrix} 0 & 0 & g \\ \beta h & \lambda h & h \\ \beta k & \lambda k & k \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$2q(q^2-1)(q-1)^2$	$F_6 = \left\{ \begin{bmatrix} 1 & d & 0 \\ d^{-1}e + \beta\lambda e & e & \lambda e \\ d^{-1}f + \beta\lambda f & f & \lambda f \end{bmatrix}, \begin{bmatrix} 1 & 0 & g \\ \delta e + g^{-1}\lambda e & e & \lambda e \\ \delta f + g^{-1}\lambda f & f & \lambda f \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$2q(q^2-1)(q-1)^2$
$E_7 = \left\{ \begin{bmatrix} 0 & d & g \\ -d^{-1}\beta g\lambda h + \beta h & \lambda h & h \\ -d^{-1}\beta g\lambda k + \beta k & \lambda k & k \end{bmatrix} \mid d \neq \lambda g, \lambda \neq 0 \right\}$	$(q^2-1)(q-1)^2(q^2-2q)$	$F_7 = \left\{ \begin{bmatrix} 1 & d & g \\ d^{-1}(1-\beta g)e + \beta\lambda e & e & \lambda e \\ d^{-1}(1-\beta g)f + \beta\lambda f & f & \lambda f \end{bmatrix} \middle\vert g \neq \lambda d, \lambda \neq 0 \right\}$	$(q^2-1)(q-1)^2(q^2-2q)$
$E_8 = \left\{ \begin{bmatrix} 0 & \lambda g & g \\ b & \lambda h & h \\ c & \lambda k & k \end{bmatrix} \lambda \neq 0 \right\}$	$(q^2-1)^2(q-1)^2$	$F_8 = \left\{ \begin{bmatrix} 1 & d & \lambda d \\ b & e & \lambda e \\ c & f & \lambda f \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$(q^2-1)^2(q-1)^2$

We have,

$$Y^0_{3,2} = B_1 \bigsqcup_{i=1}^{3} C_i \bigsqcup_{j=1}^{8} E_j$$

and

$$Y^1_{3,2} = B_2 \bigsqcup_{i=4}^{5} C_i \bigsqcup_{j=1}^{8} F_j.$$

Thus,

$$S_3 = \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\left(\sum_{\substack{X\in \mathrm{M}(3,3,2,q)\\ \mathrm{Tr}(AX)=0}} \overline{\psi_0(0)} + \sum_{\substack{X\in \mathrm{M}(3,3,2,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \overline{\psi_0(\alpha)}\right)$$

$$= \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)(\#Y^0_{3,2} - \#Y^1_{3,2})$$

$$= (q-1)(q^2-1)(q^3-1)(q^6-q^5-2q^4+q^2+q)$$

$$= (q-1)^2(q^2-1)^2(q^6-q^4-3q^3-2q^2-q).$$

From (3.1), it follows that

$$\begin{aligned}\dim_{\mathbb{C}}(\pi_{N,\psi_A}) &= \frac{1}{q^9}\{S_1+S_2+S_3+S_{4]}\}\\ &= \frac{1}{q^9}(q-1)^2(q^2-1)^2q^9\\ &= (q-1)^2(q^2-1)^2.\end{aligned}$$

□

Remark 3.4. Suppose that $B = Aw_0$. It is easy to see that $\Theta_{N,\psi_A}\left(\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\right) = \Theta_{N,\psi_B}\left(\begin{bmatrix}w_0m_1w_0 & 0\\ 0 & m_2\end{bmatrix}\right)$. Thus we have that $\dim(\pi_{N,\psi_A}) = \dim(\pi_{N,\psi_B})$.

4. Main Theorem

In this section, we prove the main result of this paper. Hereafter, we take

$$A = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

For the sake of completeness, we recall the statement below.

Theorem 4.1. *Let θ be a regular character of $F_6^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of G. Then*

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times} \otimes \mathrm{ind}_{U_A}^{H_A}\mu$$

as M_{ψ_A} modules.

The key idea of the proof is to compute the characters of the representations $\rho = \theta|_{F^\times} \otimes \mathrm{ind}_{U_A}^{H_A}\mu$ and π_{N,ψ_A} and show that they are equal at any arbitrary element in M_{ψ_A}. Before we continue, we set up some notation and record a few lemmas that we need.

Lemma 4.2. *Let $M_{\psi_A} = \{m \in M \mid \psi_A^m(n) = \psi_A(n), \forall n \in N\}$. Then we have*

$$M_{\psi_A} = \left\{\begin{bmatrix}a_{11} & a_{12} & a_{13} & & & \\ a_{21} & a_{22} & a_{23} & & & \\ 0 & 0 & a & & & \\ & & & a & y_{12} & y_{13}\\ & & & 0 & y_{22} & y_{23}\\ & & & 0 & y_{32} & y_{33}\end{bmatrix} \mid a \in F^\times\right\}.$$

Proof. Let $g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in M$. Then $g \in M_{\psi_A}$ if and only if $Ag_1 = g_2 A$. It follows that $g \in M_{\psi_A}$ if and only if $g_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a \end{bmatrix}$ and $g_2 = \begin{bmatrix} a & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix}$. ☐

Lemma 4.3. *Let $Z = Z(G)$ be the center of G. Then, we have*

$$M_{\psi_A} \simeq Z \times H_A.$$

Proof. Trivial. ☐

4.1. **Character computation for ρ.** Let $\rho_1 = \operatorname{ind}_{U_1}^{M_1} \mu_1$ and $\rho_2 = \operatorname{ind}_{U_2}^{M_2} \mu_2$. In this section, we calculate the character of the representation

$$\rho = \theta|_{F^\times} \otimes \operatorname{ind}_{U_A}^{H_A} \mu \simeq \theta|_{F^\times} \otimes (\rho_1 \otimes \rho_2).$$

4.1.1. *Character computation of ρ_1.* Let μ_1 be same as above. Consider the representation

$$\rho_1 = \operatorname{ind}_{U_1}^{M_1} \mu_1$$

of M_1. Let χ_{ρ_1} be the character of ρ_1. Let

$$S_1 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a, b \in F^\times \right\}$$

and

$$S_2 = \left\{ \begin{bmatrix} p & q & 0 \\ r & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid p \in F, q, r \in F^\times \right\}.$$

It is easy to see that $S = S_1 \cup S_2$ is a set of left coset representatives of U_1 in M_1.

Lemma 4.4. *Let*

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \in M_1, t = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S_1$$

Then, $t^{-1}mt \in U_1$ if and only if $a_{11} = a_{22} = 1$ and $a_{21} = 0$. In particular, for $m \in M_1$ with $a_{11} = a_{22} = 1$ and $a_{21} = 0$, we have

$$\sum_{t \in S_1} \mu_1(t^{-1}mt) = \sum_{a,b \in F^\times} \psi_0(a^{-1}ba_{12} + b^{-1}a_{23}).$$

Proof. For $m \in M_1$ and $t \in S_1$, we have

$$t^{-1}mt = \begin{bmatrix} a_{11} & a^{-1}ba_{12} & a^{-1}a_{13} \\ b^{-1}aa_{21} & a_{22} & b^{-1}a_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus it follows that $t^{-1}mt \in U_1$ if and only if $a_{11} = a_{22} = 1$ and $a_{21} = 0$. Clearly, we have

$$\sum_{t \in S_1} \mu_1(t^{-1}mt) = \sum_{a,b \in F^\times} \psi_0(a^{-1}ba_{12} + b^{-1}a_{23}).$$

Hence the result. ☐

Lemma 4.5. *Let*

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \in M_1 \text{ and } t = \begin{bmatrix} p & q & 0 \\ r & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S_2.$$

Suppose that $a_{21} = 0$. *Then,*

$$t^{-1}mt \in U_1 \text{ if and only if } a_{11} = a_{22} = 1 \text{ and } a_{12} = 0.$$

In particular, for $m \in M_1$ *with* $a_{11} = a_{22} = 1$ *and* $a_{21} = a_{12} = 0$, *we have*

$$\sum_{t \in S_2} \mu_1(t^{-1}mt) = \sum_{\substack{p \in F \\ r,q \in F^\times}} \psi_0(-pq^{-1}r^{-1}a_{23} + q^{-1}a_{13}).$$

Proof. Let $m \in M_1$ and $t \in S_2$. If $a_{21} = 0$, we have,

$$t^{-1}mt = \begin{bmatrix} a_{22} & 0 & r^{-1}a_{23} \\ pq^{-1}(a_{11} - a_{22}) + rq^{-1}a_{12} & a_{11} & q^{-1}a_{13} - pq^{-1}r^{-1}a_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, $t^{-1}mt \in U_1$ if and only if $a_{22} = a_{11} = 1$ and $a_{12} = 0$. Clearly, we have

$$\sum_{t \in S_2} \mu_1(t^{-1}mt) = \sum_{\substack{p \in F \\ r,q \in F^\times}} \psi_0(-pq^{-1}r^{-1}a_{23} + q^{-1}a_{13}).$$

Hence the result. □

Lemma 4.6. *Let*

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \in M_1 \text{ and } t = \begin{bmatrix} p & q & 0 \\ r & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S_2.$$

Suppose that $a_{21} \neq 0$.

a) *If* $p = 0$, *then* $t^{-1}mt \in U_1$ *if and only if* $a_{11} = a_{22} = 1$ *and* $a_{12} = 0$. *In particular, we have*

$$\sum_{t \in S_2} \mu_1(t^{-1}mt) = \sum_{r,q \in F^\times} \psi_0(qr^{-1}a_{21} + q^{-1}a_{13}).$$

b) *If* $p \neq 0$, *then* $t^{-1}mt \in U_1$ *if and only if* $a_{11} + a_{22} = 2$, $a_{12} = \frac{-(a_{11}-1)^2}{a_{21}}$ *and* $r = \left(\frac{pa_{21}}{a_{11}-1}\right)$. *In particular, we have*

$$\sum_{t \in S_2} \mu_1(t^{-1}mt) = -\sum_{q \in F^\times} \psi_0(q^{-1}(-\delta + a_{13})),$$

where $\delta = a_{21}^{-1}a_{23}(a_{11} - 1)$.

Proof. Let

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \in M_1$$

and suppose that $a_{21} \neq 0$. In case a), since $p = 0$, we have

$$t^{-1}mt = \begin{bmatrix} a_{22} & qr^{-1}a_{21} & r^{-1}a_{23} \\ rq^{-1}a_{12} & a_{11} & q^{-1}a_{13} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus it follows that $t^{-1}mt \in U_1$ if and only if $a_{22} = a_{11} = 1$ and $a_{12} = 0$. In particular, we have

$$\sum_{t \in S_2} \mu_1(t^{-1}mt) = \sum_{r,q \in F^\times} \psi_0(qr^{-1}a_{21} + q^{-1}a_{13}).$$

In case b), since $p \neq 0$, we have

$$t^{-1}mt = \begin{bmatrix} a_{22} + pr^{-1}a_{21} & qr^{-1}a_{21} & r^{-1}a_{23} \\ pq^{-1}(a_{11} - a_{22}) + rq^{-1}a_{12} - p^2q^{-1}r^{-1}a_{21} & a_{11} - pr^{-1}a_{21} & q^{-1}a_{13} - pq^{-1}r^{-1}a_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, we have $t^{-1}mt \in U_1$ if and only if $a_{11} = 1 + pr^{-1}a_{21}$, $a_{22} = 1 - pr^{-1}a_{21}$ and $pq^{-1}(a_{11} - a_{22}) + rq^{-1}a_{12} - p^2q^{-1}r^{-1}a_{21} = 0$. Using $a_{11} - a_{22} = 2pr^{-1}a_{21}$, $a_{11} + a_{22} = 2$ and $\det(t^{-1}mt) = 1$, it follows that $a_{12} = \frac{-(a_{11}-1)^2}{a_{21}}$ and $r = \left(\frac{pa_{21}}{a_{11}-1}\right)$. In particular, taking $\delta = a_{21}^{-1}a_{23}(a_{11} - 1)$ we have

$$\begin{aligned} \sum_{t \in S_2} \mu_1(t^{-1}mt) &= -\sum_{q \in F^\times} \psi_0(q^{-1}(-{a_{21}}^{-1}a_{23}(a_{11} - 1) + a_{13})) \\ &= -\sum_{q \in F^\times} \psi_0(q^{-1}(-\delta + a_{13})). \end{aligned}$$

Hence the result. □

We summarize the character values of ρ_1 in the table below.

TABLE 6. Character of ρ_1

Type of m	m	$\chi_{\rho_1}(m)$	Type of m	m	$\chi_{\rho_1}(m)$
Type-1	$\left\{\begin{bmatrix}1 & a_{12} & a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, a_{12} \in F^\times\right\}$	$(1-q)$	Type-6	$\left\{\begin{bmatrix}1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, a_{13}, a_{23} \in F^\times\right\}$	$(1-q)$
Type-2	$\left\{\begin{bmatrix}1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, a_{12}, a_{23} \in F^\times\right\}$	1	Type-7	$\left\{\begin{bmatrix}1 & 0 & 0 \\ a_{21} & 1 & a_{23} \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, a_{21} \in F^\times\right\}$	$(1-q)$
Type-3	$\left\{\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & a_{23} \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, a_{23} \in F^\times\right\}$	$(1-q)$	Type-8	$\left\{\begin{bmatrix}1 & 0 & a_{13} \\ a_{21} & 1 & a_{23} \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, a_{21}, a_{13} \in F^\times\right\}$	1
Type-4	$\left\{\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\right\}$	$(1-q)(1-q^2)$	Type-9	$\left\{\begin{bmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, \overset{a_{21} \in F^\times,\, a_{13} = \delta}{a_{12} = -{a_{21}}^{-1}(a_{11}-1)^2}\right\}$	$(1-q)$
Type-5	$\left\{\begin{bmatrix}1 & 0 & a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, a_{13} \in F^\times\right\}$	$(1-q)$	Type-10	$\left\{\begin{bmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1\end{bmatrix} \middle\vert\, \overset{a_{21} \in F^\times,\, a_{13} \neq \delta}{a_{12} = -{a_{21}}^{-1}(a_{11}-1)^2}\right\}$	1

If $m \in M_1$ is not one of the types mentioned in Table 6, then $\chi_{\rho_1}(m) = 0$.

4.1.2. *Character computation of* ρ_2. Let μ_2 be same as above. Consider the representation

$$\rho_2 = \operatorname{ind}_{U_2}^{M_2} \mu_2$$

of M_2. Let χ_{ρ_2} be the character of ρ_2. Let

$$S_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \mid a, b \in F^\times \right\}$$

and

$$S_4 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & q \\ 0 & r & 0 \end{bmatrix} \mid p \in F, q, r \in F^\times \right\}.$$

It is easy to see that $S = S_3 \cup S_4$ is a set of left coset representatives of U_2 in M_2.

Lemma 4.7. *Let*

$$m = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix} \in M_2, t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \in S_3.$$

Then, $t^{-1}mt \in U_2$ if and only if $y_{22} = y_{33} = 1$ and $y_{32} = 0$. In particular, for $m \in M_2$ with $y_{22} = y_{33} = 1$ and $y_{32} = 0$, we have

$$\sum_{t \in S_3} \mu_2(t^{-1}mt) = \sum_{a,b \in F^\times} \psi_0(ay_{12} + ba^{-1}y_{23}).$$

Proof. Similar to Lemma 4.4. □

Lemma 4.8. *Let*

$$m = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix} \in M_2 \text{ and } t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & q \\ 0 & r & 0 \end{bmatrix} \in S_4.$$

Suppose that $y_{32} = 0$. Then,

$$t^{-1}mt \in U_2 \text{ if and only if } y_{22} = y_{33} = 1 \text{ and } y_{23} = 0.$$

In particular, for $m \in M_2$ with $y_{22} = y_{33} = 1$ and $y_{32} = y_{23} = 0$, we have

$$\sum_{t \in S_4} \mu_2(t^{-1}mt) = \sum_{\substack{p \in F \\ r,q \in F^\times}} \psi_0(py_{12} + ry_{13}).$$

Proof. Similar to Lemma 4.5. □

Lemma 4.9. *Let*

$$m = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix} \in M_2 \text{ and } t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & q \\ 0 & r & 0 \end{bmatrix} \in S_4.$$

Suppose that $y_{32} \neq 0$.

a) *If $p = 0$, then $t^{-1}mt \in U_2$ if and only if $y_{22} = y_{33} = 1$ and $y_{23} = 0$. In particular, we have*

$$\sum_{t \in S_4} \mu_2(t^{-1}mt) = \sum_{r,q \in F^\times} \psi_0(qr^{-1}y_{32} + ry_{13}).$$

b) *If $p \neq 0$, then $t^{-1}mt \in U_2$ if and only if $y_{22} + y_{33} = 2$, $y_{23} = -\frac{(y_{22}-1)^2}{y_{32}}$ and $r = \left(\frac{y_{32}p}{y_{22}-1}\right)$. In particular, we have*

$$\sum_{t \in S_4} \mu_2(t^{-1}mt) = -\sum_{p \in F^\times} \psi_0(p(\gamma + y_{12})),$$

where $\gamma = y_{32}y_{13}(y_{22} - 1)^{-1}$.

Proof. Similar to Lemma 4.6. □

We record the character values of ρ_2 in the following table.

Table 7. Character of ρ_2

Type of m	m	$\chi_{\rho_2}(m)$	Type of m	m	$\chi_{\rho_2}(m)$
Type-1	$\left\{\begin{bmatrix}1 & 0 & y_{13}\\ 0 & 1 & y_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert\, y_{23} \in F^\times\right\}$	$(1-q)$	Type-6	$\left\{\begin{bmatrix}1 & y_{12} & y_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \middle\vert\, y_{12}, y_{13} \in F^\times\right\}$	$(1-q)$
Type-2	$\left\{\begin{bmatrix}1 & y_{12} & y_{13}\\ 0 & 1 & y_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert\, y_{12}, y_{23} \in F^\times\right\}$	1	Type-7	$\left\{\begin{bmatrix}1 & y_{12} & 0\\ 0 & 1 & 0\\ 0 & y_{32} & 1\end{bmatrix} \middle\vert\, y_{32} \in F^\times\right\}$	$(1-q)$
Type-3	$\left\{\begin{bmatrix}1 & y_{12} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \middle\vert\, y_{12} \in F^\times\right\}$	$(1-q)$	Type-8	$\left\{\begin{bmatrix}1 & y_{12} & y_{13}\\ 0 & 1 & 0\\ 0 & y_{32} & 1\end{bmatrix} \middle\vert\, y_{13}, y_{32} \in F^\times\right\}$	1
Type-4	$\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}$	$(1-q)(1-q^2)$	Type-9	$\left\{\begin{bmatrix}1 & y_{12} & y_{13}\\ 0 & y_{22} & y_{23}\\ 0 & y_{32} & y_{33}\end{bmatrix} \middle\vert\, \begin{matrix} y_{32} \in F^\times,\, y_{12} = -\gamma, \\ y_{23} = -{y_{32}}^{-1}(y_{22}-1)^2 \end{matrix}\right\}$	$(1-q)$
Type-5	$\left\{\begin{bmatrix}1 & 0 & y_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \middle\vert\, y_{13} \in F^\times\right\}$	$(1-q)$	Type-10	$\left\{\begin{bmatrix}1 & y_{12} & y_{13}\\ 0 & y_{22} & y_{23}\\ 0 & y_{32} & y_{33}\end{bmatrix} \middle\vert\, \begin{matrix} y_{32} \in F^\times,\, y_{12} \neq -\gamma, \\ y_{23} = -{y_{32}}^{-1}(y_{22}-1)^2 \end{matrix}\right\}$	1

If $m \in M_2$ is not one of the types mentioned above, we have $\chi_{\rho_2}(m) = 0$.

For $1 \leq i, j \leq 10$, we let

$$T(i,j) = \{k = (m_1, m_2) \in H_A \mid m_1 \in \text{Type}-i, m_2 \in \text{Type}-j\}.$$

Theorem 4.10. *Let $\rho = \theta|_{F^\times} \otimes \rho_1 \otimes \rho_2$. Let χ_ρ be the character of ρ. For $m = (a, m_1, m_2) \in Z \times M_1 \times M_2$, we have*

$$\chi_\rho(m) = \theta(a)\chi_{\rho_1}(m_1)\chi_{\rho_2}(m_2)$$

where $(m_1, m_2) \in T(i,j)$, $i, j \in \{1, \ldots, 10\}$. Otherwise, $\chi_\rho(m) = 0$.

Proof. We summarize the results from Table (6) and Table (7) below.

Table 8. Character of ρ

	Type-1	Type-2	Type-3	Type-4	Type-5	Type-6	Type-7	Type-8	Type-9	Type-10
Type-1	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-2	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)$
Type-3	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-4	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)(1-q^2)$
Type-5	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-6	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-7	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-8	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)$
Type-9	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-10	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)$

□

5. Character calculation for π_{N,ψ_A}

Lemma 5.1. *Let $m = ah \in M_{\psi_A}$, where $a \in Z$ and $h \in H_A$. Then,*

$$\Theta_{N,\psi_A}(m) = \theta(a)\Theta_{N,\psi_A}(h).$$

Proof. We have

$$\begin{aligned}\Theta_{N,\psi_A}(m) &= \Theta_{N,\psi_A}(ah)\\ &= \frac{1}{|N|}\sum_{n\in N}\Theta_\theta(ahn)\overline{\psi_A(n)}\\ &= \frac{1}{|N|}\sum_{n\in N}\mathrm{Tr}(\pi(ahn)\overline{\psi_A(n)}\\ &= \frac{1}{|N|}\sum_{n\in N}\mathrm{Tr}(\pi(a)\pi(hn)\overline{\psi_A(n)}\\ &= \omega_\pi(a)\frac{1}{|N|}\sum_{n\in N}\mathrm{Tr}(\pi(hn)\overline{\psi_A(n)}\\ &= \omega_\pi(a)\Theta_{N,\psi_A}(h)\end{aligned}$$

where ω_π is the central character of π. Explicitly, we have

$$\Theta_\theta(a) = \mathrm{Tr}(\pi(a)) = \mathrm{Tr}(\omega_\pi(a)) = \omega_\pi(a)\dim(\pi).$$

Using Theorem 2.2, it is easy to see that

$$\Theta_\theta(a) = \theta(a)\dim(\pi).$$

Thus, we have $\omega_\pi(a) = \theta(a)$ and the result follows. □

Lemma 5.2. *Let $\tau = \begin{bmatrix} 0 & w_0 \\ w_0 & 0 \end{bmatrix}$. For $1 \le i,j \le 10$, we have*

$$T(j,i) = \tau T(i,j)^\top \tau^{-1}.$$

Proof. Trivial. □

Theorem 5.3. *Let $m' \in T(j,i)$. Then there exists $m \in T(i,j)$ such that*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $m' = \begin{bmatrix} m_1' & 0 \\ 0 & m_2' \end{bmatrix} \in T(j,i)$. Since $T(j,i) = \tau T(i,j)^\top\tau^{-1}$, it follows that,

$$m' = \tau m^\top \tau^{-1}$$

for some $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,j)$. Thus we have $m_1' = w_0 {m_2}^\top {w_0}^{-1}$ and $m_2' = w_0 {m_1}^\top {w_0}^{-1}$. Since $m_1' \in M_1$, clearly $\psi_A(X) = \psi_A((w_0 {m_2}^\top w_0)^{-1}X)$. We have

$$\begin{aligned}\Theta_{N,\psi_A}(m') &= \frac{1}{|N|}\sum_{X\in \mathrm{M}(3,F)}\Theta_\theta\begin{bmatrix} w_0 {m_2}^\top {w_0}^{-1} & X \\ 0 & w_0 {m_1}^\top {w_0}^{-1}\end{bmatrix}\overline{\psi_A(X)}\\ &= \frac{1}{|N|}\sum_{X\in \mathrm{M}(3,F)}\Theta_\theta\left(\begin{bmatrix} w_0 & 0 \\ 0 & w_0\end{bmatrix}\begin{bmatrix} {m_2}^\top & {w_0}^{-1}Xw_0 \\ 0 & {m_1}^\top\end{bmatrix}\begin{bmatrix} w_0 & 0 \\ 0 & w_0\end{bmatrix}\right)\overline{\psi_A(X)}\\ &= \frac{1}{|N|}\sum_{X\in \mathrm{M}(3,F)}\Theta_\theta\left(\begin{bmatrix} {m_2}^\top & {w_0}^{-1}Xw_0 \\ 0 & {m_1}^\top\end{bmatrix}\right)\overline{\psi_A(X)}.\end{aligned}$$

On the other hand, using $\mathrm{Tr}(A({w_0}^{-1}X^\top w_0)) = \mathrm{Tr}(AX)$ we have

$$\begin{aligned}
\Theta_{N,\psi_A}(m) &= \frac{1}{|N|}\sum_{X\in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X)} \\
&= \frac{1}{|N|}\sum_{X\in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X^\top \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X^\top)} \\
&= \frac{1}{|N|}\sum_{X\in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} m_1 & {w_0}^{-1}X^\top w_0 \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({w_0}^{-1}X^\top w_0)}. \\
&= \frac{1}{|N|}\sum_{X\in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} m_1 & {w_0}^{-1}X^\top w_0 \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X)}.
\end{aligned}$$

Since

$$\begin{aligned}
\mathrm{Rank}\left(\begin{bmatrix} {m_2}^\top - 1 & w_0^{-1}Xw_0 \\ 0 & {m_1}^\top - 1 \end{bmatrix}\right) &= \mathrm{Rank}\left(\begin{bmatrix} m_2 - 1 & 0 \\ w_0^{-1}X^\top w_0 & m_1 - 1 \end{bmatrix}\right) \\
&= \mathrm{Rank}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} m_2 - 1 & 0 \\ w_0^{-1}X^\top w_0 & m_1 - 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \\
&= \mathrm{Rank}\left(\begin{bmatrix} m_1 - 1 & w_0^{-1}X^\top w_0 \\ 0 & m_2 - 1 \end{bmatrix}\right)
\end{aligned}$$

we have,

$$\dim(\ker\left(\begin{bmatrix} {m_2}^\top - 1 & w_0^{-1}Xw_0 \\ 0 & {m_1}^\top - 1 \end{bmatrix}\right) = \dim(\ker\left(\begin{bmatrix} m_1 - 1 & w_0^{-1}X^\top w_0 \\ 0 & m_2 - 1 \end{bmatrix}\right).$$

Hence,

$$\Theta_\theta \begin{bmatrix} {m_2}^\top & w_0^{-1}Xw_0 \\ 0 & {m_1}^\top \end{bmatrix} = \Theta_\theta \begin{bmatrix} m_1 & w_0^{-1}X^\top w_0 \\ 0 & m_2 \end{bmatrix}$$

and the result follows. □

Remark 5.4. We have used the fact that $\mathrm{Rank}(M) = \mathrm{Rank}(M^\top)$ and $\mathrm{Rank}(NMP) = \mathrm{Rank}(M)$ if N and P are invertible matrices.

Let $m = (m_1, m_2) \in M_1 \times M_2 = H_A$. Suppose also that m_1, m_2 are unipotent. To calculate $\Theta_{N,\psi_A}(m)$, we need to compute $\Theta_\theta(h)$, where $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$. Using Theorem 2.2, it suffices to compute $\dim \mathrm{Ker}(h-1)$. We note that the following proposition is valid even when H_A is a subgroup of $\mathrm{GL}(2n, F)$.

Proposition 5.5. *Let* $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \in \mathrm{GL}(2n, F)$*, where* $(m_1, m_2) \in H_A$*. Suppose that* m_1 *and* m_2 *are also unipotent. Let* $W' = \mathrm{Ker}(m_2 - 1)$*. Then, we have*

$\dim \mathrm{Ker}(h-1) = \dim \mathrm{Ker}(m_1-1) + \dim \mathrm{Ker}(m_2-1) - \dim(XW') + \dim\{XW' \cap \mathrm{Im}(m_1-1)\}.$

Proof. Let V be an n-dimensional vector space over F and let m_1, m_2, X be linear operators on V. Suppose that $\{e_1, \ldots, e_m\}$ is a basis for $\mathrm{Ker}(m_1 - 1)$ and $\{f_1, \ldots, f_k\}$ is a basis for $\mathrm{Ker}(m_2 - 1)$. Extending the basis of $\mathrm{Ker}(m_1 - 1)$ and $\mathrm{Ker}(m_2 - 1)$ we get ordered bases $\beta = \{e_1, \ldots, e_n\}$ and $\beta' = \{f_1, \ldots, f_n\}$ of V. Consider the ordered basis $\tilde{\beta} = \{(e_1, 0), \ldots, (e_n, 0), (0, f_1), \ldots, (0, f_n)\}$ of $V \oplus V$. We let h to be the linear operator on $V \oplus V$ defined as follows. For $1 \le i, j \le n$,

$$h((e_i, 0)) = (m_1, 0)(e_i, 0) = (m_1(e_i), 0)$$

and

$$h((0, f_j)) = (X, m_2)(0, f_j) = (X(f_j), m_2(f_j)).$$

Then,

$$[h]_{\tilde{\beta}} = \begin{bmatrix} [m_1]_\beta & [X]_{\beta'} \\ 0 & [m_2]_{\beta'} \end{bmatrix}$$

where

$$[X]_{\beta'} = [X_1 \ X_2 \ \cdots \ X_n].$$

Let

$$W_1 = \mathrm{Span}\{(m_1 - 1, 0)(e_{m+1}, 0), \ldots, (m_1 - 1, 0)(e_n, 0)\} = \mathrm{Im}(m_1 - 1),$$
$$W_2 = \mathrm{Span}\{(X, m_2 - 1)(0, f_1), \ldots, (X, m_2 - 1)(0, f_k)\} = XW'$$

and

$$W_3 = \mathrm{Span}\{(Xf_{k+1}, (m_2 - 1)f_{k+1}), \ldots, (Xf_n, (m_2 - 1)f_n).$$

Clearly,

$$\mathrm{Im}(h - 1) = W_1 + W_2 + W_3.$$

It is easy to see that

$$W_2 \cap W_3 = \{0\} = W_1 \cap W_3.$$

Since $\dim(\mathrm{Ker}(m_2 - 1)) = k$, we have that

$$\dim(W_3) = \dim(\mathrm{Im}(m_2 - 1)).$$

Therefore,

$$\dim(\mathrm{Im}(h-1)) = \dim(\mathrm{Im}(m_1-1)) + \dim(\mathrm{Im}(m_2-1)) + \dim(XW') - \dim(XW' \cap \mathrm{Im}(m_1-1)).$$

Hence the result. □

Remark 5.6. Let h be as in Proposition 5.5. We note that

$$XW' = \mathrm{Span}\{X_1, X_2, \ldots, X_k\}.$$

We will continue to use this in our character calculations at several instances to follow.

Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in H_A$, where $m_1 \in M_1$, $m_2 \in M_2$. Throughout we write $W^{'} = \mathrm{Ker}(m_2 - 1)$. For $X \in \mathrm{M}(3, F)$, we let $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$. For $\beta \in F$, we define

$$S(\beta) = \{X \in M(3, F) \mid \mathrm{Tr}(A{m_1}^{-1}X) = \mathrm{Tr}(AX) = \beta\}.$$

Let

$$E = \bigcup_{\substack{i \leq j \\ i,j \in \{1,2,4\}}} T(i, j).$$

We call E to be the fundamental set. To determine $\Theta_{N,\psi_A}(m)$ for $m \in T(i, j)$, it is enough to compute $\Theta_{N,\psi_A}(m)$ for $m \in E$.

Theorem 5.7. *Let $m \in T(1, 1)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1 - q)^2.$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X \in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

Note that $\dim \mathrm{Ker}(m_1 - 1) = 2$, $\mathrm{Im}(m_1 - 1) = \mathrm{Span}\,\{e_1\}$ and $\mathrm{Ker}(m_2 - 1) = \mathrm{Span}\,\{e_1, e_2\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(A_1 + A_2)$$

where

$$A_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}$$

and

$$A_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

For simplicity, we let $t = \dim(\mathrm{Ker}(h-1))$. To compute A_1, we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the table below.

TABLE 9. A_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1 - 1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)(a)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & 0 & l \end{bmatrix}\right\}$	0	0	4	q^3
1)(b)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & k & l \end{bmatrix} \,\Big\vert\, k \in F^\times\right\}$	1	0	3	$(q-1)q^3$
2)(a)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & l \end{bmatrix} \,\Big\vert\, ad - bc \neq 0\right\}$	2	1	3	$(q^2-1)(q-1)q^4$
2)(b)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \,\Big\vert\, c, k \in F^\times, ad - bc \neq 0\right\}$	2	0	2	$(q-1)^3 q^5$
2)(c)	$\left\{\begin{bmatrix} a & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \,\Big\vert\, k \in F^\times, ad \neq 0\right\}$	2	1	3	$(q-1)^3 q^4$
3)(a)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ 0 & 0 & l \end{bmatrix} \,\Big\vert\, c \in F^\times, \gamma \in F\right\}$	1	0	3	$(q-1)q^5$
3)(b)	$\left\{\begin{bmatrix} a & \gamma a & e \\ 0 & 0 & f \\ 0 & 0 & l \end{bmatrix} \,\Big\vert\, a \in F^\times, \gamma \in F\right\}$	1	1	4	$(q-1)q^4$
3)(c)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ 0 & k & l \end{bmatrix} \,\Big\vert\, c, k \in F^\times, \gamma \in F\right\}$	2	0	2	$(q-1)^2 q^5$
3)(d)	$\left\{\begin{bmatrix} a & \gamma a & e \\ 0 & 0 & f \\ 0 & k & l \end{bmatrix} \,\Big\vert\, k \in F^\times, \gamma \in F\right\}$	2	1	3	$(q-1)^2 q^4$
4)(a)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & 0 & l \end{bmatrix} \,\Big\vert\, d \in F^\times\right\}$	1	0	3	$(q-1)q^4$
4)(b)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & 0 & f \\ 0 & 0 & l \end{bmatrix} \,\Big\vert\, b \in F^\times\right\}$	1	1	4	$(q-1)q^3$
4)(c)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \,\Big\vert\, k \in F^\times\right\}$	1	0	3	$(q^2-1)(q-1)q^3$

Hence,

$$A_1 = K_1 + K_2 + K_3$$

where

a) $K_1 = \displaystyle\sum_{\substack{X \in S(0) \\ t=4}} \Theta_\theta \left(\begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}\right) \overline{\psi_A({m_1}^{-1}X)} = -q^5(1-q)(1-q^2)(1-q^3).$

b) $K_2 = \sum_{\substack{X \in S(0) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^5(2q+1)(q-1)^2(q^2-1).$

c) $K_3 = \sum_{\substack{X \in S(0) \\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^6(q-1)^3.$

It follows that

$$A_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^8(q-1)^3. \tag{5.1}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute A_2. We record the details in the table below.

Table 10. A_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1-1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
$1)(a)$	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ \beta & 0 & l \end{bmatrix}\right\}$	1	0	3	q^3
$1)(b)$	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert\, k \in F^\times \right\}$	1	0	3	$(q-1)q^3$
$2)(a)$	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, d \in F^\times, ad-bc \neq 0 \right\}$	2	0	2	$(q-1)^2q^5$
$2)(b)$	$\left\{\begin{bmatrix} a & b & e \\ c & 0 & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, bc \neq 0 \right\}$	2	1	3	$(q-1)^2q^4$
$2)(c)$	$\left\{\begin{bmatrix} a & b & e \\ 0 & d & f \\ \beta & k & l \end{bmatrix} \middle\vert\, k \in F^\times, ad \neq 0 \right\}$	2	0	2	$(q-1)^2(q^2-q)q^3$
$2)(d)$	$\left\{\begin{bmatrix} a & b & e \\ c & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert\, k \in F^\times, bc \neq 0 \right\}$	2	0	2	$(q-1)^2(q^2-q)q^3$
$2)(d)$	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix} \middle\vert\, c, d, k \in F^\times, ad-bc \neq 0, d = \beta^{-1}ck \right\}$	2	1	3	$(q-1)^2(q^2-q)q^3$
$2)(e)$	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix} \middle\vert\, c, d, k \in F^\times, ad-bc \neq 0, d \neq \beta^{-1}ck \right\}$	2	0	2	$(q-1)^3(q-2)q^4$
$3)(a)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, c, \gamma \in F^\times \right\}$	2	0	2	$(q-1)^2q^4$
$3)(b)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ 0 & 0 & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, a, \gamma \in F^\times \right\}$	2	1	3	$(q-1)^2q^3$
$3)(c)$	$\left\{\begin{bmatrix} a & 0 & e \\ c & 0 & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, (a, c) \neq 0 \right\}$	1	0	3	$(q^2-1)q^3$
$3)(d)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & k & l \end{bmatrix} \middle\vert\, \gamma \in F^\times, k = \gamma\beta \right\}$	1	0	3	$(q^2-1)(q-1)q^3$
$3)(e)$	$\left\{\begin{bmatrix} a & 0 & e \\ c & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert\, k, c \in F^\times \right\}$	2	0	2	$(q-1)^2q^4$
$3)(f)$	$\left\{\begin{bmatrix} a & 0 & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert\, k \in F^\times \right\}$	2	1	3	$(q-1)^2q^3$
$3)(g)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & k & l \end{bmatrix} \middle\vert\, c, k, \gamma \in F^\times, k \neq \gamma\beta \right\}$	2	0	2	$(q-1)^2(q-2)q^4$
$3)(h)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert\, a, k, \gamma \in F^\times, k \neq \gamma\beta \right\}$	2	1	3	$(q-1)^2(q-2)q^3$
$4)(a)$	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ \beta & k & l \end{bmatrix} \middle\vert\, d \in F^\times \right\}$	2	0	2	$(q-1)q^5$
$4)(b)$	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert\, b \in F^\times \right\}$	2	1	3	$(q-1)q^4$

Hence,

$$A_2 = K_4 + K_5$$

$$\text{a)}\;\; K_4 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^7(q-1)(q^2-1).$$

b) $K_5 = \sum_{\beta\in F^\times} \sum_{\substack{X\in S(\beta)\\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^7(q-1)^2.$

It follows that

$$A_2 = \sum_{\beta\in F^\times} \sum_{X\in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^8(q-1)^2. \tag{5.2}$$

From (5.1) and (5.2), it follows that

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

□

Theorem 5.8. *Let $m \in T(1,2)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q).$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X\in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

Note that $\dim \mathrm{Ker}(m_1 - 1) = 2$, $\mathrm{Im}(m_1 - 1) = \mathrm{Span}\,\{e_1\}$ and $\mathrm{Ker}(m_2 - 1) = \mathrm{Span}\,\{e_1\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(B_1 + B_2)$$

where

$$B_1 = \sum_{X\in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}$$

and

$$B_2 = \sum_{\beta\in F^\times} \sum_{X\in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

For simplicity, we let $t = \dim(\mathrm{Ker}(h-1))$. To compute B_1, we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the table below.

TABLE 11. B_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1-1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix}\right\}$	0	0	3	q^6
2)(a)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \middle\vert\, c\in F^\times\right\}$	1	0	2	$(q-1)q^7$
2)(b)	$\left\{\begin{bmatrix} a & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \middle\vert\, a\in F^\times\right\}$	1	1	3	$(q-1)q^6$

Hence,

$$B_1 = K_1 + K_2$$

where

a) $K_1 = \sum_{\substack{X\in S(0)\\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^7(1-q)(1-q^2)$.

b) $K_2 = \sum_{\substack{X\in S(0)\\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^7(1-q)^2$.

It follows that

$$B_1 = \sum_{X\in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^8(q-1)^2. \tag{5.3}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute B_2. We record the details in the table below.

Table 12. B_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1-1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix}\right\}$	1	0	2	q^8

Hence,

$$B_2 = \sum_{\beta\in F^\times} \sum_{\substack{X\in S(\beta)\\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^8(1-q). \tag{5.4}$$

From (5.3) and (5.4), it follows that

$$\Theta_{N,\psi_A}(m) = (1-q).$$

□

Theorem 5.9. *Let $m \in T(4,1)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2(1-q^2).$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X\in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

Note that $\dim \mathrm{Ker}(m_1-1) = 3$, $\mathrm{Im}(m_1-1) = \{0\}$ and $\mathrm{Ker}(m_2-1) = \mathrm{Span}\,\{e_1, e_2\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(C_1 + C_2)$$

where we have

$$C_1 = \sum_{X\in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}$$

and

$$C_2 = \sum_{\beta\in F^\times} \sum_{X\in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

For simplicity, we let $t = \dim(\mathrm{Ker}(h-1))$. To compute C_1, we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the following table.

Table 13. C_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \operatorname{Im}(m_1 - 1))$	$t = \dim(\operatorname{Ker}(h-1))$	Cardinality
$1)(a)$	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & 0 & l \end{bmatrix}\right\}$	0	0	5	q^3
$1)(b)$	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times \right\}$	1	0	4	$(q-1)q^3$
$2)(a)$	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & l \end{bmatrix} \middle\vert ad - bc \neq 0 \right\}$	2	0	3	$(q^2-1)(q^2-q)q^3$
$2)(b)$	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times, ad - bc \neq 0 \right\}$	2	0	3	$(q^2-1)(q-1)^2q^4$
$3)(a)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ 0 & 0 & l \end{bmatrix} \middle\vert \gamma \in F \right\}$	1	0	4	$(q^2-1)q^4$
$3)(b)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times, \gamma \in F \right\}$	2	0	3	$(q^2-1)(q-1)q^4$
$4)(a)$	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & 0 & l \end{bmatrix}\right\}$	1	0	4	$(q^2-1)q^3$
$4)(b)$	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times \right\}$	1	0	4	$(q^2-1)(q-1)q^3$

Hence,

$$C_1 = K_1 + K_2 + K_3$$

where

a) $K_1 = \sum\limits_{\substack{X \in S(0) \\ t=5}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^3(1-q)(1-q^2)(1-q^3)(1-q^4).$

b) $K_2 = \sum\limits_{\substack{X \in S(0) \\ t=4}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^3(1-q)^2(1-q^2)(1-q^3)(2q^2 + 2q + 1).$

c) $K_3 = \sum\limits_{\substack{X \in S(0) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^4(q^2-1)^2(1-q)(1-q^2).$

It follows that

$$C_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^8(1-q)^3(1-q^2). \tag{5.5}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute C_2. We record the details in the table below.

Table 14. C_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1 - 1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
$1)(a)$	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ \beta & 0 & l \end{bmatrix}\right\}$	1	0	4	q^3
$1)(b)$	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert k \in F^\times \right\}$	1	0	4	$(q-1)q^3$
$2)(a)$	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & 0 & l \end{bmatrix} \middle\vert ad - bc \neq 0 \right\}$	2	0	3	$(q^2-1)(q^2-q)q^3$
$2)(b)$	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix} \middle\vert k \in F^\times, ad - bc \neq 0 \right\}$	2	0	3	$(q^2-1)(q-1)^2q^4$
$3)(a)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & 0 & l \end{bmatrix} \middle\vert \gamma \in F^\times \right\}$	2	0	3	$(q^2-1)(q-1)q^3$
$3)(b)$	$\left\{\begin{bmatrix} a & 0 & e \\ c & 0 & f \\ \beta & 0 & l \end{bmatrix} \middle\vert (a,c) \neq 0 \right\}$	1	0	4	$(q^2-1)q^3$
$3)(c)$	$\left\{\begin{bmatrix} a & 0 & e \\ c & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert (a,c) \neq 0, k \in F^\times \right\}$	2	0	3	$(q^2-1)(q-1)q^3$
$3)(d)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & k & l \end{bmatrix} \middle\vert \gamma \in F^\times, k = \gamma\beta \right\}$	1	0	4	$(q^2-1)(q-1)q^3$
$3)(e)$	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & k & l \end{bmatrix} \middle\vert k, \gamma \in F^\times, k \neq \gamma\beta \right\}$	2	0	3	$(q^2-1)(q-1)(q-2)q^3$
$4)(a)$	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ \beta & 0 & l \end{bmatrix} \middle\vert (b,d) \neq 0 \right\}$	2	0	3	$(q^2-1)q^3$
$4)(b)$	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ \beta & k & l \end{bmatrix} \middle\vert (b,d) \neq 0, k \in F^\times \right\}$	2	0	3	$(q^2-1)(q-1)q^3$

We have

$$C_2 = K_4 + K_5$$

where

a) $K_4 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=4}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^6(1-q)(1-q^2)(1-q^3).$

b) $K_5 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^6(1-q)(1-q^2)^2.$

It follows that

$$C_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^8(1-q)^2(1-q^2). \tag{5.6}$$

From (5.5) and (5.6), we have

$$\Theta_{N,\psi_A}(m) = (1-q^2)(1-q)^2.$$

□

Remark 5.10. Let $m \in T(1,4)$. Since

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m')$$

for some $m' \in T(4,1)$, it is enough to compute $\Theta_{N,\psi_A}(m')$ for $m' \in T(4,1)$ to obtain the character value $\Theta_{N,\psi_A}(m)$.

Theorem 5.11. *Let $m \in T(2,2)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = 1.$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X \in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

Note that $\dim \operatorname{Ker}(m_1 - 1) = 1$, $\operatorname{Im}(m_1 - 1) = \operatorname{Span}\{e_1, e_2\}$ and $\operatorname{Ker}(m_2 - 1) = \operatorname{Span}\{e_1\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(D_1 + D_2)$$

where

$$D_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}$$

and

$$D_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

Let $t = \dim(\operatorname{Ker}(h-1))$. To compute D_1 we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the table below.

Table 15. D_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \operatorname{Im}(m_1 - 1))$	$t = \dim(\operatorname{Ker}(h-1))$	Cardinality
$1)(a)$	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \mid (a,c) \neq 0\right\}$	1	1	2	$(q^2-1)q^6$
$1)(b)$	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix}\right\}$	0	0	2	q^6

Hence,

$$D_1 = \sum_{\substack{X \in S(0) \\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^8(1-q). \tag{5.7}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute D_2. We record the details in the following table.

Table 16. D_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \operatorname{Im}(m_1 - 1))$	$t = \dim(\operatorname{Ker}(h-1))$	Cardinality
1)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix}\right\}$	1	0	1	q^8

Thus, we have

$$D_2 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=1}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^8. \tag{5.8}$$

From (5.7) and (5.8), it follows that

$$\Theta_{N,\psi_A}(m) = 1.$$

□

Theorem 5.12. *Let $m \in T(4,2)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)(1-q^2).$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X \in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

Note that $\dim \operatorname{Ker}(m_1 - 1) = 3$, $\operatorname{Im}(m_1 - 1) = \{0\}$ and $\operatorname{Ker}(m_2 - 1) = \operatorname{Span}\{e_1\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(H_1 + H_2)$$

where

$$H_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}$$

and

$$H_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

Let $t = \dim(\operatorname{Ker}(h-1))$. To compute H_1, we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the table below.

Table 17. H_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \operatorname{Im}(m_1-1))$	$t = \dim(\operatorname{Ker}(h-1))$	Cardinality
$1)(a)$	$\left\{ \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \middle\vert (a,c) \neq 0 \right\}$	1	0	3	$(q^2-1)q^6$
$1)(b)$	$\left\{ \begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \right\}$	0	0	4	q^6

Hence,

$$H_1 = K_1 + K_2$$

where

a) $K_1 = \displaystyle\sum_{\substack{X \in S(0) \\ t=4}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^6(1-q)(1-q^2)(1-q^3).$

b) $K_2 = \displaystyle\sum_{\substack{X \in S(0) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^6(1-q)(1-q^2)^2.$

It follows that

$$H_1 = \sum_{X\in S(0)} \Theta_\theta \begin{pmatrix} m_1 & X \\ 0 & m_2 \end{pmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^8(1-q)^2(1-q^2). \tag{5.9}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute H_2. We record the details in the following table.

Table 18. H_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1-1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix}\right\}$	1	0	3	q^8

Thus, we have

$$H_2 = \sum_{\beta\in F^\times} \sum_{\substack{X\in S(\beta) \\ t=3}} \Theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^8(1-q)(1-q^2). \tag{5.10}$$

From (5.9) and (5.10), it follows that

$$\Theta_{N,\psi_A} = (1-q)(1-q^2).$$

□

Remark 5.13. Let $m \in T(2,4)$. Since

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m')$$

for some $m' \in T(4,2)$, it is enough to compute $\Theta_{N,\psi_A}(m')$ for $m' \in T(4,2)$ to obtain $\Theta_{N,\psi_A}(m)$.

Theorem 5.14. *Let $m \in T(4,4)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2(1-q^2)^2.$$

Proof. Since $m \in T(4,4)$, we have $m = 1$, and the result follows from Theorem 3.3. To be precise, we have

$$\Theta_{N,\psi_A}(m) = \dim_{\mathbb{C}}(\pi_{N,\psi_A}) = (1-q)^2(1-q^2)^2.$$

□

Theorem 5.15. *Let $1 \le i \le 10$. Suppose that $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,5)$ and $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix} \in T(i,1)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$ and $h' = \begin{bmatrix} m_1 & X \\ 0 & m_2' \end{bmatrix}$ for $X \in \mathrm{M}(3,F)$. Let

$$M^{d,\beta}_{m_1,m_2} = \{X \in S(\beta) \mid \dim(\mathrm{Ker}(h-1)) = d\}$$

for $\beta \in F$. Clearly,

$$\mathrm{Ker}(m_2 - 1) = \mathrm{Ker}(m_2' - 1).$$

Hence for any $X \in \mathrm{M}(3,F)$,

$$X\,\mathrm{Ker}(m_2-1) = X\,\mathrm{Ker}(m_2'-1)$$

and

$$X\,\mathrm{Ker}(m_2-1)\cap \mathrm{Im}(m_1-1) = X\,\mathrm{Ker}(m_2'-1)\cap \mathrm{Im}(m_1-1).$$

In particular, for any $\beta \in F$, we have that

$$M^{d,\beta}_{m_1,m_2} = M^{d,\beta}_{m_1,m_2'}. \tag{5.11}$$

We have,

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(R_1+R_2)$$

where

$$\begin{aligned} R_1 &= \sum_{d=1}^{6} \sum_{X\in M^{d,0}_{m_1,m_2}} \Theta_\theta(h)\overline{\psi_A({m_1}^{-1}X)} \\ &= \sum_{d=1}^{6} (-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,0}_{m_1,m_2})\overline{\psi_0(0)} \end{aligned}$$

and

$$\begin{aligned} R_2 &= \sum_{\beta\in F^\times}\sum_{d=1}^{6} \sum_{X\in M^{d,\beta}_{m_1,m_2}} \Theta_\theta(h)\overline{\psi_A({m_1}^{-1}X)} \\ &= \sum_{\beta\in F^\times}\sum_{d=1}^{6} (-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,\beta}_{m_1,m_2})\overline{\psi_0(\beta)} \end{aligned}$$

Thus,

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,0}_{m_1,m_2} + \sum_{\beta\in F^\times} \# M^{d,\beta}_{m_1,m_2}\overline{\psi_0(\beta)}).$$

Similarly,

$$\Theta_{N,\psi_A}(m') = \frac{1}{q^9}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,0}_{m_1,m_2'} + \sum_{\beta\in F^\times} \# M^{d,\beta}_{m_1,m_2'}\overline{\psi_0(\beta)}).$$

Hence, it follows from equation (5.12) that

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

□

Proposition 5.16. *Let $1 \le i \le 10$ and $m \in T(i,3)$. Then, there exists some $m' \in T(i,5)$ such that*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,3)$, and $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Let $m_2' = w m_2 w^{-1}$ and $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix}$. Clearly, we have $m = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} m' \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}^{-1}$ and $m^{'} \in T(i,5)$. Hence the result. □

Corollary 5.17. *Let $1 \le i \le 10$ and $m \in T(i,3)$. Then,*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m')$$

for some $m' \in T(i,1)$.

Proof. Using Proposition 5.16 and Theorem 5.15, the result follows. □

Proposition 5.18. *Let $1 \leq i \leq 10$ and $m \in T(i,7)$. Then, there exists some $m' \in T(i,1)$ such that*
$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,7)$, and $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Let $m_2' = w m_2 w^{-1}$ and $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix}$. Clearly, we have $m = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} m' \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}^{-1}$ and $m' \in T(i,1)$. Hence the result. □

Proposition 5.19. *Let $1 \leq i \leq 10$ and $m \in T(i,8)$. Then, there exists some $m' \in T(i,2)$ such that*
$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,8)$, and $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Let $m_2' = w m_2 w^{-1}$ and $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix}$. Clearly, we have $m = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} m' \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}^{-1}$ and $m' \in T(i,2)$. Hence the result. □

Theorem 5.20. *Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(1,6)$ or $T(1,9)$. Then, we have*
$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

Proof. Note that $\dim \operatorname{Ker}(m_1 - 1) = 2$, $\operatorname{Im}(m_1 - 1) = \operatorname{Span}\{e_1\}$, $\dim \operatorname{Ker}(m_2 - 1) = 2$. From Remark 5.6, it follows that whenever
$$h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix},$$
we have
$$XW' = \operatorname{Span}\{X_1, X_2\}.$$
Thus, proceeding in a similar fashion as in Theorem 5.7, we get that
$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$
□

Theorem 5.21. *Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(2,6)$ or $T(2,9)$. Then, we have*
$$\Theta_{N,\psi_A}(m) = (1-q).$$

Proof. Note that $\dim \operatorname{Ker}(m_1 - 1) = 1$, $\operatorname{Im}(m_1 - 1) = \{e_1, e_2\}$, $\dim \operatorname{Ker}(m_2 - 1) = 2$. From Remark 5.6, it follows that whenever
$$h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix},$$
we have
$$XW' = \operatorname{Span}\{X_1, X_2\}.$$
Thus, the computations are similar to the case where $m \in T(2,1)$. The result follows from Theorem 5.3 and Theorem 5.8. □

Theorem 5.22. *Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(4,6)$ or $T(4,9)$. Then, we have*
$$\Theta_{N,\psi_A}(m) = (1-q)^2(1-q^2).$$

Proof. Note that $\dim \operatorname{Ker}(m_1 - 1) = 3$, $\operatorname{Im}(m_1 - 1) = \{0\}$, $\dim \operatorname{Ker}(m_2 - 1) = 2$. From Remark 5.6, it follows that whenever

$$h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix},$$

we have

$$XW' = \operatorname{Span}\{X_1, X_2\}.$$

Thus, proceeding in a similar fashion as in Theorem 5.9, we get that

$$\Theta_{N,\psi_A}(m) = (1-q)^2(1-q^2).$$

□

Theorem 5.23. *Let* $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(6,6)$ *or* $T(6,9)$*. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

Proof. Note that $\dim \operatorname{Ker}(m_1 - 1) = 2$, $\dim \operatorname{Ker}(m_2 - 1) = 2$, $\operatorname{Im}(m_1 - 1) = \operatorname{Span}\{\eta e_1 + e_2\}$ for $\eta \in F^\times$. From Remark 5.6, it follows that computing $\Theta_{N,\psi_A}(m)$ for $m \in T(6,6)$ or $m \in T(6,9)$ is the same as computing $\Theta_{N,\psi_A}(m')$ for $m' \in T(6,1)$. Using Theorem 5.3 and Theorem 5.20, it follows that

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

□

Theorem 5.24. *Let* $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(9,9)$*. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

Proof. The proof is similar to Theorem 5.23. □

Theorem 5.25. *Let* $1 \le i \le 10$*. Suppose* $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,10)$ *and* $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix} \in T(i,2)$*. Then, we have*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$ and $h' = \begin{bmatrix} m_1 & X \\ 0 & m_2' \end{bmatrix}$ for $X \in \mathrm{M}(3,F)$. Let

$$M^{d,\beta}_{m_1,m_2} = \{X \in S(\beta) \mid \dim(\operatorname{Ker}(h-1)) = d\}$$

for $\beta \in F$. Clearly,

$$\operatorname{Ker}(m_2 - 1) = \operatorname{Ker}(m_2' - 1).$$

Hence for any $X \in \mathrm{M}(3,F)$,

$$X \operatorname{Ker}(m_2 - 1) = X \operatorname{Ker}(m_2' - 1)$$

and

$$X \operatorname{Ker}(m_2 - 1) \cap \operatorname{Im}(m_1 - 1) = X \operatorname{Ker}(m_2' - 1) \cap \operatorname{Im}(m_1 - 1).$$

In particular, for any $\beta \in F$, we have that

$$M^{d,\beta}_{m_1,m_2} = M^{d,\beta}_{m_1,m_2'}. \tag{5.12}$$

Therefore,

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(R_1 + R_2)$$

where

$$\begin{aligned} R_1 &= \sum_{d=1}^{6} \sum_{X \in M^{d,0}_{m_1,m_2}} \Theta_\theta(h)\overline{\psi_A({m_1}^{-1}X)} \\ &= \sum_{d=1}^{6} (-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,0}_{m_1,m_2})\overline{\psi_0(0)} \end{aligned}$$

and

$$\begin{aligned} R_2 &= \sum_{\beta \in F^\times} \sum_{d=1}^{6} \sum_{X \in M^{d,\beta}_{m_1,m_2}} \Theta_\theta(h)\overline{\psi_A({m_1}^{-1}X)} \\ &= \sum_{\beta \in F^\times} \sum_{d=1}^{6} (-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,\beta}_{m_1,m_2}\overline{\psi_0(\beta)}) \end{aligned}$$

Thus,

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,0}_{m_1,m_2} + \sum_{\beta \in F^\times}(\# M^{d,\beta}_{m_1,m_2})\overline{\psi_0(\beta)}).$$

Similarly,

$$\Theta_{N,\psi_A}(m') = \frac{1}{q^9}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,0}_{m_1,m'_2} + \sum_{\beta \in F^\times}\#(M^{d,\beta}_{m_1,m'_2})\overline{\psi_0(\beta)}).$$

Hence, it follows from equation 5.12 that

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

□

Remark 5.26. Let $1 \leq i \leq j \leq 10$. To determine $\Theta_{N,\psi_A}(m)$ for $m \in T(i,j)$, it is enough to compute $\Theta_{N,\psi_A}(m)$ for $m \in E$. We illustrate this by an example.

Suppose that we want to compute the character value $\Theta_{N,\psi_A}(m)$ for $m \in T(3,7)$. From Proposition 5.18, it follows that $\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(k)$ for some $k \in T(3,1)$. By Theorem 5.3, we have $\Theta_{N,\psi_A}(k) = \Theta_{N,\psi_A}(x)$ for some $x \in T(1,3)$. Using Theorem 5.16, we have, $\Theta_{N,\psi_A}(x) = \Theta_{N,\psi_A}(y)$ for some $y \in T(1,1)$. Thus, using Theorem 5.7 we have

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

For clarity, we represent the chain of computations used to determine the character value of an element in $T(i,j)$ to the character value of an element in the fundamental set E in the following way.

$$(3,7) \to (3,1) \to (1,3) \to (1,1).$$

The table below summarizes the sequence of computations used to calculate $\Theta_{N,\psi_A}(m)$ for $m \in T(i,j)$, $i \leq j$.

TABLE 19. Sequence of computations for $\Theta_{N,\psi_A}(m)$

	Type-1	Type-2	Type-3	Type-4	Type-5	Type-6	Type-7	Type-8	Type-9	Type-10
Type-1	(1, 1)	(1, 2)	(1, 3) → (1, 1)	(1, 4)	(1, 5) → (1, 1)	(1, 6) → (1, 1)	(1, 7) → (1, 1)	(1, 8) → (1, 2)	(1, 9) → (1, 1)	(1, 10) → (1, 2)
Type-2	-	(2, 2)	(2, 3) → (2, 1) → (1, 2)	(2, 4)	(2, 5) → (2, 1) → (1, 2)	(2, 6) → (2, 1) → (1, 2)	(2, 7) → (2, 1) → (1, 2)	(2, 8) → (2, 2)	(2, 9) → (2, 1) → (1, 2)	(2, 10) → (2, 2)
Type-3	-	-	(3, 3) → (3, 1) → (1, 3) → (1, 1)	(4, 3) → (4, 1) → (1, 4)	(3, 5) → (3, 1) → (1, 3) → (1, 1)	(3, 6) → (6, 3) → (6, 1) → (1, 6) → (1, 1)	(3, 7) → (3, 1) → (1, 3) → (1, 1)	(3, 8) → (3, 2) → (2, 3) → (2, 1) → (1, 2)	(3, 9) → (9, 3) → (9, 1) → (1, 9) → (1, 1)	(3, 10) → (3, 2) → (2, 3) → (2, 1) → (1, 2)
Type-4	-	-	-	(4, 4)	(4, 5) → (4, 1) → (1, 4)	(4, 6) → (4, 1) → (1, 4)	(4, 7) → (4, 1) → (1, 4)	(4, 8) → (4, 2) → (2, 4)	(4, 9) → (4, 1) → (1, 4)	(4, 10) → (4, 2) → (2, 4)
Type-5	-	-	-	-	(5, 5) → (5, 1) → (1, 5) → (1, 1)	(5, 6) → (6, 5) → (6, 1) → (1, 6) → (1, 1)	(5, 7) → (5, 1) → (1, 5) → (1, 1)	(5, 8) → (5, 2) → (2, 5) → (2, 1) → (1, 2)	(5, 9) → (9, 5) → (9, 1) → (1, 9) → (1, 1)	(5, 10) → (5, 2) → (2, 5) → (2, 1) → (1, 2)
Type-6	-	-	-	-	-	(6, 6) → (6, 1) → (1, 6) → (1, 1)	(6, 7) → (6, 1) → (1, 6) → (1, 1)	(6, 8) → (6, 2) → (2, 6) → (2, 1) → (1, 2)	(6, 9) → (6, 1) → (1, 6) → (1, 1)	(6, 10) → (6, 2) → (2, 6) → (2, 1) → (1, 2)
Type-7	-	-	-	-	-	-	(7, 7) → (7, 1) → (1, 7) → (1, 1)	(7, 8) → (7, 2) → (2, 7) → (2, 1) → (1, 2)	(7, 9) → (9, 7) → (9, 1) → (1, 9) → (1, 1)	(7, 10) → (7, 2) → (2, 7) → (2, 1) → (1, 2)
Type-8	-	-	-	-	-	-	-	(8, 8) → (8, 2) → (2, 8) → (2, 2)	(8, 9) → (9, 8) → (9, 2) → (2, 9) → (2, 1) → (1, 2)	(8, 10) → (8, 2) → (2, 8) → (2, 2)
Type-9	-	-	-	-	-	-	-	-	(9, 9) → (9, 1) → (1, 9) → (1, 1)	(9, 10) → (9, 2) → (2, 9) → (2, 1) → (1, 2)
Type-10	-	-	-	-	-	-	-	-	-	(10, 10) → (10, 2) → (2, 10) → (2, 2)

Acknowledgements

We thank Professor Dipendra Prasad for suggesting this problem and for some helpful discussions. Research of Kumar Balasubramanian is supported by the SERB grant: MTR/2019/000358.

References

1. Kumar Balasubramanian and Himanshi Khurana, *A certain twisted Jacquet module of* GL(4) *over a finite field*, J. Pure Appl. Algebra **226** (2022), no. 5, Paper No. 106932, 16. MR 4328653
2. I. M. Gelfand and M. I. Graev, *Construction of irreducible representations of simple algebraic groups over a finite field*, Dokl. Akad. Nauk SSSR **147** (1962), 529–532. MR 0148765
3. S. I. Gelfand, *Representations of the full linear group over a finite field*, Mat. Sb. (N.S.) **83 (125)** (1970), 15–41. MR 0272916
4. Ofir Gorodetsky and Zahi Hazan, *On certain degenerate Whittaker models for cuspidal representations of* $\mathrm{GL}_{k\cdot n}(\mathbb{F}_q)$, Math. Z. **291** (2019), no. 1-2, 609–633. MR 3936084
5. Dipendra Prasad, *The space of degenerate Whittaker models for general linear groups over a finite field*, Internat. Math. Res. Notices (2000), no. 11, 579–595. MR 1763857
6. Jean-Pierre Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott. MR 0450380

Kumar Balasubramanian, Department of Mathematics, IISER Bhopal, Bhopal, Madhya Pradesh 462066, India

Email address: bkumar@iiserb.ac.in

Himanshi Khurana, Department of Mathematics, IISER Bhopal, Bhopal, Madhya Pradesh 462066, India

Email address: himanshi18@iiserb.ac.in