A CERTAIN TWISTED JACQUET MODULE OF GL(6) OVER A FINITE FIELD: THE RANK 2 CASE

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Abstract. Let F be a finite field and $G = \mathrm{GL}(6, F)$. In this paper, we explicitly describe the structure of the twisted Jacquet module π_{N,ψ_A} where A is a rank 2 matrix and π is an irreducible cuspidal representation of G.

1. Introduction

Let F be a finite field and $G = \mathrm{GL}(2n, F)$. Let $P = MN$ be the standard maximal parabolic subgroup of G corresponding to the partition (n, n). We fix a non-trivial character ψ_0 of F. It is easy to see that any character ψ of $N \simeq M(n, F)$ is of the form $\psi = \psi_A$, where $\psi_A(X) = \psi_0(\mathrm{Tr}(AX))$. The group $\mathrm{GL}(n, F) \times \mathrm{GL}(n, F)$ acts on the set of characters of $\mathrm{M}(n, F)$ via,

$$(g_1, g_2).\psi_A = \psi_{g_2^{-1} A g_1}$$

and the set of characters of $\mathrm{M}(n, F)$ decomposes into disjoint orbits with respect to the above action. For $0 \leq i \leq n$, we let

$$A_i = \begin{bmatrix} I_i & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{M}(n, F),$$

where I_i is the identity matrix in $\mathrm{GL}(i, F)$. The matrices $A_i, 0 \leq i \leq n$ form a set of representatives for the orbits under the above action. When $i = n$, the character ψ_{A_n} is a representative for the orbit of the non-degenerate characters of $\mathrm{M}(n, F)$.

Let π be an irreducible cuspidal representation of $\mathrm{GL}(2n, F)$. In [5], Prasad explicitly described the structure of $\pi_{N,\psi_{A_n}}$ as a module for $M_{\psi_{A_n}}$. In [1], we described the structure of $\pi_{N,\psi_{A_1}}$ as an $M_{\psi_{A_1}}$ module. The structure of $\pi_{N,\psi_{A_k}}$ as an $M_{\psi_{A_k}}$ module for $1 < k < n$ is still not known. This motivates the problem studied in this paper. We hope that understanding the structure of π_{N,ψ_A}, when $A = A_2 \in \mathrm{M}(3, F)$ and π is an irreducible cuspidal representation of $\mathrm{GL}(6, F)$ will help us gain some insight towards the general case. In this paper, we explicitly describe the structure of the twisted Jacquet module π_{N,ψ_A} of an irreducible cuspidal representation π of $\mathrm{GL}(6, F)$ when $A = A_2 \in \mathrm{M}(3, F)$.

Before we state our result, we set up some notation. Let $G = \mathrm{GL}(6, F)$ and P be the maximal parabolic subgroup of G with Levi decomposition $P = MN$, where $M \simeq \mathrm{GL}(3, F) \times \mathrm{GL}(3, F)$ and $N \simeq \mathrm{M}(3, F)$. We write F_6 for the unique field extension of F of degree 6. Let ψ_0 be a fixed non-trivial additive character of F. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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and $\psi_A : N \to \mathbb{C}^\times$ be the character of N given by

$$\psi_A\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) = \psi_0(\mathrm{Tr}(AX)).$$

Let $U_A \simeq F^2 \oplus F^2$ and $H_A = U_A \rtimes (F^\times \times \mathrm{GL}(2, F))$. Let ψ_1 be the character of U_A given by

$$\psi_1(Y, Z) = \psi_0(x + v)$$

and ψ_2 be the character of U_A given by

$$\psi_2(Y, Z) = \psi_0(x + u),$$

where $Y = \begin{bmatrix} x \\ y \end{bmatrix}$ and $Z = \begin{bmatrix} u \\ v \end{bmatrix}$. We write Mir for the Mirabolic subgroup of $\mathrm{GL}(2, F)$ and reg for the regular representation.

Theorem 1.1. *Let θ be a regular character of $F_6^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(6, F)$. Then,*

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times} \otimes [q \,\mathrm{Ind}_{U_A \rtimes F^\times}^{H_A}(\widetilde{\psi_1} \otimes \overline{\mathrm{reg}(F^\times)}) \oplus \mathrm{Ind}_{U_A \rtimes Mir}^{H_A}(\widetilde{\psi_2} \otimes \overline{\mathrm{reg}(Mir)})].$$

2. Preliminaries

In this section, we record some preliminaries that we need.

2.1. **Character of a cuspidal representation.** Let F be the finite field of order q and $G = \mathrm{GL}(m, F)$. Let F_m be the unique field extension of F of degree m. A character θ of $F_m^\times$ is called a "regular" character, if under the action of the Galois group of F_m over F, θ gives rise to m distinct characters of $F_m^\times$. It is a well known fact that the cuspidal representations of $\mathrm{GL}(m, F)$ are parametrized by the regular characters of $F_m^\times$. To avoid introducing more notation, we mention below only the relevant statements on computing the character values that we have used. We refer the reader to Section 6 in [4] for more precise statements on computing character values.

Theorem 2.1. *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(m, F)$ associated to θ. Let Θ_π be its character. If $g \in \mathrm{GL}(m, F)$ is such that the characteristic polynomial of g is not a power of a polynomial irreducible over F. Then, we have*

$$\Theta_\pi(g) = 0.$$

Theorem 2.2. *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(m, F)$ associated to θ. Let Θ_π be its character. Suppose that $g = s.u$ is the Jordan decomposition of an element g in $\mathrm{GL}(m, F)$. If $\Theta_\pi(g) \neq 0$, then the semisimple element s must come from $F_m^\times$. Suppose that s comes from $F_m^\times$. Let z be an eigenvalue of s in F_m and let t be the dimension of the kernel of $g - z$ over F_m. Then*

$$\Theta_\pi(g) = (-1)^{m-1}\left[\sum_{\alpha=0}^{d-1} \theta(z^{q^\alpha})\right](1 - q^d)(1 - (q^d)^2)\dots(1 - (q^d)^{t-1}).$$

where q^d is the cardinality of the field generated by z over F, and the summation is over the distinct Galois conjugates of z.

See Theorem 2 in [5] for this version.

2.2. **Twisted Jacquet module.** In this section, we recall the character and the dimension formula of the twisted Jacquet module of a representation π.

Let $G = \mathrm{GL}(k, F)$ and $P = MN$ be a parabolic subgroup of G. Let ψ be a character of N. For $m \in M$, let ψ^m be the character of N defined by $\psi^m(n) = \psi(mnm^{-1})$. Let

$$V(N, \psi) = \mathrm{Span}_{\mathbb{C}}\{\pi(n)v - \psi(n)v \mid n \in N, v \in V\}$$

and

$$M_\psi = \{m \in M \mid \psi^m(n) = \psi(n), \forall n \in N\}.$$

Clearly, M_ψ is a subgroup of M and it is easy to see that $V(N, \psi)$ is an M_ψ-invariant subspace of V. Hence, we get a representation $(\pi_{N,\psi}, V/V(N, \psi))$ of M_ψ. We call $(\pi_{N,\psi}, V/V(N, \psi))$ the twisted Jacquet module of π with respect to ψ. We write $\Theta_{N,\psi}$ for the character of $\pi_{N,\psi}$.

Proposition 2.3. *Let (π, V) be a representation of $\mathrm{GL}(k, F)$ and Θ_π be the character of π. We have*

$$\Theta_{N,\psi}(m) = \frac{1}{|N|} \sum_{n \in N} \Theta_\pi(mn)\overline{\psi(n)}.$$

Remark 2.4. Taking $m = 1$, we get the dimension of $\pi_{N,\psi}$. To be precise, we have

$$\dim_{\mathbb{C}}(\pi_{N,\psi}) = \frac{1}{|N|} \sum_{n \in N} \Theta_\pi(n)\overline{\psi(n)}.$$

2.3. **Representations of semidirect product of groups.** In this section, we recall some results about constructing representations of a semidirect product of groups.

Let G be a finite group and $N \trianglelefteq G$ be a normal subgroup of G. We write $\widehat{N}$ for the set of irreducible representations of N upto equivalence. For $\sigma \in \widehat{N}$ and $g \in G$, we let

$$I_G(\sigma) = \{g \in G \mid {}^g\sigma \simeq \sigma\}$$

for the inertia subgroup of $\sigma \in \widehat{N}$. For H a subgroup of G and $\sigma \in \widehat{H}$, an extension of σ to G is a representation $\widetilde{\sigma} \in \widehat{G}$ such that $\mathrm{Res}^G_H \widetilde{\sigma} = \sigma$.

Theorem 2.5. *Suppose that G is a finite group and let $N \trianglelefteq G$ be a normal subgroup. Suppose that any $\sigma \in \widehat{N}$ has an extension $\widetilde{\sigma}$ to its inertia subgroup $I_G(\sigma)$. In $\widehat{N}$, define an equivalence relation $\approx$ by setting $\sigma_1 \approx \sigma_2$ if there exists $g \in G$ such that ${}^g\sigma_1 \simeq \sigma_2$. Let Σ be a set of representatives of the corresponding quotient space $\widehat{N}/\approx$. For $\psi \in \widehat{I_G(\sigma)/N}$, denote by $\overline{\psi} \in \widehat{I_G(\sigma)}$, its inflation to $I_G(\sigma)$. Then*

$$\widehat{G} = \{\mathrm{Ind}^G_{I_G(\sigma)}(\widetilde{\sigma} \otimes \overline{\psi}) : \sigma \in \Sigma, \psi \in \widehat{I_G(\sigma)/N}\},$$

that is, the above is the list of all irreducible G-representations and, for different values of σ and ψ, we obtain inequivalent representations.

In the case when N is abelian, Theorem 2.5 above can be restated as follows.

Theorem 2.6. *Suppose that $G = N \rtimes H$ with N abelian. Given $\chi \in \widehat{N}$, its inertia group $I_G(\chi)$ coincides with $N \rtimes H_\chi$, where*

$$H_\chi = \{h \in H \mid {}^h\chi \simeq \chi\}.$$

Then any $\chi \in \widehat{N}$ may be extended to a one dimensional representation $\widetilde{\chi} \in \widehat{N \rtimes H_\chi}$ by setting

$$\widetilde{\chi}(nh) = \chi(n), \qquad \forall n \in N, h \in H_\chi.$$

Moreover, with the notation of Theorem 2.5, we have

$$\widehat{G} = \{\mathrm{Ind}_{N \rtimes H_\chi}^G (\widetilde{\chi} \otimes \overline{\psi}) \mid \chi \in \Sigma, \psi \in \widehat{H_\chi}\}.$$

We refer the reader to Section 5 in [3] for more details.

3. Dimension of the Twisted Jacquet Module

In this section, we compute the dimension of the twisted Jacquet module π_{N,ψ_A}, where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{M}(3, F)$$

is a rank 2 matrix.

Theorem 3.1. *Let θ be a regular character of $F_6^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(6, F)$. *We have*

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = q(q-1)(q^2-1)^2.$$

Proof. For $0 \leq i \leq 3$, let $\pi[i]$ be the sum of all characters of N inside π which lie in the orbit of the character ψ_{A_i} under the action of $\mathrm{GL}(n, F) \times \mathrm{GL}(n, F)$. Then,

$$\dim_{\mathbb{C}}(\pi|_N) = \dim_{\mathbb{C}}(\pi[0]) + \dim_{\mathbb{C}}(\pi[1]) + \dim_{\mathbb{C}}(\pi[2]) + \dim_{\mathbb{C}}(\pi[3]).$$

Since π is cuspidal, we have $\dim_{\mathbb{C}}(\pi[0]) = 0$. From Theorem 3.3 in [2] and Section 5 in [5], it follows that

$$\begin{aligned} \dim_{\mathbb{C}}(\pi[1]) &= (q-1)^2(q^2-1)^2 |\,\mathrm{M}(3,3,1,q)| \\ &= (q-1)(q^2-1)^2(q^3-1)^2, \end{aligned}$$

and

$$\begin{aligned} \dim_{\mathbb{C}}(\pi[3]) &= (q^3-q)(q^3-q^2) |\,\mathrm{M}(3,3,3,q)| \\ &= (q^3-1)(q^3-q)^2(q^3-q^2)^2. \end{aligned}$$

Using the fact that

$$\dim_{\mathbb{C}}(\pi|_N) = (q^5-1)(q^4-1)(q^3-1)(q^2-1)(q-1),$$

we have,

$$\dim_{\mathbb{C}}(\pi[2]) = q^2(q^2-1)^3(q-1)^2(q^2+q+1)^2.$$

Since

$$\dim_{\mathbb{C}}(\pi_{N,\psi_{A_2}}) = \dim_{\mathbb{C}}(\pi_{N,\psi_A}),$$

it follows that

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = q(q-1)(q^2-1)^2.$$

□

4. Structure of M_{ψ_A}

Let ψ_A be the character of $\mathrm{M}(n, F)$ given by

$$\psi_A(X) = \psi_0(\mathrm{Tr}(AX)),$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We let $w_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In this section, we calculate the normalizer M_{ψ_A} of ψ_A.

Lemma 4.1. *Let $M_{\psi_A} = \{m \in M \mid \psi_A^m(n) = \psi_A(n), \forall n \in N\}$. Then we have*

$$M_{\psi_A} = \left\{ \begin{bmatrix} a & X & & \\ 0 & g & & \\ & & g' & Y \\ & & 0 & f \end{bmatrix} \mid a, f \in F^\times, g, g' \in \mathrm{GL}(2), Y, X^T \in F^2 \right\},$$

where $g' = {w_0}^{-1} g w_0$.

Proof. Let $m = \begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \in M$. Then $m \in M_{\psi_A}$ if and only if $Am_1 = m_2 A$. It follows that $m \in M_{\psi_A}$ if and only if $m_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$ and $m_2 = \begin{bmatrix} a_{33} & a_{32} & y_{13} \\ a_{23} & a_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix}$. □

Let

$$U_{M_1} = \left\{ \begin{bmatrix} 1 & X \\ 0 & I \end{bmatrix} \mid X^T \in F^2 \right\} \simeq F^2$$

and

$$U_{M_2} = \left\{ \begin{bmatrix} I & Y \\ 0 & 1 \end{bmatrix} \mid Y \in F^2 \right\} \simeq F^2.$$

Let $U_A = U_{M_1} \times U_{M_2} \simeq F^2 \oplus F^2$. It is easy to see that U_A is a normal subgroup of M_{ψ_A}.

Lemma 4.2. *Let $H_A = U_A \rtimes (F^\times \times \mathrm{GL}(2, F))$. We have,*

$$M_{\psi_A} \simeq F^\times \rtimes H_A,$$

where $F^\times$ is the subgroup of scalar matrices.

Proof. Trivial. □

5. Character calculations of the twisted Jacquet module

In this section, we calculate the character values of π_{N,ψ_A} at an arbitrary element in M_{ψ_A}. Before we proceed, we record a few results we need.

Lemma 5.1. *Let $m = ah \in M_{\psi_A} \simeq F^\times \rtimes H_A$. Then,*

$$\Theta_{N,\psi_A}(m) = \theta(a)\Theta_{N,\psi_A}(h).$$

Proof. We refer the reader to Lemma 5.1 in [2]. □

Proposition 5.2. *Let $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$, where $X \in \mathrm{M}(3, F)$ and $m_1, m_2 \in \mathrm{GL}(3, F)$ are upper triangular unipotent matrices. Let $W' = \mathrm{Ker}(m_2 - 1)$. Then,*

$$\dim(\mathrm{Ker}(h-1)) = \dim(\mathrm{Ker}(m_1 - 1)) + \dim(\mathrm{Ker}(m_2 - 1)) - \dim(XW') + \dim(XW' \cap \mathrm{Im}(m_1 - 1)).$$

Proof. See Proposition 5.5 in [2]. □

Lemma 5.3. *Let $m \in M_{\psi_A}$. For a conjugacy class representative c of $\mathrm{GL}(2)$, let*

$$m_c = \begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c' & Y \\ & & 0 & f \end{bmatrix} \in M_{\psi_A}.$$

Then,

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m_c)$$

for some $m_c \in M_{\psi_A}$.

Proof. Trivial. □

Theorem 5.4. *Let $m_c \in M_{\psi_A}$, where $c = \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}$. Then,*

$$\Theta_{N,\psi_A}(m_c) = 0.$$

Proof. If $a \neq d$ or $d \neq f$, then the characteristic polynomial of m_c is not a power of a polynomial irreducible over F. Thus it follows that

$$\Theta_{N,\psi_A}(m_c) = 0.$$

Suppose that $a = d = f$. Then, $m_c = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$, where

$$m_1 = \begin{bmatrix} d & x & y \\ 0 & d & 1 \\ 0 & 0 & d \end{bmatrix}, m_2 = \begin{bmatrix} d & 0 & u \\ 1 & d & v \\ 0 & 0 & d \end{bmatrix}.$$

Let $h = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$, where

$$h_1 = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, h_2 = \begin{bmatrix} 1 & 0 & u \\ 1 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}.$$

It is enough to show that

$$\Theta_{N,\psi_A}(h) = 0.$$

Let

$$h_2{}' = \begin{bmatrix} 1 & 1 & v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix}, \ \widetilde{\omega_0} = \begin{bmatrix} w_0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{w_0} \end{bmatrix},$$

where $w_0 \in \mathrm{GL}(2, F)$ is the usual long Weyl element. Clearly, $h_2{}' = \widetilde{w_0} h_2 \widetilde{w_0}^{-1}$. Thus we have

$$\begin{aligned} \Theta_{N,\psi_A}(h) &= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \left(\begin{bmatrix} h_1 & X \\ 0 & h_2 \end{bmatrix} \right) \overline{\psi_A({h_1}^{-1} X)} \\ &= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \left(Y^{-1} \begin{bmatrix} h_1 & X \\ 0 & h_2 \end{bmatrix} Y \right) \overline{\psi_A({h_1}^{-1} X)} \\ &= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \left(\begin{bmatrix} h_1 & X \\ 0 & h_2' \end{bmatrix} \right) \overline{\psi_A({h_1}^{-1} X \widetilde{w_0}^{-1})} \end{aligned}$$

Let $u = \begin{bmatrix} h_1 & X \\ 0 & h_2' \end{bmatrix}$, where

$$X = \begin{bmatrix} a & b & c \\ d & e & f \\ p & r & s \end{bmatrix} \in \mathrm{M}(3, F).$$

Write $u = I + \nu$, where

$$\nu = \begin{bmatrix} 0 & x & y & a & b & c \\ 0 & 0 & 1 & d & e & f \\ 0 & 0 & 0 & p & r & s \\ & & & 0 & 1 & v \\ & & & 0 & 0 & u \\ & & & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $\mathrm{M}(3,F) = S_1 \oplus S_2$, where

$$S_1 = \{X = (x_{ij}) \in \mathrm{M}(3,F) \mid x_{22} = 0 = x_{32} \text{ and } x_{ij} \in F \text{ otherwise}\}$$

and

$$S_2 = \{X = (x_{ij}) \in \mathrm{M}(3,F) \mid x_{22}, x_{32} \in F \;\text{ and } x_{ij} = 0 \text{ otherwise}\}.$$

Any $X' \in \mathrm{M}(2,F)$ can be embedded inside $\mathrm{M}(3,F)$ as

$$\begin{bmatrix} 0 & 0 \\ X' & 0 \end{bmatrix} := X',$$

by abuse of notation. Let

$$X_1'^T = \begin{bmatrix} 0 & d & p \end{bmatrix} \text{ and } X_2'^T = \begin{bmatrix} 0 & e & r \end{bmatrix}.$$

Thus we have,

$$X' = \begin{bmatrix} X_1' & X_2' & 0 \end{bmatrix} \in \mathrm{M}(3,F).$$

Let $B = (b_{ij}) \in \mathrm{M}(3,F)$ be such that the first column of B is $-X_2'$ and $b_{ij} = 0$ otherwise. Let

$$I_B = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}.$$

Then,

$$I_B \cdot \nu = \begin{bmatrix} 0 & x & y & b & a & c \\ 0 & 0 & 1 & d & 0 & f - ev \\ 0 & 0 & 0 & p & 0 & s - ru \\ & & & 0 & 1 & v \\ & & & 0 & 0 & u \\ & & & 0 & 0 & 0 \end{bmatrix} := u_B - I.$$

Note that $\dim(\mathrm{Ker}(u - I)) = 6 - \mathrm{Rank}(u - I))$. Since $u = I + \nu$,

$$\mathrm{Rank}(u - I) = \mathrm{Rank}(\nu) = \mathrm{Rank}(I_B \cdot \nu) = \mathrm{Rank}(u_B - I).$$

Thus

$$\dim(\mathrm{Ker}(u - I)) = \dim(\mathrm{Ker}(u_B - I)),$$

and we have

$$\Theta_\pi(u) = \Theta_\pi(u_B).$$

Therefore,

$$\begin{aligned} \Theta_{N,\psi_A}(h) &= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi(u) \overline{\psi_A({h_1}^{-1} X \widetilde{w_0}^{-1})} \\ &= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi(u_B) \overline{\psi_A({h_1}^{-1} X \widetilde{w_0}^{-1})}. \end{aligned}$$

Since $X \in \mathrm{M}(3,F)$, we can write $X = Y_1 + Y_2, Y_1 \in S_1, Y_2 \in S_2$ uniquely. Therefore,

$$\Theta_{N,\psi_A}(h) = \frac{1}{|N|} \sum_{Y_1 \in S_1} \Theta_\theta(u_B) \overline{\psi_A({h_1}^{-1} Y_1 \widetilde{w_0}^{-1})} \left(\sum_{Y_2 \in S_2} \overline{\psi_A({h_1}^{-1} Y_2 \widetilde{w_0}^{-1})} \right).$$

Now, we claim that

$$\sum_{Y_2 \in S_2} \overline{\psi_A({h_1}^{-1} Y_2 \widetilde{w_0}^{-1})} = 0.$$

Since $\mathrm{Tr}(A{h_1}^{-1}Y_2\widetilde{w_0}^{-1}) = r$, we have

$$\begin{aligned}\sum_{Y_2\in S_2}\overline{\psi_A({h_1}^{-1}Y_2\widetilde{w_0}^{-1})} &= \sum_{Y_2\in S_2}\overline{\psi_0(\mathrm{Tr}(A{h_1}^{-1}Y_2\widetilde{w_0}^{-1}))}\\ &= \sum_{e,r\in F}\overline{\psi_0(r)}\\ &= 0.\end{aligned}$$

The result follows. ◻

For $c = \begin{bmatrix} a & 0\\ 0 & a\end{bmatrix}$, a conjugacy class representative of $\mathrm{GL}(2,F)$, consider

$$m_c = \begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c & Y\\ & & 0 & a\end{bmatrix} \in M_{\psi_A}.$$

We have,

$$\Theta_{N,\psi_A}(m_c) = \theta(a)\Theta_{N,\psi_A}(k)$$

where $k = \begin{bmatrix} k_1 & 0\\ 0 & k_2\end{bmatrix} \in M_{\psi_A}$. Thus, the computation of $\Theta_{N,\psi_A}(m_c)$ reduces to the computation of $\Theta_{N,\psi_A}(k)$.

Theorem 5.5. *Let* $k = \begin{bmatrix} k_1 & 0\\ 0 & k_2\end{bmatrix} \in M_{\psi_A}$*, where* $k_1 = \begin{bmatrix} 1 & X\\ 0 & I\end{bmatrix}$ *and* $k_2 = \begin{bmatrix} I & Y\\ 0 & 1\end{bmatrix}$. *Suppose that* $X \neq 0$ *and* $Y = 0$*. Then,*

$$\Theta_{N,\psi_A}(k) = -(q^2-1)(q^2-q).$$

Proof. We have

$$\Theta_{N,\psi_A}(k) = \frac{1}{|N|}\sum_{X\in M(3,F)}\Theta_\pi\begin{bmatrix} k_1 & X\\ 0 & k_2\end{bmatrix}\overline{\psi_A({k_1}^{-1}X)}.$$

It is easy to see that

$$\dim(\mathrm{Ker}(k_1-1)) = 2, \mathrm{Im}(k_1-1) = \mathrm{Span}\{e_1\}$$

and

$$W' = \mathrm{Ker}(k_2-1) = \mathrm{Span}\{e_1,e_2,e_3\}.$$

For $\beta \in F$, let

$$S(\beta) = \{X \in \mathrm{M}(3,F) \mid \mathrm{Tr}(AX) = \beta\}.$$

To calculate the character value, we write

$$\Theta_{N,\psi_A}(k) = \frac{1}{q^9}(C_1 + C_2)$$

where

$$C_1 = \sum_{X\in S(0)}\Theta_\pi\begin{bmatrix} k_1 & X\\ 0 & k_2\end{bmatrix}\overline{\psi_A({k_1}^{-1}X)}$$

and

$$C_2 = \sum_{\beta\in F^\times}\sum_{X\in S(\beta)}\Theta_\pi\begin{bmatrix} k_1 & X\\ 0 & k_2\end{bmatrix}\overline{\psi_A({k_1}^{-1}X)}.$$

To compute C_1 and C_2 we need some calculations which we summarize in Table 1 and Table 2 below. We write $t = \dim(\mathrm{Ker}(h-1))$. For

$$X = \begin{bmatrix} a & b & c \\ d & e & f \\ p & r & s \end{bmatrix}$$

we let

$$X' = \begin{bmatrix} d & e \\ p & r \end{bmatrix}.$$

Using the computations, we have

$$C_1 = K_1 + K_2 + K_3,$$

where

a) $K_1 = \sum_{\substack{X\in S(0) \\ t=5}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right)\overline{\psi_A({k_1}^{-1}X)} = -q^3(1-q)(1-q^2)(1-q^3)(1-q^4).$

b) $K_2 = \sum_{\substack{X\in S(0) \\ t=4}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right)\overline{\psi_A({k_1}^{-1}X)} = -q^3(q-1)(q+1)^2(1-q)(1-q^2)(1-q^3).$

c) $K_3 = \sum_{\substack{X\in S(0) \\ t=3}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right)\overline{\psi_A({k_1}^{-1}X)} = -q^4(q-1)(q^3+q^2-1)(1-q)(1-q^2).$

It follows that

$$C_1 = \sum_{X\in S(0)} \Theta_\pi \begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix} \overline{\psi_A({k_1}^{-1}X)} = -q^9(q-1)^3(q+1). \tag{5.1}$$

Similarly, we have

$$C_2 = K_4 + K_5,$$

where

a) $K_4 = \sum_{\beta\in F^\times} \sum_{\substack{X\in S(\beta) \\ t=4}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right)\overline{\psi_A({k_1}^{-1}X)} = q^5(q+1)(1-q)(1-q^2)(1-q^3)$

b) $K_5 = \sum_{\beta\in F^\times} \sum_{\substack{X\in S(\beta) \\ t=3}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right)\overline{\psi_A({k_1}^{-1}X)} = q^5(q^3-q-1)(1-q)(1-q^2).$

It follows that

$$C_2 = \sum_{\beta\in F^\times} \sum_{X\in S(\beta)} \Theta_\pi \begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix} \overline{\psi_A({k_1}^{-1}X)} = -q^9(q-1)^2(q+1). \tag{5.2}$$

From (5.1) and (5.2), it follows that

$$\Theta_{N,\psi_A}(k) = -(q^2-q)(q^2-1).$$

□

TABLE 1. Calculations for computing C_1

Partition of $S(0)$	Type of matrix	$\dim(\mathrm{Ker}(h-1))$	Cardinality
$\mathrm{Rank}(X)=0$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$2+3-0+0=5$	1
$\mathrm{Rank}(X)=1$ $\mathrm{Rank}(X')=0$	$\begin{bmatrix}a&b&c\\0&0&0\\0&0&0\end{bmatrix}$	$2+3-1+1=5$	(q^3-1)
	$\begin{bmatrix}0&0&c\\0&0&f\\0&0&s\end{bmatrix}, (f,s)\neq(0,0)$	$2+3-1+0=4$	$(q^2-1)q$
$\mathrm{Rank}(X)=1$ $\mathrm{Rank}(X')=1$	$\begin{bmatrix}0&b&c\\0&0&0\\0&r&s\end{bmatrix}, r\in F^\times$	$2+3-1+0=4$	$(q-1)q^2$
	$\left\{\begin{bmatrix}a&b&c\\d&e&f\\-e&r&s\end{bmatrix}\middle\vert \substack{d\in F^\times\\ dr+e^2=0}\right\}$	$2+3-1+0=4$	$(q-1)q^3$
$\mathrm{Rank}(X)=2$ $\mathrm{Rank}(X')=0$	$\begin{bmatrix}a&b&c\\0&0&f\\0&0&s\end{bmatrix}$	$2+3-2+1=4$	$(q^2-1)^2q$
$\mathrm{Rank}(X)=2$ $\mathrm{Rank}(X')=1$	$\begin{bmatrix}a&b&c\\0&0&0\\0&r&s\end{bmatrix}, r\in F^\times$	$2+3-2+1=4$	$(q-1)q(q^3-q)$
	$\begin{bmatrix}a&b&c\\0&0&f\\0&r&s\end{bmatrix}, r,f\in F^\times$	$2+3-2+0=3$	$(q-1)^2q(q^2)$
	$\left\{\begin{bmatrix}a&b&c\\d&e&f\\-e&r&s\end{bmatrix}\middle\vert \substack{d\in F^\times\\ dr+e^2=0\\ (f,s)\in\mathrm{Span}\,(e,r)}\right\}$	$2+3-2+1=4$	$(q-1)q^2(q^3-q)$
	$\left\{\begin{bmatrix}a&b&c\\d&0&f\\0&0&s\end{bmatrix}\middle\vert \substack{d\in F^\times\\ (f,s)\in\mathrm{Span}\,(d,0)}\right\}$	$2+3-2+1=4$	$(q-1)^2q^2(q+1)$
	$\left\{\begin{bmatrix}a&b&c\\d&e&f\\-e&r&s\end{bmatrix}\middle\vert \substack{d,e\in F^\times\\ dr+e^2=0\\ (f,s)\notin\mathrm{Span}\,(e,r)}\right\}$	$2+3-2+0=3$	$(q-1)^2q^2(q^2-q-1)$
$\mathrm{Rank}(X)=2$ $\mathrm{Rank}(X')=2$	$\left\{\begin{bmatrix}a&b&c\\d&e&f\\-e&0&s\end{bmatrix}\middle\vert e\in F^\times\right\}$	$2+3-2+0=3$	$(q-1)q^5$
	$\left\{\begin{bmatrix}a&b&c\\d&e&f\\-e&r&s\end{bmatrix}\middle\vert \substack{r\in F^\times\\ dr+e^2\neq0}\right\}$	$2+3-2+0=3$	$(q-1)^2q^5$
$\mathrm{Rank}(X)=3$ $\mathrm{Rank}(X')=1$	$\left\{\begin{bmatrix}a&b&c\\0&0&f\\0&r&s\end{bmatrix}\middle\vert r\in F^\times\right\}$	$2+3-3+1=3$	$(q-1)^2(q^2-q)q^2$
	$\left\{\begin{bmatrix}a&b&c\\d&e&f\\-e&r&s\end{bmatrix}\middle\vert \substack{d\in F^\times\\ dr+e^2=0}\right\}$	$2+3-3+1=3$	$(q-1)^3q^4$
$\mathrm{Rank}(X)=3$ $\mathrm{Rank}(X')=2$	$\left\{\begin{bmatrix}a&b&c\\d&e&f\\-e&0&s\end{bmatrix}\middle\vert e\neq0\right\}$	$2+3-3+1=3$	$(q-1)^2q^5$
	$\left\{\begin{bmatrix}a&b&c\\d&e&f\\-e&r&s\end{bmatrix}\middle\vert \substack{r\in F^\times\\ dr+e^2\neq0}\right\}$	$2+3-3+1=3$	$(q-1)^3q^5$

TABLE 2. Computations for calculating C_2

Partition of $S(\beta)$	Type of matrix	$\dim(\mathrm{Ker}(h-1))$	Cardinality
$\mathrm{Rank}(X)=1$ $\mathrm{Rank}(X')=1$	$\begin{bmatrix} a & b & c \\ 0 & \beta & 0 \\ 0 & r & s \end{bmatrix}$	$2+3-1+0=4$	q^3
	$\begin{bmatrix} a & b & c \\ 0 & 0 & f \\ \beta & r & s \end{bmatrix}$	$2+3-1+0=4$	q^3
	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta-e & r & s \end{bmatrix} \;\middle\vert\; \substack{d\neq 0 \\ r=e(\beta-e)d^{-1}} \right\}$	$2+3-1+0=4$	$(q-1)q^3$
$\mathrm{Rank}(X)=2$ $\mathrm{Rank}(X')=1$	$\begin{bmatrix} 0 & b & c \\ 0 & \beta & f \\ 0 & r & s \end{bmatrix}$	$2+3-2+0=3$	$q^3(q^2-q)$
	$\begin{bmatrix} a & b & c \\ 0 & \beta & f \\ 0 & r & s \end{bmatrix}$	$2+3-2+1=4$	$q^2(q^3-q)$
	$\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ \beta & r & s \end{bmatrix}$	$2+3-2+1=4$	$q^2(q^3-q)$
	$\left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & f \\ \beta & r & s \end{bmatrix} \;\middle\vert\; f\in F^\times \right\}$	$2+3-2+0=3$	$q^4(q-1)$
	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta-e & r & s \end{bmatrix} \;\middle\vert\; \substack{d\in F^\times \\ r=e(e-\beta)d^{-1} \\ (f,s)\in\mathrm{Span}\,(e,r)} \right\}$	$2+3-2+1=4$	$q^2(q-1)(q^3-q)$
	$\left\{ \begin{bmatrix} a & b & c \\ d & 0 & f \\ \beta & 0 & s \end{bmatrix} \;\middle\vert\; \substack{d\in F^\times \\ (f,s)\in\mathrm{Span}\,(d,\beta)} \right\}$	$2+3-2+1=4$	$(q-1)^2q^2(q+1)$
	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta-e & r & s \end{bmatrix} \;\middle\vert\; \substack{d,e\in F^\times \\ r=e(e-\beta)d^{-1} \\ (f,s)\notin\mathrm{Span}\,(e,r)} \right\}$	$2+3-2+0=3$	$(q-1)^2q^2(q^2-q-1)$
$\mathrm{Rank}(X)=2$ $\mathrm{Rank}(X')=2$	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta-e & 0 & s \end{bmatrix} \;\middle\vert\; e\in F^\times, e\neq\beta \right\}$	$2+3-2+0=3$	$(q-2)q^5$
	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta-e & r & s \end{bmatrix} \;\middle\vert\; r\in F^\times \right\}$	$2+3-2+0=3$	$(q-1)^2q^5$
$\mathrm{Rank}(X)=3$ $\mathrm{Rank}(X')=1$	$\begin{bmatrix} a & b & c \\ 0 & \beta & f \\ 0 & r & s \end{bmatrix}$	$2+3-3+1=3$	$2q(q^2-q)(q^3-q^2)$
	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta-e & r & s \end{bmatrix} \;\middle\vert\; \substack{d\in F^\times \\ r=e(e-\beta)d^{-1}} \right\}$	$2+3-3+1=3$	$(q-1)^3q^4$
$\mathrm{Rank}(X)=3$ $\mathrm{Rank}(X')=2$	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta-e & 0 & s \end{bmatrix} \;\middle\vert\; e\neq 0, e\neq\beta \right\}$	$2+3-3+1=3$	$(q-2)(q-1)q^5$
	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta-e & r & s \end{bmatrix} \;\middle\vert\; r\in F^\times \right\}$	$2+3-3+1=3$	$(q-1)^3q^5$

Theorem 5.6. *Let* $k = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \in M_{\psi_A}$, *where* $k_1 = \begin{bmatrix} 1 & X \\ 0 & I \end{bmatrix}$ *and* $k_2 = \begin{bmatrix} I & Y \\ 0 & 1 \end{bmatrix}$. *Suppose that* $X = 0$ *and* $Y \neq 0$. *Then,*

$$\Theta_{N,\psi_A}(k) = -(q^2 - 1)(q^2 - q).$$

Proof. Let $m_1 = w_0 {k_2}^T {w_0}^{-1}$ and $m_2 = w_0 {k_1}^T {w_0}^{-1} = I$, where w_0 is the usual long Weyl element of $\mathrm{GL}(3, F)$. Using Theorem 5.5, it is enough to show that

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(k).$$

We have,

$$\begin{aligned} |N|\Theta_{N,\psi_A}(k) &= \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \begin{bmatrix} w_0 {m_2}^T {w_0}^{-1} & X \\ 0 & w_0 {m_1}^T {w_0}^{-1} \end{bmatrix} \overline{\psi_A(X)} \\ &= \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \left(\begin{bmatrix} w_0 & 0 \\ 0 & w_0 \end{bmatrix} \begin{bmatrix} {m_2}^T & {w_0}^{-1} X w_0 \\ 0 & {m_1}^T \end{bmatrix} \begin{bmatrix} w_0^{-1} & 0 \\ 0 & w_0^{-1} \end{bmatrix} \right) \overline{\psi_A(X)} \\ &= \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \left(\begin{bmatrix} {m_2}^T & {w_0}^{-1} X w_0 \\ 0 & {m_1}^T \end{bmatrix} \right) \overline{\psi_A(X)}. \end{aligned}$$

Using the fact that $\mathrm{Tr}(A {w_0}^{-1} X^T w_0) = \mathrm{Tr}(AX)$, and $\psi_A({m_1}^{-1} X) = \psi_A(X)$, it follows that

$$\begin{aligned} |N|\Theta_{N,\psi_A}(m) &= \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X)} \\ &= \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \begin{bmatrix} m_1 & X^T \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X^T)} \\ &= \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \begin{bmatrix} m_1 & {w_0}^{-1} X^T w_0 \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({w_0}^{-1} X^T w_0)} \\ &= \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi \begin{bmatrix} m_1 & {w_0}^{-1} X^T w_0 \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X)}. \end{aligned}$$

Since,

$$\begin{aligned} \mathrm{Rank}\left(\begin{bmatrix} {m_2}^T - I & {w_0}^{-1} X w_0 \\ 0 & {m_1}^T - I \end{bmatrix} \right) &= \mathrm{Rank}\left(\begin{bmatrix} m_2 - I & 0 \\ {w_0}^{-1} X^T w_0 & m_1 - I \end{bmatrix} \right) \\ &= \mathrm{Rank}\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} m_2 - I & 0 \\ {w_0}^{-1} X^T w_0 & m_1 - I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right) \\ &= \mathrm{Rank}\left(\begin{bmatrix} m_1 - I & {w_0}^{-1} X^T w_0 \\ 0 & m_2 - I \end{bmatrix} \right) \end{aligned}$$

it follows that

$$\Theta_\pi \left(\begin{bmatrix} m_1 & {w_0}^{-1} X^T w_0 \\ 0 & m_2 \end{bmatrix} \right) = \Theta_\pi \left(\begin{bmatrix} {m_2}^T & {w_0}^{-1} X w_0 \\ 0 & {m_1}^T \end{bmatrix} \right).$$

Hence the result. □

Theorem 5.7. *Let* $k = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \in M_{\psi_A}$, *where* $k_1 = \begin{bmatrix} 1 & X \\ 0 & I \end{bmatrix}$ *and* $k_2 = \begin{bmatrix} I & Y \\ 0 & 1 \end{bmatrix}$. *Suppose that* $X \neq 0$ *and* $Y \neq 0$. *Then,*

$$\Theta_{N,\psi_A}(k) = (q^2 - q).$$

Proof. We have

$$\Theta_{N,\psi_A}(k) = \frac{1}{|N|} \sum_{X\in M(3,F)} \Theta_\pi \begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix} \overline{\psi_A({k_1}^{-1}X)}.$$

It is easy to see that

$$\dim(\mathrm{Ker}(k_1-1)) = 2, \mathrm{Im}(k_1-1) = \mathrm{Span}\{e_1\}$$

and

$$W' = \mathrm{Ker}(k_2-1) = \mathrm{Span}\{e_1, e_2\}.$$

To calculate the character value, we write

$$\Theta_{N,\psi_A}(k) = \frac{1}{q^9}(B_1+B_2)$$

where

$$B_1 = \sum_{X\in S(0)} \Theta_\pi \begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix} \overline{\psi_A({k_1}^{-1}X)}$$

and

$$B_2 = \sum_{\beta\in F^\times} \sum_{X\in S(\beta)} \Theta_\pi \begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix} \overline{\psi_A({k_1}^{-1}X)}.$$

To compute B_1 and B_2 we need some calculations which we summarize in Table 3 and Table 4 below. We write $t = \dim(\mathrm{Ker}(h-1))$. For

$$X = \begin{bmatrix} a & b & c \\ d & e & f \\ p & r & s \end{bmatrix}$$

we let

$$X' = \begin{bmatrix} d & e \\ p & r \end{bmatrix}.$$

Using the computations, we have

$$B_1 = K_1 + K_2 + K_3,$$

where

a) $K_1 = \sum_{\substack{X\in S(0)\\ t=4}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right) \overline{\psi_A({k_1}^{-1}X)} = -q^5(1-q)(1-q^2)(1-q^3).$

b) $K_2 = \sum_{\substack{X\in S(0)\\ t=3}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right) \overline{\psi_A({k_1}^{-1}X)} = -q^5(q^2-1)(1-q)(1-q^2).$

c) $K_3 = \sum_{\substack{X\in S(0)\\ t=2}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right) \overline{\psi_A({k_1}^{-1}X)} = -q^7(q-1)(1-q).$

Hence,

$$B_1 = \sum_{X\in S(0)} \Theta_\pi \begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix} \overline{\psi_A({k_1}^{-1}X)} = q^9(q-1)^2. \tag{5.3}$$

Similarly, we have

$$B_2 = K_4 + K_5,$$

where

a) $K_4 = \sum_{\beta\in F^\times} \sum_{\substack{X\in S(\beta)\\ t=3}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right) \overline{\psi_A({k_1}^{-1}X)} = q^6(q+1)(1-q)(1-q^2)$

b) $K_5 = \sum_{\beta\in F^\times} \sum_{\substack{X\in S(\beta)\\ t=2}} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2\end{bmatrix}\right) \overline{\psi_A({k_1}^{-1}X)} = q^6(q^2-q-1)(1-q).$

It follows that

$$B_2 = \sum_{\beta\in F^\times} \sum_{X\in S(\beta)} \Theta_\pi \begin{bmatrix} k_1 & X \\ 0 & k_2\end{bmatrix} \overline{\psi_A({k_1}^{-1}X)} = q^9(q-1). \quad (5.4)$$

From Equations (5.3) and (5.4), it follows that

$$\Theta_{N,\psi_A}(k) = (q^2-q).$$

TABLE 3. Computations for calculating B_1

Partition of $S(0)$	Type of matrix	dim(Ker($h-1$))	Cardinality
Rank$(X')=0$	$\begin{bmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & s\end{bmatrix}$	$2+2-0+0=4$	q^3
	$\left\{\begin{bmatrix} a & b & c \\ 0 & 0 & f \\ 0 & 0 & s\end{bmatrix} \middle\vert (a,b)\neq 0\right\}$	$2+2-1+1=4$	$(q^2-1)q^3$
Rank$(X')=1$	$\left\{\begin{bmatrix} a & b & c \\ d & e & f \\ -e & r & s\end{bmatrix} \middle\vert \substack{e^2+dr=0 \\ \text{Rank}\begin{pmatrix} a & b \\ d & e\end{pmatrix}\leq 1}\right\}$	$2+2-1+0=3$	$(q^2-1)q^4$
	$\left\{\begin{bmatrix} a & b & c \\ 0 & 0 & f \\ 0 & r & s\end{bmatrix} \middle\vert r,a\neq 0\right\}$	$2+2-2+1=3$	$(q-1)^2q^4$
	$\left\{\begin{bmatrix} a & b & c \\ d & e & f \\ -e & r & s\end{bmatrix} \middle\vert \substack{e^2+dr=0, d\neq 0 \\ \text{Rank}\begin{pmatrix} a & b \\ d & e\end{pmatrix}=2}\right\}$	$2+2-2+1=3$	$(q-1)^2q^5$
Rank$(X')=2$	$\left\{\begin{bmatrix} a & b & c \\ d & e & f \\ -e & r & s\end{bmatrix} \middle\vert dr+e^2\neq 0\right\}$	$2+2-2+0=2$	$(q-1)q^7$

TABLE 4. Computations for calculating B_2

Partition of $S(\beta)$	Type of matrix	$\dim(\mathrm{Ker}(h-1))$	Cardinality
$\mathrm{Rank}(X') = 1$	$\begin{bmatrix} 0 & b & c \\ 0 & \beta & f \\ 0 & r & s \end{bmatrix}$	$2+2-1+0=3$	q^5
	$\begin{bmatrix} a & b & c \\ 0 & \beta & f \\ 0 & r & s \end{bmatrix}$	$2+2-2+1=3$	$q(q^2-q)q^3$
	$\begin{bmatrix} a & b & c \\ 0 & 0 & f \\ \beta & r & s \end{bmatrix}$	$2+2-1+0=3$	q^5
	$\begin{bmatrix} a & b & c \\ 0 & 0 & f \\ \beta & r & s \end{bmatrix}$	$2+2-2+1=3$	$q(q^2-q)q^3$
	$\left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ \beta - e & r & s \end{bmatrix} \middle\vert \substack{d \neq 0, r = e(\beta - e)d^{-1} \\ \mathrm{Rank}\begin{pmatrix} a & b \\ d & e \end{pmatrix} = 1} \right\}$	$2+2-1+0=3$	$(q-1)q^5$
	$\left\{ \begin{bmatrix} a & b & cd & e & f \\ \beta - e & r & s & & \end{bmatrix} \middle\vert \substack{d \neq 0, r = e(\beta - e)d^{-1} \\ \mathrm{Rank}\begin{pmatrix} a & b \\ d & e \end{pmatrix} = 2} \right\}$	$2+2-2+1=3$	$(q-1)(q^2-q)q^4$
$\mathrm{Rank}(X') = 2$	$\begin{bmatrix} a & b & c \\ d & e & f \\ \beta - e & r & s \end{bmatrix}$	$2+2-2+0=2$	$q^5(q^3-q^2-q)$

□

5.1. **Steps for computing the character values of π_{N,ψ_A}.** In this section, we explain the steps that we need to compute the character values for π_{N,ψ_A}. Let

$$m_c = \begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c & Y \\ & & 0 & f \end{bmatrix} \in M_{\psi_A}$$

where c is a conjugacy class representative of GL(2). Let $h_c = m_c n, n \in N$. By Proposition 2.3, we have

$$\Theta_{N,\psi_A}(m_c) = \frac{1}{|N|} \sum_{n \in N} \Theta_\pi(h_c)\overline{\psi_A(n)}.$$

If the characteristic polynomial of h_c is not a power of a polynomial irreducible over F, we can conclude using Theorem 2.1 that

$$\Theta_{N,\psi_A}(m_c) = 0.$$

Suppose that the characteristic polynomial of h_c is a power of a polynomial irreducible over F. We proceed in the following manner.

(1) Let $c = \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}$. Suppose also that $a = d = f$. Using Theorem 5.4, we have

$$\Theta_{N,\psi_A}(m_c) = 0.$$

(2) Let $c = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Suppose also that $a = f$. From Lemma 5.1, we have

$$\Theta_{N,\psi_A}(m_c) = \theta(a)\Theta_{N,\psi_A}(k),$$

where $k = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ with $k_1 = \begin{bmatrix} 1 & Z \\ 0 & I \end{bmatrix}$ and $k_2 = \begin{bmatrix} I & Y \\ 0 & 1 \end{bmatrix}$, $Z^T, Y \in F^2$.

(a) We partition the set

$$\mathrm{M}(3,F) = S(0) \bigsqcup_{\beta \in F^\times} S(\beta),$$

where

$$S(0) = \{X \in \mathrm{M}(3,F) \mid \mathrm{Tr}(A{k_1}^{-1}X) = 0\}$$

and

$$S(\beta) = \{X \in \mathrm{M}(3,F) \mid \mathrm{Tr}(A{k_1}^{-1}X) = \beta\}.$$

(b) We write

$$\begin{aligned} \Theta_{N,\psi_A}\left(\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\right) &= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right) \overline{\psi_A({k_1}^{-1}X)} \\ &= \frac{1}{|N|}(C_1 + C_2) \end{aligned}$$

where

$$C_1 = \sum_{X \in S(0)} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right) \overline{\psi_0(0)}$$

and

$$C_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\pi\left(\begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}\right) \overline{\psi_0(\beta)}.$$

(c) Let $h = \begin{bmatrix} k_1 & X \\ 0 & k_2 \end{bmatrix}$. We compute $\dim(\mathrm{Ker}(k_1 - 1), \dim(\mathrm{Ker}(k_2 - 1))$ and the space of $W' = \mathrm{Ker}(k_2 - 1))$ and $\mathrm{Im}(k_1 - 1)$.

(d) For $X = (x_{ij}) \in \mathrm{M}(3,F)$, we denote $X' \in \mathrm{M}(2,F)$ to be the submatrix

$$X' = \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$$

of X. For $0 \le j \le i \le 3, j < 3$ and $\beta \in F$, we let

$$S(\beta)_i^j = \{X \in S(\beta) \mid \mathrm{Rank}(X) = i, \mathrm{Rank}(X') = j\}.$$

We show that

$$S(0) = \bigsqcup_{\substack{0 \le i \le 3 \\ j \le i, j < 3}} S(0)_i^j.$$

(e) For $X \in S(0)_i^j$, using Proposition 5.2 we compute $\dim(\mathrm{Ker}(h - 1))$. Using Theorem 2.2, we compute

$$C_1 = \sum_{t=1}^{6} \sum_{\substack{X \in S(0) \\ \dim(\mathrm{Ker}(h-1)) = t}} (-1)\theta(1)(1-q)(1-q^2)\cdots(1-q^{t-1})\overline{\psi_0(0)}.$$

In a similar way can can compute C_2.

(f) Using (b) above we can compute $\Theta_{N,\psi_A}\left(\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\right)$.

Table 5. Character values for the twisted Jacquet module

Cl(GL(2))	m	$\Theta_{N,\psi_A}(m)$
$c = \begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix}, p \neq r$	$\begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c' & Y \\ & & 0 & f \end{bmatrix}$	0
$c = \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}$	$\begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c' & Y \\ & & 0 & f \end{bmatrix}$	$\left\{ \begin{array}{ll} 0 & ;\text{if } d \neq a \\ & \text{or } d \neq f \\ 0 & ;\text{if } a = d = f \end{array} \right\}$
$c = \begin{bmatrix} p & \delta r \\ r & p \end{bmatrix}$	$\begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c' & Y \\ & & 0 & f \end{bmatrix}$	0
$c = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$	$\begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c' & Y \\ & & 0 & f \end{bmatrix}$	$\left\{ \begin{array}{ll} 0 & ;\text{if } p \neq a \\ & \text{or } p \neq f \end{array} \right\}$
$c = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$	$\begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c' & Y \\ & & 0 & a \end{bmatrix}$	$\left\{ \begin{array}{ll} -\theta(a)(q^2-1)(q^2-q) & ;\text{if } X^T \neq \mathbf{0}, \mathbf{Y} = \mathbf{0} \\ & \text{or } Y \neq \mathbf{0}, \mathbf{X} = \mathbf{0} \end{array} \right\}$
$c = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$	$\begin{bmatrix} a & X & & \\ 0 & c & & \\ & & c' & Y \\ & & 0 & a \end{bmatrix}$	$\theta(a)(q^2-q)$ if $X^T, Y \neq \mathbf{0}$

6. The Representation τ_1

Let ψ_1 be the character of U_A given by

$$\psi_1\left(\begin{bmatrix} 1 & X & & \\ 0 & I & & \\ & & I & Y \\ & & 0 & 1 \end{bmatrix}\right) = \psi_0(x+v)$$

where $X^T = \begin{bmatrix} x \\ y \end{bmatrix}$ and $Y = \begin{bmatrix} u \\ v \end{bmatrix}$. The inertia subgroup of ψ_1 in H_A is given by

$$\begin{aligned} I_1 = I_{H_A}(\psi_1) &= \{h \in H_A \mid {}^h\psi_1 \simeq \psi_1\} \\ &= \{h \in H_A \mid \psi_1(h^{-1}uh) = \psi_1(u), \forall u \in U_A\} \\ &= U_A \rtimes F^\times. \end{aligned}$$

To be precise, we have

$$I_1 = I_{H_A}(\psi_1) = \left\{ \begin{bmatrix} 1 & x & y & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & a & & & \\ & & & a & 0 & u \\ & & & 0 & 1 & v \\ & & & 0 & 0 & 1 \end{bmatrix} \mid a \in F^\times, x, y, u, v \in F \right\}.$$

Using Theorem 2.5, Theorem 2.6 above, we have that

$$\tau_1 = \mathrm{Ind}_{F^\times \ltimes U_A}^{H_A}(\widetilde{\psi_1} \otimes \overline{\mathrm{reg}(F^\times)})$$

is a representation of H_A, where

$$\widetilde{\psi_1}(au) = \psi_1(u)$$

and

$$\overline{\mathrm{reg}(F^\times)}(au) = \mathrm{reg}(F^\times)(a)$$

(reg is the regular representation of $F^\times$).

Lemma 6.1. *Let $\eta \in \widehat{F^\times}$. Then,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_A}(\pi_{N,\psi_A}|_{H_A}, \mathrm{Ind}_{I_1}^{H_A}(\widetilde{\psi_1} \otimes \overline{\eta})) = q.$$

Proof. We have

$$\begin{aligned}
\langle \Theta_{N,\psi_A}|_{I_1}, \chi_{\widetilde{\psi_1}\otimes\overline{\eta}} \rangle &= \frac{1}{|I_1|} \sum_{k \in I_1} \Theta_{N,\psi_A}(k) (\overline{\chi_{\widetilde{\psi_1}\otimes\overline{\eta}})(k)} \\
&= \frac{1}{|I_1|} \left[\sum_{k \in I_1 \setminus U_A} \Theta_{N,\psi_A}(k)(\overline{\chi_{\widetilde{\psi_1}\otimes\overline{\eta}})(k)} + \sum_{k \in U_A} \Theta_{N,\psi_A}(k)\overline{\psi_1(k)}\,\overline{\eta(1)} \right] \\
&= \frac{1}{|I_1|} \left[0 + \sum_{k \in U_A} \Theta_{N,\psi_A}(k)\overline{\psi_1(k)} \right] \\
&= \frac{1}{|I_1|}(K_1 + K_2 + K_3)
\end{aligned}$$

where

$$K_1 = \sum_{\substack{k \in U_A; \\ X \neq 0, Y = 0}} \Theta_{N,\psi_A}(k)\overline{\psi_1(k)} \quad + \quad \sum_{\substack{k \in U_A; \\ Y \neq 0, X = 0}} \Theta_{N,\psi_A}(k)\overline{\psi_1(k)},$$

$$K_2 = \sum_{\substack{k \in U_A; \\ X, Y \neq 0}} \Theta_{N,\psi_A}(k)\overline{\psi_1(k)} \quad \text{and} \quad K_3 = \Theta_{N,\psi_A}(1)\overline{\psi_1(1)}.$$

Clearly,

$$K_3 = (q^2 - 1)^2(q^2 - q).$$

To compute K_1, it is enough to show that

$$\sum_{\substack{k \in U_A; \\ X \neq 0, Y = 0}} \overline{\psi_1(k)} = -1. \tag{6.1}$$

Indeed, we have

$$\begin{aligned}
\sum_{\substack{k \in U_A; \\ X \neq 0, Y = 0}} \overline{\psi_1(k)} &= \sum_{x = 0, y \neq 0} \overline{\psi_0(0)} + \sum_{x \neq 0, y \in F} \overline{\psi_0(x)} \\
&= (q - 1) + q(-1) \\
&= -1.
\end{aligned}$$

Similarly, we have

$$\sum_{\substack{k \in U_A; \\ Y \neq 0, X = 0}} \overline{\psi_1(k)} = -1. \tag{6.2}$$

From Equations 6.1 and 6.2 and row 5 of Table 5, it follows that

$$\begin{aligned} K_1 &= -(q^2-1)(q^2-q)\left[\sum_{\substack{k\in U_A;\\ X\neq 0, Y=0}} \overline{\psi_1(k)} \;+\; \sum_{\substack{k\in U_A;\\ Y\neq 0, X=0}} \overline{\psi_1(k)}\right] \\ &= 2(q^2-1)(q^2-q). \end{aligned}$$

To compute K_2, we need

$$\sum_{\substack{k\in U_A;\\ X,Y\neq 0}} \overline{\psi_1(k)} = \sum_{X,Y\neq 0} \overline{\psi_0(x+v)} = A+B,$$

where

$$\begin{aligned} A &= \sum_{\substack{X,Y\neq 0;\\ x+v=0}} \overline{\psi_0(0)} \\ &= \sum_{\substack{x,v=0\\ y,u\neq 0}} \overline{\psi_0(0)} \;+\; \sum_{\substack{x\neq 0, v=-x\\ y,u\in F}} \overline{\psi_0(0)} \\ &= (q-1)(q^2+q-1) \end{aligned}$$

and

$$\begin{aligned} B &= \sum_{\alpha\in F^\times} \sum_{\substack{X,Y\neq 0;\\ x+v=\alpha}} \overline{\psi_0(\alpha)} \\ &= \sum_{\alpha\in F^\times}\left[\sum_{\substack{x=0,v=\alpha\\ y\neq 0,u\in F}} \overline{\psi_0(\alpha)} \;+\; \sum_{\substack{v=0,x=\alpha\\ u\neq 0,y\in F}} \overline{\psi_0(\alpha)} \;+\; \sum_{\substack{x\neq 0,x\neq\alpha,\\ v=\alpha-x,y,u\in F}} \overline{\psi_0(\alpha)}\right] \\ &= \sum_{\alpha\in F^\times} \overline{\psi_0(\alpha)}\left[2(q-1)q+(q-2)q^2\right] \\ &= -(q^3-2q) \end{aligned}$$

Thus, we have

$$\sum_{\substack{k\in U_A;\\ X,Y\neq 0}} \overline{\psi_1(k)} = 1. \tag{6.3}$$

From Equation 6.3 and row 6 of Table 5, it follows that

$$K_2 = (q^2-q).$$

Hence,

$$\begin{aligned} \langle \Theta_{N,\psi_A}|_{I_1}, \chi_{\widetilde{\psi_1}\otimes\overline{\eta}}\rangle &= \frac{1}{|I_1|}(K_1+K_2+K_3) \\ &= \frac{1}{(q-1)q^4}(K_1+K_2+K_3) \\ &= \frac{1}{(q-1)q^4}(q^2-q)q^4 \\ &= q. \end{aligned}$$

Using Frobenius reciprocity, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_A}(\pi_{N,\psi_A}|_{H_A}, \mathrm{Ind}_{I_1}^{H_A}(\widetilde{\psi_1}\otimes\overline{\eta})) = q.$$

□

7. The Representation τ_2

Let ψ_2 be the character of U_A given by

$$\psi_2\left(\begin{bmatrix} 1 & X & & \\ 0 & I & & \\ & & I & Y \\ & & 0 & 1 \end{bmatrix}\right) = \psi_0(x+u)$$

where $X^T = \begin{bmatrix} x \\ y \end{bmatrix}$ and $Y = \begin{bmatrix} u \\ v \end{bmatrix}$. The inertia subgroup of ψ_2 in H_A is given by

$$\begin{aligned} I_2 = I_{H_A}(\psi_2) &= \{h \in H_A \mid {}^h\psi_2 \simeq \psi_2\} \\ &= \{h \in H_A \mid \psi_2(h^{-1}uh) = \psi_2(u), \quad \forall u \in U_A\} \\ &= U_A \rtimes \mathit{Mir}, \end{aligned}$$

where Mir is the mirabolic subgroup of GL(2). To be precise, we have

$$I_2 = I_{H_A}(\psi_2) = \left\{ \begin{bmatrix} a & x & y & & & \\ 0 & a & r & & & \\ 0 & 0 & 1 & & & \\ & & & 1 & 0 & u \\ & & & r & a & v \\ & & & 0 & 0 & 1 \end{bmatrix} \mid a \in F^\times, r, x, y, u, v \in F \right\}.$$

Using Theorem 2.5, we have that

$$\tau_2 = \mathrm{Ind}_{U_A \rtimes \mathit{Mir}}^{H_A}(\widetilde{\psi_2} \otimes \overline{\mathrm{reg}(\mathit{Mir})})$$

is a representation of H_A, where

$$\widetilde{\psi_2}(gu) = \psi_2(u)$$

and

$$\overline{\mathrm{reg}(\mathit{Mir})}(gu) = \mathrm{reg}(\mathit{Mir})(g), \quad \forall g \in \mathit{Mir}, \ u \in U_A.$$

(here reg is the regular representation of Mir).

Let ρ be the unique irreducible representation of Mir of degree $q-1$. To compute the multiplicity of τ_2 in $\pi_{N,\psi_A|_{H_A}}$, we have to find the multiplicity of $\widetilde{\psi_2} \otimes \overline{\rho}$ and $\widetilde{\psi_2} \otimes \overline{\eta}$ for each $\eta \in \widehat{F^\times}$ inside the restriction of π_{N,ψ_A} to I_2. We record it in the following lemmas.

Lemma 7.1. *We have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{I_2}(\pi_{N,\psi_A}|_{I_2}, \widetilde{\psi_2} \otimes \overline{\rho}) = q-1.$$

Proof. We have

$$\begin{aligned} \langle \Theta_{N,\psi_A}|_{I_2}, \chi_{\widetilde{\psi_2}\otimes\overline{\rho}} \rangle &= \frac{1}{|I_2|} \sum_{k \in I_2} \Theta_{N,\psi_A}(k)(\overline{\chi_{\widetilde{\psi_2}\otimes\overline{\rho}}})(k) \\ &= \frac{1}{|I_2|}\left[\sum_{k \in I_2 \setminus U_A} \Theta_{N,\psi_A}(k)(\overline{\chi_{\widetilde{\psi_2}\otimes\overline{\rho}})(k)} + \sum_{k \in U_A} \Theta_{N,\psi_A}(k)\overline{\psi_2(k)\chi_\rho(1)} \right] \\ &= \frac{1}{|I_2|}\left[0 + (q-1) \sum_{k \in U_A} \Theta_{N,\psi_A}(k)\overline{\psi_2(k)} \right] \\ &= \frac{(q-1)}{|I_2|}(C_1 + C_2 + C_3), \end{aligned}$$

where

$$C_1 = \sum_{\substack{k\in U_A;\\ X\neq 0, Y=0}} \Theta_{N,\psi_A}(k)\overline{\psi_2(k)} + \sum_{\substack{k\in U_A;\\ Y\neq 0, X=0}} \Theta_{N,\psi_A}(k)\overline{\psi_2(k)},$$

$$C_2 = \sum_{\substack{k\in U_A;\\ X,Y\neq 0}} \Theta_{N,\psi_A}(k)\overline{\psi_2(k)} \quad \text{and} \quad C_3 = \Theta_{N,\psi_A}(1)\overline{\psi_2(1)}.$$

Clearly,

$$C_3 = (q^2-1)^2(q^2-q).$$

To compute C_1, it is enough to show that

$$\sum_{\substack{k\in U_A;\\ X\neq 0, Y=0}} \overline{\psi_2(k)} = -1. \tag{7.1}$$

Indeed, we have

$$\begin{aligned}\sum_{\substack{k\in U_A;\\ X\neq 0, Y=0}} \overline{\psi_2(k)} &= \sum_{x=0,y\neq 0} \overline{\psi_0(0)} + \sum_{x\neq 0, y\in F} \overline{\psi_0(x)} \\ &= (q-1) + q(-1) \\ &= -1.\end{aligned}$$

Similarly, we have

$$\sum_{\substack{k\in U_A;\\ Y\neq 0, X=0}} \overline{\psi_2(k)} = -1. \tag{7.2}$$

From Equations 7.1 and 7.2 and row 5 of Table 5, it follows that

$$\begin{aligned}C_1 &= -(q^2-1)(q^2-q)\left[\sum_{\substack{k\in U_A;\\ X\neq 0, Y=0}} \overline{\psi_2(k)} + \sum_{\substack{k\in U_A;\\ Y\neq 0, X=0}} \overline{\psi_2(k)}\right] \\ &= 2(q^2-1)(q^2-q).\end{aligned}$$

To compute C_2, we need

$$\sum_{\substack{k\in U_A;\\ X,Y\neq 0}} \overline{\psi_2(k)} = \sum_{X,Y\neq 0} \overline{\psi_0(x+u)} = A + B,$$

where

$$\begin{aligned}A &= \sum_{\substack{X,Y\neq 0;\\ x+u=0}} \overline{\psi_0(0)} \\ &= \sum_{\substack{x,u=0\\ y,v\neq 0}} \overline{\psi_0(0)} + \sum_{\substack{x\neq 0, u=-x\\ y,v\in F}} \overline{\psi_0(0)} \\ &= (q-1)(q^2+q-1)\end{aligned}$$

and

$$
\begin{aligned}
B &= \sum_{\alpha \in F^\times} \sum_{\substack{X,Y \neq 0; \\ x+u=\alpha}} \overline{\psi_0(\alpha)} \\
&= \sum_{\alpha \in F^\times} \left[\sum_{\substack{x=0,u=\alpha \\ y\neq 0, v \in F}} \overline{\psi_0(\alpha)} \; + \sum_{\substack{u=0,x=\alpha \\ v\neq 0, y\in F}} \overline{\psi_0(\alpha)} \; + \sum_{\substack{x\neq 0, x\neq \alpha, \\ u=\alpha-x, y, v \in F}} \overline{\psi_0(\alpha)} \right] \\
&= \sum_{\alpha \in F^\times} \overline{\psi_0(\alpha)} \left[2(q-1)q + (q-2)q^2 \right] \\
&= -(q^3 - 2q)
\end{aligned}
$$

Thus, we have

$$
\sum_{\substack{k \in U_A; \\ X,Y \neq 0}} \overline{\psi_2(k)} = 1. \tag{7.3}
$$

From Equation 7.3 and row 6 of Table 5, it follows that

$$
C_2 = (q^2 - q).
$$

Hence,

$$
\begin{aligned}
\langle \Theta_{N,\psi_A}|_{I_2}, \chi_{\widetilde{\psi_2} \otimes \overline{\rho}} \rangle &= \frac{(q-1)}{|I_2|}(C_1 + C_2 + C_3) \\
&= \frac{(q-1)}{(q-1)q^5}(C_1 + C_2 + C_3) \\
&= \frac{1}{q^5}(q^2 - q)q^4 \\
&= q - 1.
\end{aligned}
$$

□

Lemma 7.2. *For each $\eta \in \widehat{F^\times}$, we have*

$$
\dim_{\mathbb{C}} \operatorname{Hom}_{I_2}(\pi_{N,\psi_A}|_{I_2}, \widetilde{\psi_2} \otimes \overline{\eta}) = 1.
$$

Proof. We have

$$
\begin{aligned}
\langle \Theta_{N,\psi_A}|_{I_2}, \chi_{\widetilde{\psi_2} \otimes \overline{\eta}} \rangle &= \frac{1}{|I_2|} \sum_{k \in I_2} \Theta_{N,\psi_A}(k) (\overline{\chi_{\widetilde{\psi_2} \otimes \overline{\eta}}})(k) \\
&= \frac{1}{|I_2|} \left[\sum_{k \in I_2 \setminus U_A} \Theta_{N,\psi_A}(k) (\overline{\chi_{\widetilde{\psi_2} \otimes \overline{\eta}})(k)} + \sum_{k \in U_A} \Theta_{N,\psi_A}(k) \overline{\psi_2(k)} \cdot \overline{\eta(1)} \right] \\
&= \frac{1}{|I_2|} \left[0 + \sum_{k \in U_A} \Theta_{N,\psi_A}(k) \overline{\psi_2(k)} \right] \\
&= \frac{1}{|I_2|}(C_1 + C_2 + C_3).
\end{aligned}
$$

Proceeding as in Lemma 7.1, we have that

$$\begin{aligned}\langle \Theta_{N,\psi_A}|_{I_2}, \chi_{\widetilde{\psi_2}\otimes\overline{\eta}}\rangle &= \frac{1}{|I_2|}(C_1+C_2+C_3)\\ &= \frac{1}{(q-1)q^5}(C_1+C_2+C_3)\\ &= \frac{1}{(q-1)q^5}(q^2-q)q^4\\ &= 1.\end{aligned}$$

□

8. Main Theorem

In this section, we prove the main result of this paper. We continue with the notation of the previous sections. For $1 \leq i \leq q-1$, we let

$$\sigma_i = \mathrm{Ind}_{I_1}^{H_A}(\widetilde{\psi_1}\otimes\overline{\eta_i}) \quad \text{and} \quad \phi_i = \mathrm{Ind}_{I_2}^{H_A}(\widetilde{\psi_2}\otimes\overline{\eta_i}),$$

where η_i's are the distinct characters of $F^\times$. Thus we have

$$\begin{aligned}\tau_1 &= \mathrm{Ind}_{U_A\rtimes F^\times}^{H_A}(\widetilde{\psi_1}\otimes\overline{\mathrm{reg}(F^\times)})\\ &= \bigoplus_{i=1}^{q-1}\sigma_i\end{aligned}$$

and

$$\begin{aligned}\tau_2 &= \mathrm{Ind}_{U_A\rtimes Mir}^{H_A}(\widetilde{\psi_2}\otimes\overline{\mathrm{reg}(Mir)})\\ &= (q-1)\,\mathrm{Ind}_{I_2}^{H_A}(\widetilde{\psi_2}\otimes\overline{\rho}) \ \oplus\ \bigoplus_{i=1}^{q-1}\phi_i.\end{aligned}$$

Theorem 8.1. *Let θ be a regular character of $F_6^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(6,F)$*. Then,*

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times}\otimes(q\tau_1\oplus\tau_2)$$

as M_{ψ_A} modules.

Proof. Using Lemma 6.1, 7.1 and 7.2, we have

$$\pi_{N,\psi_A}|_{H_A} = q\bigoplus_{i=1}^{q-1}\sigma_i \quad\oplus\quad (q-1)\,\mathrm{Ind}_{I_2}^{H_A}(\widetilde{\psi_2}\otimes\overline{\rho}) \quad\oplus\quad \bigoplus_{i=1}^{q-1}\phi_i.$$

Since $\theta|_{F^\times}$ is the central character of π, it follows that

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times}\otimes(q\tau_1\oplus\tau_2).$$

□

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