# DUALIZING INVOLUTIONS ON THE METAPLECTIC GL(2)

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Abstract. Let $F$ be a non-Archimedean local field of characteristic zero. Let $G = \mathrm{GL}(2,F)$ and $\widetilde{G} = \widetilde{\mathrm{GL}}(2,F)$ be the metaplectic group. Let $\tau$ be the standard involution on $G$. A well known theorem of Gelfand and Kazhdan says that the standard involution takes any irreducible admissible representation of $G$ to its contragredient. In such a case, we say that $\tau$ is a dualizing involution. In this paper, we show that any lift of the standard involution to $\widetilde{G}$ is also a dualizing involution.

## 1. Introduction

Let $F$ be a non-Archimedean local field of characteristic 0 and $G = \mathrm{GL}(n,F)$. For $g \in G$, we let $g^\top$ denote the transpose of the matrix $g$, and $w_0$ to be the matrix with anti-diagonal entries equal to one. Let $\tau : G \to G$ be the map $\tau(g) = w_0 g^\top w_0$. It is easy to see that $\tau$ is an anti-automorphism of $G$ such that $\tau^2 = 1$. We call $\tau$ the *standard involution* on $G$. Let $(\pi, V)$ be an irreducible smooth complex representation of $G$. We write $(\pi^\vee, V^\vee)$ for the smooth dual or the contragredient of $(\pi, V)$. For $\beta$ an anti-automorphism of $G$ such that $\beta^2 = 1$, we let $\pi^\beta$ to be the twisted representation defined by

$$\pi^\beta(g) = \pi(\beta(g^{-1})).$$

The following theorem is an old result of Gelfand and Kazhdan.

**Theorem 1.1** (Gelfand-Kazhdan). *Let $\tau$ be the standard involution on $G$. Then*

$$\pi^\tau \simeq \pi^\vee.$$

We refer the reader to Theorem 2 in [2] for a proof of the above result.

If $\beta$ is any anti-automorphism of $G$ such that $\beta^2 = 1$, and satisfies $\pi^\beta \simeq \pi^\vee$, then we call $\beta$ a *dualizing involution.* The above result implies that the standard involution $\tau$ on $G$ is a dualizing involution.

Let $\widetilde{G}$ be the metaplectic cover of $G$ (see Chapter 0 in [5] for the general definition). It is well known that (see Proposition 3.1 in [6]) the standard involution $\tau$ on $G$ has at least one lift to the metaplectic group. A natural and interesting question that one can ask is whether the lifts of the standard involution are themselves dualizing involutions.

In this paper, we address this question when $\widetilde{G} = \widetilde{GL}(2,F)$ is the metaplectic double cover of $G$ (explained later). In this case, it follows from Proposition 1 in [4] that there are many lifts of the standard involution to $\widetilde{G}$. In particular, for each $\alpha \in F^\times$, we have a lift $\sigma_\alpha$ (see Section 5 for the definition) of $\tau$ to $\widetilde{G}$. We show

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that all these involutions are dualizing involutions. To be precise, we prove

**Theorem 1.2** (Main Theorem). *Let $\pi$ be any irreducible admissible genuine representation of $\widetilde{G}$. Then*

$$\pi^{\sigma_\alpha} \simeq \pi^\vee.$$

The above result has prompted us to consider further generalizations of this problem. To be precise, we would like to explore if a similar result can be proved for the $r$-fold covering of $\mathrm{GL}(n)$ for $r, n \geq 2$. However, explicit description of a lift of the standard involution in the general case seems difficult. We hope to address some of these questions in the near future.

The paper is organized as follows. In Section 2, we set up some preliminaries on the Hilbert symbol and the metaplectic group. In Section 3, we recall a result of Harish-Chandra about the character of an admissible representation and give references for analogous results in the setting of metaplectic groups. We also recall a few results about lifts of the standard involution. In Section 4, we explicitly define a lift of the standard involution and discuss some properties. In Section 5, we prove the main result of this paper.

## 2. PRELIMINARIES

In this section, we set up some preliminaries which we need and recall a few results which will be used throughout this paper.

2.1. **Quadratic Hilbert Symbol and its properties.** Let $F$ be a local field and $F^\times$ be the group of non-zero elements in $F$ and let $\mu_2 = \{\pm 1\}$. The quadratic Hilbert symbol is a map

$$\langle\, ,\, \rangle : F^\times \times F^\times \to \mu_2$$

defined by

$$\langle a, b\rangle = \begin{cases} +1, & \text{if } z^2 - ax^2 - by^2 = 0 \text{ has a non-trivial solution in } F^3 \\ -1, & \text{otherwise} \end{cases}.$$

The following basic properties of the Hilbert symbol are well known. We record it in the proposition below.

**Proposition 2.1.** *The Hilbert symbol satisfies*

1) $\langle a, b\rangle = \langle b, a\rangle$ *and* $\langle a, c^2\rangle = 1$.
2) $\langle a, -a\rangle = 1$ *and* $\langle a, 1-a\rangle = 1$ *if* $a \neq 1$.
3) $\langle a, b\rangle = 1$ *implies* $\langle aa', b\rangle = \langle a', b\rangle$.
4) $\langle a, b\rangle = \langle a, -ab\rangle = \langle a, (1-a)b\rangle$.
5) $\langle a, b\rangle = 1$ *for all* $a \in F^\times$*, then* $b \in (F^\times)^2$.

We refer the reader to Chapter 3, Section 1 in [11] for the details.

2.2. **Metaplectic Groups.** In this section, we recall a few basic facts about central extensions and define "the" metaplectic group $\widetilde{G}$.

Throughout, we write $\mathfrak{o}$ for the ring of integers in $F$, $\mathfrak{p}$ for the unique maximal ideal in $\mathfrak{o}$ and $\varpi$ for the generator of $\mathfrak{p}$. We write $k_F$ for the finite residue field and assume throughout that $char(k_F) \neq 2$. We write $ord$ for the valuation on $F$.

2.2.1. *Central extensions.* Let $G$ be a group and $A$ an abelian group. A group $E$ is called a *central extension* of $G$ by $A$ if there exists a short-exact sequence of groups,

$$S:\ 1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

such that the image of $i$ is contained in the center of $E$. If

$$S' : 1 \to A \xrightarrow{i'} E' \xrightarrow{p'} G \to 1.$$

is another central extension, we say that $S$ and $S'$ are *isomorphic* if there exists an isomorphism $f : E \to E'$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & \big\downarrow 1_A & & \big\downarrow f & & \big\downarrow 1_G & & \\ 1 & \longrightarrow & A & \xrightarrow{i'} & E' & \xrightarrow{p'} & G & \longrightarrow & 1 \end{array}$$

In fact, by the Five Lemma, if $f$ is a homomorphism with this property, then it is automatically a group isomorphism.

The central extension

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

is called a *topological central extension* of $G$ by $A$, if $A$, $E$ and $G$ are Hausdorff (locally compact) topological groups such that

1) $i$ is continuous and $i(A)$ is a closed subgroup of the center of $E$ and
2) $p$ is continuous and induces a topological isomorphism $E/i(A) \simeq G$.

When $A$ and $G$ are locally compact topological groups, it is a well known fact that the elements of the second cohomology group $\mathrm{H}^2(G, A)$ (see Section 2 in [1] for the definition) are in bijection with the isomorphism classes of topological central extensions of $G$ by $A$. We explain a part of this identification which will be relevant to us in defining the metaplectic group which we work with.

For a 2-cocycle $\alpha$, we denote by $[\alpha]$ the class of $\alpha$ in $\mathrm{H}^2(G, A)$. Given $[\alpha] \in \mathrm{H}^2(G, A)$, we can construct a central extension of $G$ by $A$ using the following recipe. Let $G_\alpha = G \times A$ with the multiplication

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \alpha(g_1, g_2)\epsilon_1\epsilon_2), \quad (g_1, \epsilon_1), (g_2, \epsilon_2) \in G_\alpha.$$

Then, $G_\alpha$ is a group with the above multiplication. Further, the maps $i : A \to G_\alpha$ and $p : G_\alpha \to G$ given by $\epsilon \mapsto (1, \epsilon)$ and $(g, \epsilon) \mapsto g$ respectively are group homomorphisms such that the sequence

$$1 \to A \xrightarrow{i} G_\alpha \xrightarrow{p} G \to 1$$

is a central extension of $G$ by $A$ associated with cohomology class $[\alpha]$.

If $G$, $A$ are locally compact groups, a theorem of Mackey (see Theorem 2 in [9]) implies that there is a natural topology on $G_\alpha$ with respect to which it is a locally compact group and defines a topological central extension of $G$ by $A$.

2.2.2. *The Metaplectic group.* The first explicit construction of a metaplectic cover of $\mathrm{GL}(2,F)$ was given by Kubota in [7] by concretely describing a 2-cocyle. For $g_1, g_2, m \in G$, the following simpler version of the Kubota cocycle $c : G \times G \to \mu_2$ defined as

$$c(g_1, g_2) = \left\langle \frac{X(g_1 g_2)}{X(g_1)}, \frac{X(g_1 g_2)}{X(g_2)\Delta(g_1)} \right\rangle,$$

where

$$X(m) = \begin{cases} m_{21} & \text{if } m_{21} \neq 0 \\ m_{22} & \text{otherwise} \end{cases}$$

was given by Kazhdan and Patterson in [6]. We take $\widetilde{G}$ to be the central extension of $G$ by $\mu_2$ determined by 2-cocycle $c$. Since $G$, $\mu_2$ are locally compact groups, Mackey's theorem (see Theorem 2 in [9]) implies that $\widetilde{G}$ is a locally compact topological group and defines a topological central extension of $G$ by $\mu_2$. The group $\widetilde{G}$ constructed above is called "the" metaplectic group.

It can be shown that the topology on $\widetilde{G}$ has a neighborhood base at the identity consisting of compact open subgroups (see Lemma 3 in [4]). Before we give the construction of this basis, we recall a few preliminaries.

A map $\ell : G \to \widetilde{G}$ is called a *section* if $p \circ \ell = 1_G$. Given a subgroup $H$ of $G$, we say that $\widetilde{G}$ *splits* over $H$ if there exists a homomorphism $h : H \to \widetilde{G}$ such that $p \circ h = 1_H$.

Let $\ell : G \to \widetilde{G}$ be the map $\ell(g) = (g, 1)$. Then $\ell$ is a section and is called the natural or preferred section. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, let $\Delta(g) = \det(g)$ and define $s : G \to \mu_2$ as

$$s(g) = \begin{cases} \langle c, d\Delta(g) \rangle, & \text{if } cd \neq 0 \text{ and } ord(c) \text{ is odd} \\ \quad 1, & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $K = \mathrm{GL}(2, \mathfrak{o})$ be the maximal compact subgroup in $G$, and for $\lambda \geq 1$, let $K_\lambda = 1 + \varpi^\lambda \mathrm{M}(n, \mathfrak{o})$. It is known that $\{K_\lambda\}_{\lambda \geq 0}$ is a neighborhood base at the identity element in $G$ consisting of compact open subgroups. We can use this base to define a neighborhood base at the identity in $\widetilde{G}$. Define $\kappa : K \to \widetilde{G}$ as $\kappa(k) = (k, s(k))$. It can be shown that $\kappa : K \to \widetilde{G}$ is a homomorphism such that $p \circ \kappa = 1_K$, (i.e., $\widetilde{G}$ splits $K$). Let $K^* = \kappa(K)$ and for $\lambda \geq 1$, $K^*_\lambda = K^* \cap p^{-1}(K_\lambda)$. It can be shown that $\{K^*_\lambda\}_{\lambda \geq 0}$ is a neighborhood base at the identity in $\widetilde{G}$.

## 3. Some results we need

3.1. **Character of an admissible representation.** Let $F$ be a non-Archimedean local field of characteristic 0 and $G = G(F)$ be a connected reductive algebraic group defined over $F$. We let $(\pi, V)$ be an irreducible smooth complex representation of $G$. It can be shown that such representations are always admissible. For an admissible representation $(\pi, V)$, we can define a suitable notion of a character. Before we proceed further, we set up some notation and recall an important result of Harish-Chandra.

Throughout we let $G = G(F)$ and $(\pi, V)$ to be an irreducible smooth representation of $G$. We let $C_c^\infty(G)$ to be the space of all locally constant complex valued functions on $G$ with compact support. For $f \in C_c^\infty(G)$, we let $\pi(f) : V \to V$ denote the linear operator given by

$$\pi(f)v = \int_G f(g)\pi(g)vdg, \quad v \in V,$$

where the integral is with respect to a Haar measure on $G$ which we fix throughout. If $(\pi, V)$ is an admissible representation, it can be shown that the trace of the operator $\pi(f)$ is finite for all $f \in C_c^\infty(G)$. The resulting linear functional

$$\Theta_\pi : C_c^\infty(G) \longrightarrow \mathbb{C}$$

given by

$$\Theta_\pi(f) = \mathrm{Tr}(\pi(f))$$

is called the *distribution character* of $\pi$. It determines the irreducible representation $\pi$ up to equivalence, i.e., if $\Theta_{\pi_1}(f) = \Theta_{\pi_2}(f)$, $\forall f \in C_c^\infty(G)$, then $\pi_1 \simeq \pi_2$.

We now state a theorem of Harish-Chandra which is used to define the character of the representation $\pi$.

Let $G_{\mathrm{reg}}$ be the subset of regular semi-simple elements in $G$. It can be shown that $G_{\mathrm{reg}}$ is an open dense subset of $G$ whose complement has measure zero. The following is a deep result of Harish-Chandra.

**Theorem 3.1** (Harish-Chandra)**.** *There exists a locally integrable complex valued function $\Theta_\pi$ on $G$ such that $\Theta_\pi|_{G_{\mathrm{reg}}}$ is a locally constant function on $G_{\mathrm{reg}}$ and satisfies*

$$\Theta_\pi(f) = \int_G f(g)\Theta_\pi(g)dg, \quad \forall f \in C_c^\infty(G).$$

*Also, for $x \in G_{\mathrm{reg}}, y \in G$, we have*

$$\Theta_\pi(yxy^{-1}) = \Theta_\pi(x).$$

We refer the reader to [3] for a proof of the above result. The locally constant function $\Theta_\pi$ on $G_{\mathrm{reg}}$ in the above theorem is called the *character* of $\pi$.

Let $\widetilde{G}$ be a locally compact topological central extension of $G$ by $\mu_2$, where $\mu_2$, is the group of square roots of unity in $F$. Let $\xi : \mu_2 \to \mathbb{C}^\times$ be the non-trivial character of $\mu_2$. Let $(\pi, V)$ be an irreducible admissible representation of $\widetilde{G}$. $\pi$ is called a *genuine* representation, if for $\epsilon \in \mu_2$, $g \in \widetilde{G}$, we have

$$\pi(\epsilon g) = \xi(\epsilon)\pi(g).$$

The above result of Harish-Chandra also holds in this setting. To be more precise, it can be shown that the distribution character of an irreducible admissible genuine representation of $\widetilde{G}$ is represented by a locally integrable function $\Theta_\pi$ on $\widetilde{G}$ which is locally constant on $\widetilde{G}_{\mathrm{reg}} = p^{-1}(G_{\mathrm{reg}})$ and satisfies

$$\Theta_\pi(y^{-1}xy) = \Theta_\pi(x), \text{ for } x \in \widetilde{G}_{\mathrm{reg}}, y \in G.$$

We refer the reader to Theorem 4.3.2 and Corollary 4.3.3 in [8] where results about character theory for metaplectic groups is discussed. See also Theorem I.5.1 in [6] where the result is just mentioned.

3.2. **Some Results about lifts of the standard involution.** We recall a few results from [4] which we need in proving our main result. Let $\widetilde{G}$ be a central extension of a group $G$ by an abelian group $A$. Let $p : \widetilde{G} \to G$ be the projection map, $s : G \to \widetilde{G}$ be a section of $p$ and $\tau$ be the 2-cocycle representing the class of this central extension in $\mathrm{H}^2(G, A)$ with respect to the section $s$. If $f : G \to G$ is an automorphism (anti-automorphism) of $G$, then a lift of $f$ is an automorphism (anti-automorphism) $\tilde{f} : \widetilde{G} \to \widetilde{G}$ such that

$$p(\tilde{f}(g)) = f(p(g)), \forall g \in \widetilde{G}.$$

Let $\mathcal{L}(f)$ denote the set of all lifts of $f$. The group $\mathrm{Aut}(G)$ acts on $\mathrm{H}^2(G, A)$ by $f[\sigma] = [\sigma \circ (f^{-1} \times f^{-1})]$ for any 2-cocycle $\sigma$.

**Proposition 3.2.** *The set $\mathcal{L}(f)$ is precisely described in terms of this action by the following:*

1) *The set $\mathcal{L}(f)$ is non-empty if and only of $f[\tau] = [\tau]$.*
2) *If $\mathcal{L}(f)$ is non-empty, then $\mathcal{L}(f)$ is a principal homogeneous space for the group $\mathrm{Hom}(G, A)$ under the action*

$$(\phi.\tilde{f})(g) = \phi(p(g))\tilde{f}(g).$$

*Remark* 3.3. Let $G = \mathrm{GL}(2, F)$ and $\widetilde{G} = \widetilde{\mathrm{GL}}(2, F)$ be the metaplectic group with respect to $[c] \in \mathrm{H}^2(G, \mu_2)$. Let $f \in \mathrm{Aut}(G)$ be the automorphism $f(g) = w_0(g^\top)^{-1}w_o$. Since $f$ is an involution, we have $f^{-1} = f$ and it is easy to see that $f[c]$=$[c]$. Hence there is a lift $\tilde{f}$ of $f$ to $\widetilde{G}$.

We also need the following result (see Corollary 1 in [4] for a proof) which discusses the continuity properties of the lift in the case when $G = \mathrm{GL}(n, F)$. We state it below for clarity.

**Proposition 3.4.** *Let $F$ be a non-Archimedean local field and suppose that the group of $n^{th}$ roots of unity in $F$ has order $n$. Let $\langle\, ,\, \rangle$ be the $n^{th}$ order Hilbert symbol on $F$ and $\widetilde{\mathrm{GL}}(n)$ the corresponding metaplectic group. Then the lift of any topological automorphism of $\mathrm{GL}(n)$ to $\widetilde{\mathrm{GL}}(n)$ is also a topological automorphism.*

*Remark* 3.5. In the case when $n = 2$, clearly we have $|\mu_2| = 2$. Let $G = \mathrm{GL}(2, F)$ and $f \in \mathrm{Aut}(G)$ be the continuous automorphism of $G$ described above. Then for the metaplectic group $\widetilde{G} = \widetilde{GL}(2, F)$, it is clear that any lift $\tilde{f}$ of $f$ should also be a topological automorphism.

## 4. A Lift of the standard involution

Let $G = \mathrm{GL}(2, F)$ and $\widetilde{G} = \widetilde{\mathrm{GL}}(2, F)$ be the metaplectic double cover of $G$. Let $\tau$ be the standard involution on $G$. In this section, we explicitly define a lift $\sigma$ and show that it is an involution. We also discuss an important property of this lift (see Theorem 4.11 below) which is crucial in proving the main result of this paper.

For $\lambda \in F^\times$ and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, we let $u(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$ and $\Delta(g) = \det(g)$. It is easy to see that

$$\tau(g) = w_0 g^\top w_0 = u(\Delta(g))g^{-1}u(1).$$

For $\epsilon \in \mu_2$, we again denote the element $(1, \epsilon) \in \widetilde{G}$ as $\epsilon$. Let $\tilde{u}(\lambda) = (u(\lambda), 1)$ and $\tilde{z}(\lambda) = (\lambda I_2, 1)$ where $I_2$ is the $2 \times 2$ identity matrix. We extend $\Delta$ to $\widetilde{G}$ by $\Delta((g, \epsilon)) = \Delta(g)$. For $h \in \widetilde{G}$ define

$$\sigma(h) = \tilde{u}(\Delta(h))h^{-1}\tilde{u}(1).$$

Before we discuss properties of $\sigma$, we need a proposition which is useful in verifying its properties. We state it below without proof. The details can be checked easily using basic properties of the Hilbert symbol mentioned in Proposition 2.1.

**Proposition 4.1.** *For $h \in \widetilde{G}$, we have*

(1) $h\tilde{z}(\lambda) = \langle \lambda, \Delta(h) \rangle \tilde{z}(\lambda) h$.
(2) $\tilde{z}(\lambda_1)\tilde{z}(\lambda_2) = \langle \lambda_1, \lambda_2 \rangle \tilde{z}(\lambda_1\lambda_2)$.
(3) $\tilde{u}(\lambda_1)\tilde{u}(\lambda_2) = \langle \lambda_1, -\lambda_2 \rangle \tilde{z}(\lambda_1\lambda_2)$.
(4) $\tilde{u}(\lambda)^{-1} = \tilde{u}(\lambda^{-1})$.
(5) $\tilde{u}(\lambda_1)\tilde{z}(\lambda_2) = \langle \lambda_1, \lambda_2 \rangle \tilde{u}(\lambda_1\lambda_2)$.

**Lemma 4.2.** *$\sigma$ is an anti-automorphism.*

*Proof.* For $h_1, h_2 \in \widetilde{G}$, we have

$$\begin{aligned}
\sigma(h_1h_2) &= \tilde{u}(\Delta(h_1h_2))(h_1h_2)^{-1}\tilde{u}(1) \\
&= \tilde{u}(\Delta(h_2)\Delta(h_1))(h_1h_2)^{-1}\tilde{u}(1) \\
&= \langle \Delta(h_1), \Delta(h_2) \rangle \, \tilde{u}(\Delta(h_2))\tilde{z}(\Delta(h_1))h_2^{-1}h_1^{-1}\tilde{u}(1) \\
&= \tilde{u}(\Delta(h_2))h_2^{-1}\tilde{z}(\Delta(h_1))h_1^{-1}\tilde{u}(1) \\
&= \tilde{u}(\Delta(h_2))h_2^{-1}\tilde{u}(1)\tilde{u}(\Delta(h_1))h_1^{-1}\tilde{u}(1) \\
&= \sigma(h_2)\sigma(h_1).
\end{aligned}$$

□

**Lemma 4.3.** *$\sigma(\epsilon) = \epsilon$.*

*Proof.* It is enough to see this when $\epsilon$ is the non-trivial element in $\mu_2$. Indeed for $1 \neq \epsilon \in \mu_2$ we have,

$$\begin{aligned}
\sigma(\epsilon) &= \tilde{u}(\Delta(\epsilon))\epsilon^{-1}\tilde{u}(1) \\
&= \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, 1 \right) \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1 \right) \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, 1 \right) \\
&= \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, -1 \right) \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, 1 \right) \\
&= \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1 \right) \\
&= \epsilon.
\end{aligned}$$

□

*Remark* 4.4. We have used the following cocycle computations in the above proof.

1) $c\left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 1.$

2) $c\left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = 1.$

**Lemma 4.5.** *$\sigma$ is a lift of $\tau$.*

*Proof.* It is enough to see this for $h = (g, 1)$, where $g \in G$. We consider the cases when $c$ is either nonzero or zero. If $c \neq 0$, we have

$$\begin{aligned}\sigma(h) &= (u(\Delta(g)), 1)(g^{-1}, 1)(u(1), 1)\\ &= (u(\Delta(g))g^{-1}, \langle c, \Delta(g)\rangle)(u(1), 1)\\ &= (u(\Delta(g)g^{-1}u(1), \langle c, \Delta(g)\rangle)\\ &= (\tau(g), \langle c, \Delta(g)\rangle)\\ &= (\tau(g), \langle X(g), \Delta(g)\rangle),\end{aligned}$$

and if $c = 0$, we have

$$\begin{aligned}\sigma(h) &= (u(ad), 1)(g^{-1}, \langle a, d\rangle)(u(1), 1)\\ &= (u(ad)g^{-1}, \langle -1, d\rangle)(u(1), 1)\\ &= (u(ad)g^{-1}u(1), 1))\\ &= (\tau(g), 1).\end{aligned}$$

Thus, in both the cases, it is clear that $(p \circ \sigma)(h) = \tau(g) = (\tau \circ p)(h)$ and hence the result. $\square$

*Remark* 4.6. For $\epsilon \in \mu_2$, we know that $\sigma(\epsilon) = \epsilon$. Hence, to compute $\sigma(g, \epsilon)$ for arbitrary $(g, \epsilon) \in \widetilde{G}$, it suffices to determine $\sigma(g, 1)$. Suppose $\sigma(g, 1) = (x, \xi)$, then $\sigma(g, \epsilon) = (x, \epsilon\xi)$ and hence $x = (p \circ \sigma)(g, \epsilon) = (p \circ \sigma)(g, 1)$. Now, it is easy to see that $\sigma$ is a lift of $\tau$ by applying the above computation to a more explicit form of $\sigma$ given above.

*Remark* 4.7. We have also used the following computations. For $g \in G$, it is easy to see that

$$c(g, g^{-1}) = \begin{cases} \langle c, -c\rangle, & c \neq 0\\ \langle a, d\rangle, & c = 0\end{cases}$$

and hence

$$(g, 1)^{-1} = \begin{cases} (g^{-1}, 1), & c \neq 0\\ (g^{-1}, \langle a, d\rangle), & c = 0.\end{cases}$$

**Lemma 4.8.** *$\sigma$ is an involution.*

*Proof.* For $h \in \widetilde{G}$, observe that we have $\Delta(\sigma(h)) = \Delta(h)$. Using the properties mentioned in Proposition 4.1, it is easy to see that $\sigma$ is an involution. Indeed,

$$\begin{aligned}\sigma((\sigma(h)) &= \sigma(\tilde{u}(\Delta(h))h^{-1}\tilde{u}(1))\\ &= \tilde{u}(\Delta(h))\tilde{u}(1)h\tilde{u}(\Delta(h)^{-1})\tilde{u}(1)\\ &= \langle \Delta(h), -1\rangle \tilde{z}(\Delta(h))h\langle \Delta(h)^{-1}, -1\rangle \tilde{z}(\Delta(h)^{-1})\\ &= \langle \Delta(h), \Delta(h)\rangle h\tilde{z}(\Delta(h))\tilde{z}(\Delta(h)^{-1})\\ &= \langle \Delta(h), \Delta(h)\rangle h\langle \Delta(h), \Delta(h)^{-1}\rangle\\ &= h.\end{aligned}$$

$\square$

**Lemma 4.9.** *$\sigma(h^{-1}) = \sigma(h)^{-1}$ for all $h \in \widetilde{G}$.*

*Proof.* It suffices to check this for $h = (g, 1)$ where $g \in G$. We consider the cases when $c$ is either non-zero or zero. It is easy to see that

$$\sigma((g, 1)^{-1}) = \begin{cases} \sigma(g^{-1}, 1) = (\tau(g^{-1}), \langle X(g^{-1}), \Delta(g^{-1})\rangle), & c \neq 0\\ \sigma(g^{-1}, \langle a, d\rangle) = (\tau(g^{-1}), \langle a, d\rangle), & c = 0\end{cases}$$

and

$$\sigma(g,1)^{-1} = \begin{cases} (\tau(g), \langle X(g), \Delta(g)\rangle)^{-1} = (\tau(g)^{-1}, \langle X(g), \Delta(g)\rangle), & c \neq 0 \\ (\tau(g), 1)^{-1} = ((\tau(g)^{-1}, \langle d, a\rangle), & c = 0. \end{cases}$$

Clearly, $\tau(g^{-1}) = \tau(g)^{-1}$ and $\langle a, d\rangle = \langle d, a\rangle$. Therefore, it is enough to show that $\langle X(g^{-1}), \Delta(g^{-1})\rangle = \langle X(g), \Delta(g)\rangle$. A simple computation verifies this. Indeed,

$$\begin{aligned} \langle X(g^{-1}), \Delta(g^{-1})\rangle &= \left\langle -\frac{X(g)}{\Delta(g)}, \frac{1}{\Delta(g)}\right\rangle \\ &= \langle X(g), \Delta(g)\rangle\langle -\Delta(g), \Delta(g)\rangle \\ &= \langle X(g), \Delta(g)\rangle. \end{aligned}$$

Hence the result. $\square$

Having established the basic properties of $\sigma$, we now discuss a preliminary lemma which we need to prove the main result of this section.

**Lemma 4.10.** *Let $\mathcal{S} = \left\{ h \in \widetilde{G} \mid \sigma(h) = xhx^{-1}, \text{ for some } x \in \widetilde{G}\right\}$. Then $\mathcal{S}$ is invariant under multiplication by $\epsilon$ and conjugation.*

*Proof.* Let $h \in \mathcal{S}$. Choose $x \in \widetilde{G}$ such that $\sigma(h) = xhx^{-1}$. Then,

$$\sigma(\epsilon h) = \sigma(\epsilon)\sigma(h) = \epsilon xhx^{-1} = x(\epsilon h)x^{-1}.$$

Hence it follows that $\epsilon h \in \mathcal{S}$. Also, for $y \in \widetilde{G}$, we have

$$\sigma(yhy^{-1}) = \sigma(y^{-1})\sigma(h)\sigma(y) = \sigma(y)^{-1}xhx^{-1}\sigma(y) = zhz^{-1}$$

with $z = (\sigma(y)x)^{-1}$. Hence, $yhy^{-1} \in \mathcal{S}$. $\square$

**Theorem 4.11.** *For $g \in \widetilde{G}$, we have $\sigma(g) = zgz^{-1}$ for some $z \in \widetilde{G}$.*

*Proof.* Suppose that $g = \begin{bmatrix} 0 & v \\ 1 & w \end{bmatrix} \in G$. Note that $v \neq 0$ and let $y = \begin{bmatrix} 1 & 0 \\ -\frac{w}{v} & 1 \end{bmatrix}$. Let $h = (g, 1)$ and $\tilde{y} = (y, 1)$. We have $\tau(g) = \begin{bmatrix} w & v \\ 1 & 0 \end{bmatrix}$ and $ygy^{-1} = \tau(g)$. Now,

$$\sigma(h) = (\tau(g), \langle X(g), \Delta(g)\rangle) = (\tau(g), 1),$$

since $X(g) = 1$. A simple computation shows that $c(y, g) = c(yg, y^{-1}) = 1$. It follows that $\tilde{y}h\tilde{y}^{-1} = \sigma(h)$. Indeed,

$$\begin{aligned} \tilde{y}h\tilde{y}^{-1} &= (y,1)(g,1)(y,1)^{-1} \\ &= (yg, 1)(y^{-1}, 1) \\ &= (ygy^{-1}, 1) \\ &= (\tau(g), 1) \\ &= \sigma(h). \end{aligned}$$

Hence, $h \in \mathcal{S}$. Suppose that $g = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in G$ and $k = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$. Then, $\tau(g) = kgk^{-1}$. Let $h = (g, 1)$ and $\tilde{k} = (k, 1)$. Then, $\sigma(h) = (\tau(g), 1)$ and

$$\begin{aligned} \tilde{k}h\tilde{k}^{-1} &= (k,1)(g,1)(k,1)^{-1} \\ &= (kg, \langle a, d\rangle)(k^{-1}, 1) \\ &= (kgk^{-1}, \langle a, d\rangle\,\langle a, d\rangle) \\ &= (\tau(g), 1) \\ &= \sigma(h). \end{aligned}$$

Therefore, $h \in \mathcal{S}$. Finally, suppose that $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \in G$ and $h = (g, 1)$. Then $\sigma(h) = (\tau(g), 1) = (g, 1) = h$, so again $h \in \mathcal{S}$.

Let $h = (g, 1) \in \widetilde{G}$. The element $g$ is conjugate to one of the three types of elements considered above. Thus, there is some $x \in G$ such that $(xgx^{-1}, 1) \in \mathcal{S}$. Let $\tilde{x} = (x, 1)$. Then $\tilde{x}h\tilde{x}^{-1} = (xgx^{-1}, \zeta)$ with $\zeta \in \mu_2$ and so $\tilde{x}(h\zeta)\tilde{x}^{-1} = (xgx^{-1}, 1) \in \mathcal{S}$. It follows that $h\zeta \in \mathcal{S}$ and hence $h \in \mathcal{S}$. Finally, this implies that $\epsilon h = (h, \epsilon) \in \mathcal{S}$ and hence $\mathcal{S} = \widetilde{G}$. ☐

## 5. Dualizing involutions on $\widetilde{G}$

For $\alpha \in F^\times$, $h \in \widetilde{G}$, let $\sigma_\alpha(h) = \langle \alpha, \Delta(h) \rangle \sigma(h)$. It is easy to see that $\sigma_\alpha$ is a lift of $\tau$ for each $\alpha \in F^\times$ and is also an involution. In fact from Proposition 3.2, it follows that any lift of $\tau$ is of the form $\sigma_\alpha$ for $\alpha \in F^\times$. In this section, we show that all the lifts $\sigma_\alpha$ of $\tau$ are dualizing involutions.

For the sake of clarity, we verify that $\sigma_\alpha$ is a lift of $\tau$ which is also an involution. We also prove a technical lemma which we need to establish an important property of $\sigma_\alpha$.

**Lemma 5.1.** *$\sigma_\alpha$ is an anti-automorphism of $\widetilde{G}$.*

*Proof.* For $g, h \in \widetilde{G}$, we have

$$\begin{aligned}
\sigma_\alpha(gh) &= \langle \alpha, \Delta(gh) \rangle \sigma(gh) \\
&= \langle \alpha, \Delta(g)\Delta(h) \rangle \sigma(h)\sigma(g) \\
&= \langle \alpha, \Delta(g) \rangle \langle \alpha, \Delta(h) \rangle \sigma(h)\sigma(g) \\
&= \langle \alpha, \Delta(h) \rangle \sigma(h) \langle \alpha, \Delta(g) \rangle \sigma(g) \\
&= \sigma_\alpha(g)\sigma_\alpha(h).
\end{aligned}$$

☐

**Lemma 5.2.** *For $\alpha \in F^\times$, $\sigma_\alpha$ is an involution.*

*Proof.* Let $\sigma_\alpha(h) = y$. We have

$$\begin{aligned}
\sigma_\alpha(\sigma_\alpha(h)) &= \sigma_\alpha(y) \\
&= \langle \alpha, \Delta(y) \rangle \sigma(y) \\
&= \langle \alpha, \Delta(h) \rangle \sigma(\langle \alpha, \Delta(h) \rangle \sigma(h)) \\
&= \langle \alpha, \Delta(h) \rangle \sigma(\epsilon\sigma(h)) \\
&= \langle \alpha, \Delta(h) \rangle \sigma(\sigma(h))\sigma(\epsilon) \\
&= \langle \alpha, \Delta(h) \rangle \langle \alpha, \Delta(h) \rangle h \\
&= h.
\end{aligned}$$

☐

*Remark* 5.3. Let $h = (h, \xi_h)$, $\sigma(h) = g = (g, \xi_g)$. Then we get

$$y = \sigma_\alpha(h) = \langle \alpha, \Delta(h) \rangle \sigma(h) = (g, \langle \alpha, \Delta(h) \rangle \xi_g).$$

Using the extension of $\Delta$ to $\widetilde{G}$ as defined earlier, we see that

$$\Delta(y) = \Delta(g) = \Delta(\sigma(h)).$$

Now using the fact that $\sigma(h) = zhz^{-1}$ for some $z \in \widetilde{G}$, we see that

$$\Delta(\sigma(h)) = \Delta(h)$$

and hence it follows that

$$\Delta(y) = \Delta(h).$$

**Lemma 5.4.** *Let $h \in \widetilde{G}$ be such that $\Delta(h) \notin (F^{\times})^2$. Then there exists $u \in \widetilde{G}$ such that*

$$\epsilon h = uhu^{-1}.$$

*Proof.* Since $\Delta(h) \notin (F^{\times})^2$, using non-degeneracy of the Hilbert symbol, it follows that there exists $\lambda \in F^{\times}$ such that $\langle \lambda, \Delta(h) \rangle = -1$. Let $u \in \widetilde{G}$ be defined by

$$u = \Big( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, 1 \Big).$$

A simple computation shows that

$$\epsilon h = uhu^{-1}.$$

$\square$

*Remark* 5.5. Let $u = \Big( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, 1 \Big)$ and $h = \Big( \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \xi \Big)$. To prove the above lemma, we consider the cases when $c$ is either zero or non-zero. In both these cases we show that $uh = \epsilon hu$.

The following cocycles,

1) $c\Big( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big) = \begin{cases} \langle c, \lambda \rangle & \text{if } c \neq 0 \\ \langle d, \lambda \rangle & \text{if } c \neq 0 \end{cases}$

2) $c\Big( \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \Big) = \begin{cases} \langle \lambda, c\Delta(h) \rangle & \text{if } c \neq 0 \\ \langle \lambda, d\Delta(h) \rangle & \text{if } c \neq 0 \end{cases}$

along with the fact that $\langle \lambda, \Delta(h) \rangle = -1$ are useful in verifying the above computations.

**Theorem 5.6.** *$\sigma_\alpha(h)$ is conjugate to $h$ for any $h \in \widetilde{G}$.*

*Proof.* Suppose $\Delta(h) \in (F^{\times})^2$, then $\sigma_\alpha(h) = \sigma(h)$ and hence it follows that $\sigma_\alpha(h)$ is conjugate to $h$. It is enough to consider the case when $\Delta(h) \notin (F^{\times})^2$ and $\langle \alpha, \Delta(h) \rangle = -1$. The result now follows from Theorem 4.11 and Lemma 5.4. For completeness, we give the details below.

$$\begin{aligned} \sigma_\alpha(h) &= \langle \alpha, \Delta(h) \rangle \sigma(h) \\ &= \epsilon \sigma(h) \\ &= \sigma(\epsilon h) \\ &= z(\epsilon h) z^{-1} \\ &= (zu)h(zu)^{-1}. \end{aligned}$$

$\square$

Throughout, we let $\mu_{\widetilde{G}}$ denote the Haar measure on $\widetilde{G}$. We write $\mathrm{Aut}_c(\widetilde{G})$ for the group of continuous automorphisms of $\widetilde{G}$ and $\mathbb{R}^{\times}_{>0}$ for the multiplicative group of positive real numbers.

**Lemma 5.7.** *Let $\gamma \in \mathrm{Aut}_c(\widetilde{G})$. There exists $c_\gamma > 0$ such that*

$$\mu_{\widetilde{G}} \circ \gamma = c_\gamma \mu_{\widetilde{G}}.$$

*Proof.* Let $\nu_{\widetilde{G}} = \mu_{\widetilde{G}} \circ \gamma$. Clearly $\nu_{\widetilde{G}}$ is a left invariant Haar measure on $\widetilde{G}$. Indeed, for $g \in G$, and $U \subset G$, we have

$$\begin{aligned}
\nu_{\widetilde{G}}(gU) &= \mu_{\widetilde{G}}(\gamma(gU)) \\
&= \mu_{\widetilde{G}}(\gamma(g)\gamma(U)) \\
&= \mu_{\widetilde{G}}(\gamma(U)) \\
&= \nu_{\widetilde{G}}(U).
\end{aligned}$$

□

**Lemma 5.8.** *For $\gamma \in \mathrm{Aut}_c(\widetilde{G})$, the map*

$$\gamma \mapsto c_\gamma : \mathrm{Aut}_c(\widetilde{G}) \to \mathbb{R}^\times_{>0}$$

*is a homomorphism.*

*Proof.* For $U \subset \widetilde{G}$, we have

$$\begin{aligned}
c_{\gamma_1\gamma_2}\mu_{\widetilde{G}}(U) &= (\mu_{\widetilde{G}} \circ \gamma_1\gamma_2)(U) \\
&= (\mu_{\widetilde{G}} \circ \gamma_1)(\gamma_2(U)) \\
&= c_{\gamma_1}(\mu_{\widetilde{G}} \circ \gamma_2)(U) \\
&= c_{\gamma_1}c_{\gamma_2}\mu_{\widetilde{G}}(U).
\end{aligned}$$

□

**Lemma 5.9.** *Let $\gamma$ be any continuous automorphism of $\widetilde{G}$, and $X$ a measurable subset of $\widetilde{G}$. Suppose also that $\gamma^2 = 1$. Then*

$$(\mu_{\widetilde{G}} \circ \gamma)(X) = \mu_{\widetilde{G}}(X),$$

*i.e., $\gamma$ preserves the Haar measure on $\widetilde{G}$.*

*Proof.* Let $K$ be any compact open subset of $\widetilde{G}$. Since $K$ is open, using properties of the Haar measure it follows that $\mu_{\widetilde{G}}(K) > 0$. Since $\gamma^2 = 1$, it is easy to see that $c_\gamma = 1$. Indeed,

$$\begin{aligned}
c_\gamma\mu_{\widetilde{G}}(K) &= (\mu_{\widetilde{G}} \circ \gamma)(K) \\
&= (\mu_{\widetilde{G}} \circ \gamma^{-1})(K) \\
&= c_{\gamma^{-1}}\mu_{\widetilde{G}}(K) \\
&= c_\gamma^{-1}\mu_{\widetilde{G}}(K)
\end{aligned}$$

It follows that $c_\gamma = 1$ and hence the result. That is, for any $X \subset \widetilde{G}$ we have

$$\mu_{\widetilde{G}}(\gamma(X)) = \mu_{\widetilde{G}}(X).$$

□

Let $\pi$ be an irreducible admissible genuine representation of $\widetilde{G}$. For $f \in C_c^\infty(\widetilde{G})$, $\rho \in \mathrm{Aut}(\widetilde{G})$ we define $f^\rho(g) = f(\rho(g))$ and $\pi^\rho(g) = \pi(\rho(g))$. Suppose that $\rho$ is also continuous and satisfies

1) $\rho$ is an involution, i.e., $\rho^2 = 1$ and
2) $\rho(g)$ is conjugate to $g^{-1}$ for any $g \in \widetilde{G}$.

In this case, we show that $\rho$ is a dualizing involution. Before we proceed further, we record a few simple observations which we need.

**Lemma 5.10.** *For $f \in C_c^\infty(\widetilde{G})$, we have*

$$\Theta_{\pi^\rho}(f) = \Theta_\pi(f^\rho).$$

*Proof.* It is enough to show that $\pi^\rho(f) = \pi(f^\rho)$. Indeed, for $v \in V$, we have

$$\begin{aligned}
\pi^\rho(f)v &= \int_{\widetilde{G}} f(g)\pi^\rho(g)vdg \\
&= \int_{\widetilde{G}} f(g)\pi(\rho(g))vdg \\
&= \int_{\widetilde{G}} f(\rho(g))\pi(g)vdg \\
&= \int_{\widetilde{G}} f^\rho(g)\pi(g)vdg \\
&= \pi(f^\rho)v.
\end{aligned}$$

From this it follows that $\mathrm{Tr}(\pi^\rho(f)) = \mathrm{Tr}(\pi(f^\rho))$ and hence the result. $\square$

**Lemma 5.11.** *For $g \in \widetilde{G}_{\mathrm{reg}}$, we have*

$$\Theta_{\pi^\vee}(g) = \Theta_\pi(g^{-1}).$$

*Proof.* For $f \in C_c^\infty(\widetilde{G})$, let $f^\vee(g) = f(g^{-1})$. It is easy to see that

$$\pi^\vee(f) = \pi^{tr}(f^\vee),$$

where $\pi^{tr}(f)$ is the transpose of the operator $\pi(f)$. Since the trace is invariant under taking transpose, it is clear that

$$\Theta_{\pi^\vee}(f) = \mathrm{Tr}(\pi^\vee(f)) = \mathrm{Tr}(\pi^{tr}(f^\vee)) = \mathrm{Tr}(\pi(f^\vee)) = \Theta_\pi(f^\vee).$$

The result follows. Indeed,

$$\begin{aligned}
\Theta_{\pi^\vee}(f) &= \int_{\widetilde{G}} f(g)\Theta_{\pi^\vee}(g)dg \\
&= \int_{\widetilde{G}} f^\vee(g)\Theta_\pi(g)dg \\
&= \int_{\widetilde{G}} f(g^{-1})\Theta_\pi(g)dg \\
&= \int_{\widetilde{G}} f(g)\Theta_\pi(g^{-1})dg \\
&= \Theta_\pi(f^\vee).
\end{aligned}$$

$\square$

**Proposition 5.12.** *Let $\rho$ be a continuous automorphism of $\widetilde{G}$ such that $\rho(g)$ is conjugate to $g^{-1}$ for any $g \in \widetilde{G}$. Suppose also that $\rho^2 = 1$. Then for any irreducible admissible genuine representation $\pi$ of $\widetilde{G}$, we have $\pi^\rho \simeq \pi^\vee$.*

*Proof.* For $f \in C_c^\infty(\widetilde{G})$, we have

$$\begin{aligned}
\Theta_{\pi^\rho}(f) &= \Theta_\pi(f^\rho) \\
&= \int_{\widetilde{G}} f(g)\Theta_{\pi^\rho}(g)dg \\
&= \int_{\widetilde{G}} f^\rho(g)\Theta_\pi(g)dg \\
&= \int_{\widetilde{G}} f(g)\Theta_\pi(\rho(g))dg \\
&= \int_{\widetilde{G}} f(g)\Theta_\pi(xg^{-1}x^{-1})dg \\
&= \int_{\widetilde{G}} f(g)\Theta_\pi(g^{-1})dg
\end{aligned}$$

From the above computation, it follows that $\Theta_{\pi^\rho}(g) = \Theta_\pi(g^{-1})$, for all $g \in \widetilde{G}_{\mathrm{reg}}$. Thus $\Theta_{\pi^\rho} = \Theta_{\pi^\vee}$ and $\pi^\rho \simeq \pi^\vee$. □

*Remark* 5.13. The above Proposition 5.12 also appears as Lemma 8.1 in [10]. We have merely used an additional assumption that the continuous automorphism which appears in that proof is also an involution. This leads to some minor simplification of some of the arguments which are used in that proof.

***Proof of the Main Theorem***: For $\alpha \in F^\times$ and $g \in \widetilde{G}$, let $\rho_\alpha(g) = \sigma_\alpha(g^{-1})$. Since $\sigma_\alpha$ is continuous (see Proposition 3.4 above), it follows that $\rho_\alpha \in \mathrm{Aut}_c(\widetilde{G})$ for all $\alpha \in F^\times$. Using the fact that $\sigma_\alpha(g)$ is a conjugate of $g$, it is clear that $\rho_\alpha$ is a conjugate of $g^{-1}$. The result now follows from Proposition 5.12 above.

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