A NOTE ON THE CARDINALITY OF A CERTAIN SUBSET OF $n \times n$ MATRICES OVER A FINITE FIELD

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Abstract. Let F be the finite field of order q and $\mathrm{M}(n, r, F)$ be the set of $n \times n$ matrices of rank r over the field F. For $\alpha \in F$ and $A \in \mathrm{M}(n, F)$, let

$$Z_{A,r}^{\alpha} = \{X \in \mathrm{M}(n, r, F) \mid \mathrm{Tr}(AX) = \alpha\}.$$

In this note, we compute the cardinality of $Z_{A,r}^{\alpha}$.

1. Introduction

Let F be the finite field of order q. For n a positive integer and $0 \leq r \leq n$, we let $\mathrm{M}(n, F)$ to be the set of $n \times n$ matrices with entries in F and $\mathrm{M}(n, r, F)$ to be the subset of $\mathrm{M}(n, F)$ consisting of rank r matrices. For $1 \leq k \leq n$, we define the *k-trace* of a matrix as the sum of its first k diagonal entries. For $\alpha \in F$, let $Y_{n,r,k}^{\alpha}$ be the set of $n \times n$ matrices of rank r such that its k-trace is α. The main result of this paper is to compute the cardinality of $Y_{n,r,k}^{\alpha}$. It is easy to see that the sets $Z_{A,r}^{\alpha}$ and $Y_{n,r,k}^{\alpha}$ where k is the rank of A, have the same cardinality.

In [2], Buckhiester counted the number of $n \times n$ matrices of rank r and trace α over F. This cardinality was also counted independently by Prasad in [5] and was used to compute the dimension of a certain representation of $\mathrm{GL}(2n, F)$. In [1], Balasubramanian, Khurana and Dangodara computed the cardinality of $n \times n$ matrices of rank r and 1-trace α over F, and used it to calculate the dimension of a certain representation of $\mathrm{GL}(2n, F)$. More recently in [4], a similar type of calculation was done, generalizing the work of Prasad, to compute the dimension of a certain representation of $\mathrm{GL}(kn, F)$. Motivated by these works, in this article, we count the number of $n \times n$ matrices of rank r and k-trace α over F for $1 \leq k \leq n$. The main reason behind this calculation is to compute the dimension formula for a certain twisted Jacquet module of a cuspidal representation of $\mathrm{GL}(2n, F)$. We will discuss the dimension formula in a forthcoming paper.

For the sake of clarity, we mention below the statement of the main theorem. We let $a(n, r, q)$ be the cardinality of $\mathrm{M}(n, r, q)$ and $f_{n,r,k}^{\alpha}$ be the cardinality of $Y_{n,r,k}^{\alpha}$. The Gaussian binomial coefficient is the number of r-dimensional linear subspaces of the n-dimensional space F^n and is denoted as $\begin{bmatrix} n \\ r \end{bmatrix}_q$.

Theorem 1.1 (Main Theorem). *We have*

$$f_{n,r,k}^{0} - f_{n,r,k}^{1} = \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2} + k(r-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q a(n-k, r-i, q).$$

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Using the above theorem, we can explicitly determine the cardinality of $Y^{\alpha}_{n,r,k}$ for all $\alpha \in F$.

2. Preliminaries

Throughout, we let F to be the finite field of order q and $\mathrm{M}(n, F)$ for the set of $n \times n$ matrices with entries in F. For $0 \le r \le n$, we let $\mathrm{M}(n, r, F)$ denote the subset of $\mathrm{M}(n, F)$ consisting of matrices of rank r. For $1 \le k \le n$, and $\alpha \in F$, define

$$Y^{\alpha}_{n,r,k} = \left\{ X = (x_{ij}) \in \mathrm{M}(n, r, F) \mid \sum_{i=1}^{k} x_{ii} = \alpha \right\}.$$

We write $f^{\alpha}_{n,r,k}$ for the cardinality of $Y^{\alpha}_{n,r,k}$. If $k = n$, to simplify notation, we will denote $Y^{\alpha}_{k,r,k} = Y^{\alpha}_{k,r}$ and $f^{\alpha}_{k,r,k} = f^{\alpha}_{k,r}$.

Proposition 2.1. *Let n, r be non-negative integers. For $A \in \mathrm{M}(n, F)$ and $\alpha \in F$, let*

$$Z^{\alpha}_{A,r} = \{X \in \mathrm{M}(n, r, F) \mid \mathrm{Tr}(AX) = \alpha\}.$$

If $\mathrm{Rank}(A) = k$, *then*

$$|Z^{\alpha}_{A,r}| = f^{\alpha}_{n,r,k}.$$

Proof. Since $A \in \mathrm{M}(n, k, F)$, there exists $g_1, g_2 \in \mathrm{GL}(n, F)$ such that $g_1 A g_2^{-1} = B$, where

$$B = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}.$$

Consider the map

$$\phi : Z^{\alpha}_{A,r} \to Y^{\alpha}_{n,r,k}$$

given by $\phi(X) = g_2 X g_1^{-1}$. Clearly $\phi(X) \in Y^{\alpha}_{n,r,k}$. Indeed, we have

$$\begin{aligned} \mathrm{Tr}(B\phi(X)) &= \mathrm{Tr}(B g_2 X g_1^{-1}) \\ &= \mathrm{Tr}(g_1^{-1} B g_2 X) \\ &= \mathrm{Tr}(AX) \\ &= \alpha. \end{aligned}$$

Trivially, ϕ is a bijection and the result follows. ☐

Before we proceed further, we set up some more notation and recall some more results that we need. We write $a(n, q)$ (respectively $a(n, r, q)$) for the cardinality of $\mathrm{M}(n, F)$ (respectively $\mathrm{M}(n, r, F)$). We let $\begin{bmatrix} n \\ r \end{bmatrix}_q$ for the Gaussian binomial coefficient and $(X; q)_n$ for the q-Pochhammer symbol. For the sake of clarity, we record their definitions below.

Definition 2.2. Let n, r be non-negative integers. We define

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \begin{cases} \prod\limits_{i=0}^{r-1} \frac{(q^{n-i}-1)}{(q^{r-i}-1)}, & \text{if } 0 \le r \le n \\ \\ 0, & \text{if } , r > n \end{cases}.$$

Definition 2.3. Let n be a non-negative integer. We define

$$(X;q)_n = \begin{cases} \prod_{i=0}^{n-1}(1-Xq^i), & \text{if } n>0 \\ \\ 1, & \text{if } n=0 \end{cases}.$$

Theorem 2.4 (q-binomial theorem)**.** *We have*

$$(X;q)_n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\binom{r}{2}} X^r.$$

Proof. See Theorem 3.2 in [3]. □

Proposition 2.5. *Let n be a positive integer and let $0 \le r \le n$. Then*

$$a(n,r,q) = \begin{bmatrix} n \\ r \end{bmatrix}_q^2 |\operatorname{GL}(r,F)|.$$

Proof. We have

$$\begin{aligned} a(n,r,q) &= \prod_{i=0}^{r-1} \frac{(q^n-q^i)^2}{(q^r-q^i)} \\ &= q^{\binom{r}{2}} \frac{(q^n-1)^2\cdots(q^{n-r+1}-1)^2}{(q^r-1)^2\cdots(q-1)^2}((q^r-1)\cdots(q-1)) \\ &= \begin{bmatrix} n \\ r \end{bmatrix}_q^2 |\operatorname{GL}(r,F)|. \end{aligned}$$

□

Proposition 2.6. *Let n be a positive integer and let $0 \le r \le n$. Then*

$$f^0_{n,r,k} + (q-1)f^1_{n,r,k} = a(n,r,q).$$

Proof. Let $\alpha \neq 0 \in F$. It is enough to show that $f^\alpha_{n,r,k} = f^1_{n,r,k}$. Let

$$Z = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in Y^\alpha_{n,r,k}.$$

Then $P \in \operatorname{M}(k,F)$ and $\operatorname{Tr}(P) = \alpha \neq 0$. Clearly the map $\phi : Y^\alpha_{n,r,k} \to Y^1_{n,r,k}$ defined by $\phi(Z) = KZ$, where

$$K = \begin{bmatrix} \alpha^{-1}I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \operatorname{GL}(n,F)$$

is a bijection. The result follows. □

The following result is due to Prasad where he computes $f^0_{k,r} - f^1_{k,r}$. We can use this result in combination with Proposition 2.6 to explicitly compute the numbers $f^0_{k,r}$ and $f^1_{k,r}$.

Lemma 2.7 (Prasad)**.**

$$f^0_{k,r} - f^1_{k,r} = (-1)^r q^{\binom{r}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q.$$

Proof. For a proof, we refer the reader to Lemma 2 in [5]. □

We can reformulate Lemma 2.7, using generating functions and the q-binomial theorem as follows.

Lemma 2.8. *Consider the polynomial generating function*

$$f^{\alpha}_{k,k}(X) = \sum_{r\geq 0} f^{\alpha}_{k,r} X^r \in \mathbb{Q}[X].$$

We have

$$f^0_{k,k}(X) - f^1_{k,k}(X) = (1-X)(1-qX)\cdots(1-q^{k-1}X) = (X;q)_k. \tag{2.1}$$

3. MAIN THEOREM

In this section, we prove the main result of this paper. We continue with the same notation as in the previous section. Before we proceed further, we prove some preliminary results that we need.

Lemma 3.1. *Let $k = n-1$. For $0 \leq r \leq k$, we have*

$$f^0_{k+1,r,k} - f^1_{k+1,r,k} = (-1)^{r-2} q^{\binom{r}{2}} \begin{bmatrix} k \\ k-r \end{bmatrix}_q \left\{ \frac{(q^{k-r+2}-1) + q^{k+1}(1-q)}{(q^{k+1-r}-1)} \right\}.$$

Proof. Let $X \in Y^0_{k+1,r,k}$. Then, X is of the form

$$X = \begin{bmatrix} D & v \\ w & x \end{bmatrix}$$

where $x \in F$, $w^T, v \in F^k$ and $D \in \mathrm{M}(k, F)$. We denote the $(k+1)\times 1$ column vector $\begin{bmatrix} v \\ x \end{bmatrix}$ by $\tilde{v}$. We write the $(k+1)\times k$ matrix

$$\begin{bmatrix} D \\ w \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix}$$

where $v_i \in F^{k+1}$ for all $1 \leq i \leq k$.

Let V be the column span of D and $W = \mathrm{Span}\{v_1, v_2, \ldots, v_k\}$. Since $X \in Y^0_{k+1,r,k}$, the rank of D has three possibilities, either $r, r-1$ or $r-2$. We consider these cases separately.

Case 1) Suppose that $\mathrm{Rank}(D) = r$. Then we have that $D \in Y^0_{k,r}$. Since $\mathrm{Rank}(X) = r$, it follows that $w \in V$ and $\tilde{v} \in W$. It implies that $\dim(V) = r = \dim(W)$. Therefore, the number of choices for the vector w are q^r and for the vector $\tilde{v}$ are q^r.

Case 2) Suppose that $\mathrm{Rank}(D) = r-1$. Then we have that $D \in Y^0_{k,r-1}$ and $\dim(V) = r-1$. There are two possibilities for the vector w, either $w \in V$ or $w \notin V$.

a) If $w \in V$, it implies that $\dim(W) = r-1$. Since $\mathrm{Rank}(X) = r$, it follows that $\tilde{v} \notin W$. Then the number of choices for w are q^{r-1} and for $\tilde{v}$ are $(q^{k+1} - q^{r-1})$.

b) If $w \notin V$, it implies that $\dim(W) = r$. Since $\mathrm{Rank}(X) = r$, it follows that $\tilde{v} \in W$. Then the number of choices for w are $(q^k - q^{r-1})$ and for $\tilde{v}$ are q^r.

Case 3) Suppose that $\mathrm{Rank}(D) = r-2$. Then we have that $D \in Y^0_{k,r-2}$. Since $\mathrm{Rank}(X) = r$, it follows that $w \notin V$ and $\dim(W) = r-1$. It also implies that $\tilde{v} \notin W$. Therefore, the number of choices for w are $(q^k - q^{r-2})$ and for $\tilde{v}$ are $(q^{k+1} - q^{r-1})$.

It follows that

$$\begin{aligned} f^0_{k+1,r,k} &= q^{2r} f^0_{k,r} + f^0_{k,r-1}\big(q^{r-1}(q^{k+1}-q^{r-1}) + q^r(q^k - q^{r-1})\big) \\ &\quad + f^0_{k,r-2}(q^k - q^{r-2})(q^{k+1}-q^{r-1}). \end{aligned} \tag{3.1}$$

Similarly, the number $f^1_{k+1,r,k}$ satisfies the recursion

$$\begin{aligned} f^1_{k+1,r,k} &= q^{2r} f^1_{k,r} + f^1_{k,r-1}\big(q^{r-1}(q^{k+1}-q^{r-1}) + q^r(q^k - q^{r-1})\big) \\ &\quad + f^1_{k,r-2}(q^k - q^{r-2})(q^{k+1}-q^{r-1}). \end{aligned}$$

Thus, we have

$$\begin{aligned} f^0_{k+1,r,k} - f^1_{k+1,r,k} &= q^{2r}\big(f^0_{k,r} - f^1_{k,r}\big) \\ &\quad + \big(f^0_{k,r-1} - f^1_{k,r-1}\big)\big(q^{r-1}(q^{k+1}-q^{r-1}) + q^r(q^k - q^{r-1})\big) \\ &\quad + \big(f^0_{k,r-2} - f^1_{k,r-2}\big)(q^k - q^{r-2})(q^{k+1}-q^{r-1}) \\[2ex] &= q^{2r}(-1)^r q^{\binom{r}{2}} \begin{bmatrix} k \\ k-r \end{bmatrix}_q \\ &\quad + (q^k - q^{r-2})(q^{k+1}-q^{r-1})(-1)^{r-2} q^{\binom{r-2}{2}} \begin{bmatrix} k \\ k-(r-2) \end{bmatrix}_q \\ &\quad + \big(q^{r-1}(q^{k+1}-q^{r-1}) + q^r(q^k-q^{r-1})\big)(-1)^{r-1} q^{\binom{r-1}{2}} \begin{bmatrix} k \\ k-(r-1) \end{bmatrix}_q \\[2ex] &= (-1)^{r-2} q^{\binom{r}{2}} \begin{bmatrix} k \\ k-r \end{bmatrix}_q \Bigg[q^{2r} \\ &\quad + \frac{(q^{k-r+2}-1)^2(q^{r-1}-1)(q^r-1)}{(q^{k+1-r}-1)(q^{k-r+2}-1)} \\ &\quad - \frac{q^{-r+1}(q^r-1)}{(q^{k+1-r}-1)} \Big(q^{2r-1}(q^{k+1-r}-1) + q^{2r-2}(q^{k-r+2}-1)\Big)\Bigg] \\[2ex] &= \frac{(-1)^{r-2} q^{\binom{r}{2}} | \begin{bmatrix} k \\ k-r \end{bmatrix}_q}{(q^{k+1-r}-1)} \Big[q^{2r}(q^{k+1-r}-1) \\ &\quad + (q^{k-r+2}-1)(q^{r-1}-1)(q^r-1) \\ &\quad - q^{r-1}(q^r-1)(2q^{k-r+2}-q-1)\Big] \\[2ex] &= (-1)^{r-2} q^{\binom{r}{2}} \begin{bmatrix} k \\ k-r \end{bmatrix}_q \left\{ \frac{(q^{k-r+2}-1) + q^{k+1}(1-q)}{(q^{k+1-r}-1)} \right\}. \end{aligned}$$

$\square$

Lemma 3.2. *Let $k = n-1$. We have*

$$f^0_{k+1,k+1,k} - f^1_{k+1,k+1,k} = (-1)^k q^{\binom{k+1}{2}}(q-1).$$

Proof. Continuing with the same notations as in Lemma 3.1, we have that if $X \in Y^0_{k+1,k+1,k}$, then the rank of D has only two possibilities, either k or $k-1$.

Case 1) Suppose that $\text{Rank}(D) = k$. Then $D \in Y^0_{k,k}$ and $\dim(V) = k$. Since w is an $1 \times k$ vector, we have that $w \in V$. It follows that $\dim(W) = k$. Since $\text{Rank}(X) = k+1$, it follows that the column vector $\tilde{v} \notin W$. Then, it follows that number of choices for w is q^k and for the vector $\tilde{v}$ is $(q^{k+1} - q^k)$.

Case 2) Suppose that $\text{Rank}(D) = k-1$. Then $D \in Y^0_{k,k-1}$. Since $\text{Rank}(X) = k+1$, it follows that $w \notin V$ which implies that $\dim(W) = k$. Furthermore, we have that the column vector $\tilde{v} \notin W$. Therefore, the number of choices for the row vector w are $(q^k - q^{k-1})$ and for the vector $\tilde{v}$ are $(q^{k+1} - q^k)$.

Hence, it follows that

$$f^0_{k+1,k+1,k} = q^k(q^{k+1} - q^k)f^0_{k,k} + (q^k - q^{k-1})(q^{k+1} - q^k)f^0_{k,k-1}.$$

Similarly, computing $f^1_{k+1,k+1,k}$ we have

$$f^1_{k+1,k+1,k} = q^k(q^{k+1} - q^k)f^1_{k,k} + (q^k - q^{k-1})(q^{k+1} - q^k)f^1_{k,k-1}.$$

Thus, we get

$$\begin{aligned}
f^0_{k+1,k+1,k} - f^1_{k+1,k+1,k} &= q^k(q^{k+1} - q^k)(f^0_{k,k} - f^1_{k,k}) \\
&+ (q^k - q^{k-1})(q^{k+1} - q^k)(f^0_{k,k-1} - f^1_{k,k-1}) \\
&= (-1)^{k-1} q^{\binom{k-1}{2}} q^{k-1} q^k (q-1)^2 \begin{bmatrix} k \\ 1 \end{bmatrix}_q \\
&+ (-1)^k (q^k)^2 (q-1) q^{\binom{k}{2}} \begin{bmatrix} k \\ 0 \end{bmatrix}_q \\
&= (-1)^{k-1} q^{\binom{k-1}{2}} (q-1)[(q^k - 1)q^{2k-1} - q^{3k-1}] \\
&= (-1)^k q^{\binom{k+1}{2}} (q-1).
\end{aligned}$$

□

Lemma 3.3. *$f^\alpha_{n,r,k}$ satisfies the the recursion*

$$f^\alpha_{n,r,k} = f^\alpha_{n-1,r,k} q^{2r} + f^\alpha_{n-1,r-1,k} q^{2r-2}(2q^{n-r+1} - 1 - q) + f^\alpha_{n-1,r-2,k} q^{2r-3}(q^{n-r+1} - 1)^2$$

for $n > k$.

Proof. Let $X \in Y^\alpha_{n,r,k}$. Then, X is of the form

$$X = \begin{bmatrix} D & v \\ w & x \end{bmatrix}$$

where $x \in F$, $w^T, v \in F^{n-1}$ and $D \in \text{M}(n-1, F)$. We denote the $n \times 1$ column vector $\begin{bmatrix} v \\ x \end{bmatrix}$ by $\tilde{v}$. We also write the $n \times (n-1)$ matrix

$$\begin{bmatrix} D \\ w \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix}$$

where $v_i \in F^n$ for all $1 \le i \le n-1$.

Let V be the column span of D and $W = \mathrm{Span}\{v_1, v_2, \ldots, v_{n-1}\}$. Since $X \in Y^{\alpha}_{n,r,k}$, the rank of D has three possibilities, either $r, r-1$ or $r-2$.

Since $k < n$, it is clear that $\sum_{i=1}^{k} x_{ii} = \alpha$ implies that $\sum_{i=1}^{k} d_{ii} = \alpha$ where $D = (d_{ij})$. One can proceed as in Lemma 3.1 and obtain a recursion formula for $f^{\alpha}_{n,r,k}$ similar to Equation (3.1). To be precise, $f^{\alpha}_{n,r,k}$ satisfies the following recursion for $n > k$:

$$f^{\alpha}_{n,r,k} = f^{\alpha}_{n-1,r,k} q^{2r} + f^{\alpha}_{n-1,r-1,k} q^{2r-2}(2q^{n-r+1} - 1 - q) + f^{\alpha}_{n-1,r-2,k} q^{2r-3}(q^{n-r+1} - 1)^2.$$

□

Lemma 3.4. *Consider the polynomial generating function*

$$f^{\alpha}_{n,k}(X) = \sum_{r \geq 0} f^{\alpha}_{n,r,k} X^r \in \mathbb{Q}[X].$$

For $n > k$, we have

$$f^{\alpha}_{n,k}(X) = f^{\alpha}_{n-1,k}(q^2X)(1-X)(1-qX) + f^{\alpha}_{n-1,k}(qX)2X(1-X)q^n + f^{\alpha}_{n-1,k}(X)X^2 q^{2n-1}.$$

Proof. It is easy to see that

$$f^{\alpha}_{n,k}(X) = \sum_{r \geq 0} \Big(f^{\alpha}_{n-1,r,k}(q^2X)^r + X f^{\alpha}_{n-1,r-1,k}(q^2X)^{r-1}(2q^{n-r+1} - 1 - q) + qX^2 f^{\alpha}_{n-1,r-2,k}(q^2X)^{r-2}(q^{n-r+1} - 1)^2 \Big). \tag{3.2}$$

By applying the change of variable $r - 1 = t$ to the second term in the sum, we get

$$\begin{aligned} X \sum_{r \geq 1} f^{\alpha}_{n-1,r-1,k}(q^2X)^{r-1}(2q^{n-r+1} - 1 - q) &= 2X \sum_{t \geq 0} f^{\alpha}_{n-1,t,k}(q^2X)^t(q^{n-t}) \\ &\quad - (1+q)X \sum_{t \geq 0} f^{\alpha}_{n-1,t,k}(q^2X)^t \\ &= 2Xq^n \sum_{t \geq 0} f^{\alpha}_{n-1,t,k}(qX)^t \\ &\quad - (1+q)X \sum_{t \geq 0} f^{\alpha}_{n-1,t,k}(q^2X)^t \\ &= 2Xq^n f^{\alpha}_{n-1,k}(qX) - (1+q)X f^{\alpha}_{n-1,k}(q^2X). \end{aligned}$$

Similarly, we can simplify the third term in the sum by applying the change of variable $r-2=y$ to obtain

$$
\begin{aligned}
qX^2\sum_{r\geq 2} f^{\alpha}_{n-1,r-2,k}(q^2X)^{r-2}(q^{n-r+1}-1)^2 &= \sum_{y\geq 0}\Big(q^{2n-1}X^2 f^{\alpha}_{n-1,y,k}(X)^y \\
&\quad + X^2 q f^{\alpha}_{n-1,y,k}(q^2X)^y \\
&\quad - 2q^n X^2 f^{\alpha}_{n-1,y,k}(qX)^y\Big) \\
&= X^2q^{2n-1}f^{\alpha}_{n-1,k}(X) + qX^2 f^{\alpha}_{n-1,k}(q^2X) \\
&\quad - 2q^n X^2 f^{\alpha}_{n-1,k}(qX).
\end{aligned}
$$

Hence, it follows that Equation (3.2) becomes

$$f^{\alpha}_{n,k}(X) = (1-X)(1-qX)f^{\alpha}_{n-1,k}(q^2X)+2Xq^n(1-X)f^{\alpha}_{n-1,k}(qX)+X^2q^{2n-1}f_{n-1,k}(X).$$

□

The numbers $a(n,r,q)$ being independent of k, also obey the same recurrence as $f^{\alpha}_{n,r,k}$ for all $0\leq r\leq n$ (i.e., without the restriction that $n>k$) because the cases on the submatrices $D, w, \bar{v}$ remain the same. Therefore, if

$$A_n(X) = \sum_{r\geq 0} a(n,r,q)X^r \in \mathbb{Q}[X]$$

is the polynomial generating function of the numbers $a(n,r,q)$ (for fixed n), it satisfies the following recurrence. We have

$$A_n(X) = A_{n-1}(q^2X)(1-X)(1-qX)+2X(1-X)q^nA_{n-1}(qX)+A_{n-1}(X)X^2q^{2n-1}. \quad (3.3)$$

Let $g_{n,r,k} = f^0_{n,r,k} - f^1_{n,r,k}$. Consider the polynomial generating function

$$g_{n,k}(X) = \sum_{r\geq 0} g_{n,r,k}X^r = f^0_{n,k}(X) - f^1_{n,k}(X) \in \mathbb{Q}[X].$$

Then, $g_{n,k}(X) = f^0_{n,k}(X) - f^1_{n,k}(X)$ also obeys the same recursion as $A_n(X)$. To be precise, we have

$$
\begin{aligned}
g_{n,k}(X) &= g_{n-1,k}(q^2X)(1-X)(1-qX) + 2X(1-X)q^n g_{n-1,k}(qX) \\
&\quad + g_{n-1,k}(X)X^2q^{2n-1}.
\end{aligned}
\quad (3.4)
$$

Since the base case of the recursion $n=k$ is known for $g_{n,k}(X)$ (see Lemma 2.7), we can solve the recurrence. For example if $n=k+1$, we solve the recurrence for the function $g_{k+1,k}(X)$. We record it in the Lemma below.

Lemma 3.5. *We have*

$$\frac{g_{k+1,k}(X)}{(X;q)_k} = 1+(q-1)Xq^k = A_1(q^kX).$$

and

$$\frac{g_{k+2,k}(X)}{(X;q)_k} = 1+(q^2-1)\left[(Xq^k)(q+1)+(Xq^k)^2(q^2-q)\right] = A_2(q^kX).$$

Proof. Using the fact that $g_{k,k}(X) = (X;q)_k$ (the base case of the recursion), Equation (3.4) becomes

$$\frac{g_{k+1,k}(X)}{(X;q)_k} = \frac{1}{(X;q)_k}\Big\{g_{k,k}(q^2X)(1-X)(1-qX) + 2X(1-X)q^{k+1}g_{k,k}(qX) + g_{k,k}(X)X^2q^{2k+1}\Big\}$$

$$= (1-q^kX)(1-q^{k+1}X) + 2Xq^{k+1}(1-q^kX) + X^2q^{2k+1}$$

$$= 1 + Xq^k(q-1).$$

Substituting $n=1$ in $A_n(X) = \sum_{r\geq 0} a(n,r,q)X^r$, we obtain that

$$A_1(X) = 1 + (q-1)X.$$

Continuing in a similar way, we get

$$\frac{g_{k+2,k}(X)}{(X;q)_k} = \frac{1}{(X;q)_k}\{A_1(q^{k+2}X)(q^2X;q)_k(1-X)(1-qX) + 2X(1-X)q^{k+2}A_1(q^{k+1}X)(qX;q)_k + X^2q^{2k+3}A_1(qX)(X;q)_k\}$$

$$= A_1(q^{k+2}X)(1-q^kX)(1-q(q^kX)) + 2Xq^{k+2}A_1(q^{k+1}X)(1-q^kX) + q^3A_1(q^kX)(q^kX)^2$$

$$= A_2(q^kX).$$

□

This suggests the following theorem.

Theorem 3.6. *We have,*

$$g_{n,k}(X) = (X;q)_k\ A_{n-k}(q^kX).$$

Proof. The base case is $k=n$ which is true by Lemma 2.7. Assume inductively that the theorem is true for $g_{n-1,k}(X)$. We have

$$g_{n,k}(X) = g_{n-1,k}(q^2X)(1-X)(1-qX) + 2X(1-X)q^ng_{n-1,k}(qX) + g_{n-1,k}(X)X^2q^{2n-1}$$

$$= (q^2X;q)_kA_{n-k-1}(q^{k+2}X)(1-X)(1-qX) + 2X(1-X)q^n(qX;q)_kA_{n-k-1}(q^{k+1}X) + (X;q)_k\ A_{n-k-1}(q^kX)X^2q^{2n-1}$$

$$= (X;q)_k\big((1-q^kX)(1-q^{k+1}X)A_{n-k-1}(q^{k+2}X) + 2Xq^n(1-q^kX)A_{n-k-1}(q^{k+1}X) + A_{n-k-1}(q^kX)X^2q^{2n-1}\big).$$

In terms of the recurrence relation (3.3) for $A_m(Y)$ where $Y = q^k X$ and $m = n-k$, we see that the above expression is just

$$A_{n-k}(q^k X).$$

This completes the proof of the theorem. □

Theorem 3.7. *We have*

$$f^0_{n,r,k} - f^1_{n,r,k} = g_{n,r,k} = \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}+k(r-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q a(n-k, r-i, q).$$

Proof. From Theorem 3.6, Equation(2.1) and the definition of $A_{n-k}(X)$, it follows that

$$g_{n,k}(X) = \sum_{i\geq 0}\sum_{j\geq 0} (-1)^i q^{\binom{i}{2}+kj} \begin{bmatrix} k \\ i \end{bmatrix}_q a(n-k, j, q) X^{j+i}.$$

Let $j + i = r$. Using the fact that $i, j \geq 0$, we get that $i \leq r$ and $r \geq 0$ and hence it follows that

$$g_{n,k}(X) = \sum_{r\geq 0}\sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}+k(r-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q a(n-k, r-i, q) X^r.$$

Thus,

$$g_{n,r,k} = \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}+k(r-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q a(n-k, r-i, q).$$

□

One can obtain $f^1_{n,r,k}$ and $f^0_{n,r,k}$ by using Theorem 3.7 and Proposition 2.6. We mention it in the corollary below.

Corollary 3.8. *We have that*

$$f^1_{n,r,k} = \frac{1}{q}\big(a(n, r, q) - g_{n,r,k})\big)$$

and

$$f^0_{n,r,k} = a(n, r, q) - (q-1) f^1_{n,r,k},$$

where $g_{n,r,k}$ *is as in Theorem 3.7.*

Proof. From Proposition 2.6, we have that

$$f^0_{n,r,k} - f^1_{n,r,k} + q f^1_{n,r,k} = a(n, r, q).$$

Since $g_{n,r,k} = f^0_{n,r,k} - f^1_{n,r,k}$, it follows that

$$f^1_{n,r,k} = \frac{1}{q}\big(a(n, r, q) - g_{n,r,k}\big).$$

and

$$f^0_{n,r,k} = a(n, r, q) - (q-1) f^1_{n,r,k}.$$

□

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