MTH 406 : Differential Geometry of Curves and Surfaces

Quiz 2 11th April, 2024

Question. Show that if a surface contains a straight line, then it has non-positive Gauss curvature at all the points of this line.

Points 10.

Solution.

Solution. Let S denote the surface, $I \subseteq \mathbb{R}$ denote an open interval and $\alpha : I \to S$ be a segment of a straight line in $\mathbb{R}^3$ (i.e., α is a restriction of an affine linear map). Pre-composing by a suitable diffeomorphism, we may assume that α is p.b.a.l.

Let $\alpha(t_0) \in \alpha(I)$ be arbitrary. Since we are required to examine the Gauss curvature at this point, we may assume that there is a local Gauss map N defined at all points of $\alpha(I)$.

By definition of Gauss map we have

$$\langle (N \circ \alpha)(s), \alpha'(s) \rangle = 0 \qquad (s \in I).$$

Differentiating this w.r.t. s we have

$$\langle (N \circ \alpha)'(s), \alpha'(s) \rangle + \langle (N \circ \alpha)(s), \alpha''(s) \rangle = 0$$

and using chain rule of derivatives we have

$$\langle (N \circ \alpha)(s), \alpha''(s) \rangle = -\langle (dN)_{\alpha(s)}(\alpha'(s)), \alpha'(s) \rangle = \sigma_{\alpha(s)}(\alpha'(s), \alpha'(s))$$

for all $s \in I$. Since α is affine linear, we have $\alpha''(s) = 0$ for all $s \in I$. This implies that $\sigma_{\alpha(s)}$ cannot be either positive definite, or negative definite.

Now suppose that $K(\alpha(s)) > 0$ for some $s \in I$. Then the eigenvalues of the matrix of $\sigma_{\alpha(s)}$ are either both positive, or both negative. This happens if and only if $\sigma_{\alpha(s)}$ is either positive definite, or negative definite, a contradiction.