MTH 406 : Differential Geometry of Curves and Surfaces

Quiz 1 17th January, 2024

Question. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve and $\mathbf{a} \in \mathbb{R}^2 \setminus \alpha(I)$. Let $t_0 \in I$ such that

$$|\alpha(t) - \mathbf{a}| \geq |\alpha(t_0) - \mathbf{a}|$$

for every $t \in I$.

Show that the straight line joining the point $\mathbf{a}$ with $\alpha(t_0)$ is the normal line of α at t_0.

Prove that the same is true if we reverse the inequality.

Solution.

We use the notation $\alpha(t) = (x(t), y(t))$ $(t \in I)$ and $\mathbf{a} = (a_1, a_2)$. Now define $f : I \to \mathbb{R}$ by

$$f(t) = |\alpha(t) - \mathbf{a}|^2 = (x(t) - a_1)^2 + (y(t) - a_2)^2 \quad (t \in I).$$

From hypothesis, it follows that f is a positive valued C^∞-function and $t = t_0$ is a minima of f. Hence $f'(t_0) = 0$. Now we have

$$f'(t) = 2\big(x(t) - a_1\big)x'(t) + 2\big(y(t) - a_2\big)y'(t) = 2\big(\alpha(t) - \mathbf{a}\big)\cdot\alpha'(t).$$

Now from $f'(t_0) = 0$, it follows that $\alpha(t_0) - \mathbf{a}$ is perpendicular to the vector $\alpha'(t_0)$. Hence the normal line to α at t_0 has direction vector $\alpha(t_0) - \mathbf{a} = (x(t_0) - a_1, y(t_0) - a_2)$. Then the equation of the normal line at $\alpha(t_0)$ is of the form

$$\big(y(t_0) - a_2\big)x - \big(x(t_0) - a_1\big)y + \lambda = 0$$

for some constant $\lambda \in \mathbb{R}$. Since $\alpha(t_0) = (x(t_0), y(t_0))$ belongs to this line, we have

$$\big(y(t_0) - a_2\big)x(t_0) - \big(x(t_0) - a_1\big)y(t_0) + \lambda = 0.$$

Subtracting the above two equations (to free the unknown constant λ) we obtain

$$\big(y(t_0) - a_2\big)\big(x - x(t_0)\big) - \big(x(t_0) - a_1\big)\big(y - y(t_0)\big) = 0 \quad (*)$$

It is easy to verify that this equation satisfies (a_1, a_2), which means that $(*)$ represents the line that passes through $\alpha(t_0)$ and $\mathbf{a}$.

If we reverse the inequality, $t = t_0$ would be a maxima of f, and then again we have $f'(t_0)$. Now the rest of the calculation would be identical as before.