# MTH 406 : Differential Geometry of Curves and Surfaces

**Mid semester Exam** **27th February, 2024**
**Full score: 50** **Time: 120 minutes**

**Question 1.** Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ a plane curve p.b.a.l. (i.e. $|\alpha'(s)| = 1$ for all $s \in \mathbb{R}$) such that $\alpha(\mathbb{R})$ is included in a closed disc of radius $r > 0$ and center $\mathbf{a}$, that satisfies $|k(s)| \leq 1/r$ for every $s \in \mathbb{R}$. Prove that:

(i) The function $f(s) := |\alpha(s) - \mathbf{a}|^2$ ($s \in \mathbb{R}$) is bounded from above and satisfies $f''(s) \geq 0$ for all $s \in \mathbb{R}$.

(ii) Any differentiable function $g : \mathbb{R} \to \mathbb{R}$ that is both bounded from above and satisfies $g'' \geq 0$ is necessarily a constant function.

Using (i) and (ii) deduce that $\alpha$ is a circle centred at $\mathbf{a}$ with radius $r > 0$.

**Points. 5+5+2**

**Question 2.** Let $\alpha$ and $\beta$ be two regular plane curves defined, respectively, on two open intervals of $\mathbb{R}$ containing the origin. Suppose that $\alpha(0) = \beta(0) = p$ and that $\alpha'(0) = \beta'(0)$.

We say that $\alpha$ **is over** $\beta$ **at** $p$ if there is a neighbourhood of 0 in $\mathbb{R}$ where $\langle \alpha - \alpha(0), N_\alpha(0) \rangle \geq \langle \beta - \beta(0), N_\beta(0) \rangle$. Prove that:

(i) If $\alpha$ is over $\beta$ at $p$, then $k_\alpha(0) \geq k_\beta(0)$.

(ii) If $k_\alpha(0) > k_\beta(0)$, then $\alpha$ is over $\beta$ at $p$.

**Points. 6+6**

**Question 3.** Let $O_1, O_2 \subseteq \mathbb{R}^3$ be Euclidean open sets and let $\phi : O_1 \to O_2$ be a diffeomorphism. If $S \subseteq O_1$ is a surface, show that $\phi(S)$ is a surface. Prove that the restriction map $\phi : S \to \phi(S)$ is a diffeomorphism.

**Points. 6+6**

**Question 4.** Let $S$ be the right cylinder of radius $r > 0$, whose axis is the line passing through the origin with direction $\mathbf{a}$ (with $|\mathbf{a}| = 1$), given by

$$S := \left\{ p \in \mathbb{R}^3 \ : \ |p|^2 - \langle p, \mathbf{a} \rangle^2 = r^2 \right\}.$$

Prove that

$$T_pS = \left\{ v \in \mathbb{R}^3 \ : \ \langle p, v \rangle - \langle p, a \rangle \langle a, v \rangle = 0 \right\}.$$

Conclude that all the normal lines of $S$ cut the axis perpendicularly.

**Points. 7+7**

## Solution.

**Question 1.** Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ a plane curve p.b.a.l. (i.e. $|\alpha'(s)| = 1$ for all $s \in \mathbb{R}$) such that $\alpha(\mathbb{R})$ is included in a closed disc of radius $r > 0$ and center $\mathbf{a}$, that satisfies $|k(s)| \leq 1/r$ for every $s \in \mathbb{R}$. Prove that:

(i) The function $f(s) := |\alpha(s) - \mathbf{a}|^2$ ($s \in \mathbb{R}$) is bounded from above and satisfies $f''(s) \geq 0$ for all $s \in \mathbb{R}$.

(ii) Any differentiable function $g : \mathbb{R} \to \mathbb{R}$ that is bounded and satisfies $g'' \geq 0$ is necessarily a constant function.

Using (i) and (ii) deduce that $\alpha$ is a circle centred at $\mathbf{a}$ with radius $r > 0$.

**Solution.** (i) We write $\alpha(s) = (x(s), y(s))$ ($s \in \mathbb{R}$) and $\mathbf{a} = (a_1, a_2)$. Since $\alpha(\mathbb{R})$ is included in a closed disc of radius $r > 0$ and center $\mathbf{a}$, we have

$$|\alpha(s) - \mathbf{a}| \leq r \quad (s \in \mathbb{R}) \qquad (*).$$

This shows that $0 \leq f(s) \leq r^2$ for all $s \in \mathbb{R}$. Next,

$$f(s) = (x(a) - a_1)^2 + (y(s) - a_2)^2 \quad (s \in \mathbb{R}).$$

Then, for every $s \in \mathbb{R}$ we have

$$\begin{aligned} f'(s) &= 2\big(x(s) - a_1\big)x'(s) + 2\big(y(s) - a_2\big)y'(s) = 2\big(\alpha(s) - \mathbf{a}\big)\cdot\alpha'(s), \\ f''(s) &= 2\big(x(s) - a_1\big)x''(s) + 2x'(s)^2 + 2\big(y(s) - a_2\big)y''(s) + 2y'(s)^2 \\ &= 2\big(\alpha(s) - \mathbf{a}\big)\cdot\alpha''(s) + 2 \end{aligned}$$

since $\alpha$ is p.b.a.l.

Now from Frenet's first equation $T'(s) = k(s)N(S)$, we have

$$x''(s) = -k(s)y'(s), \qquad y''(s) = k(s)x'(s)$$

and hence

$$x''(s)^2 + y''(s)^2 = k(s)^2 \leq \frac{1}{r^2} \quad (s \in \mathbb{R}) \qquad (**).$$

Next, using the dot product inequality we have

$$\big|\big(\alpha(s) - \mathbf{a}\big)\cdot\alpha''(s)\big| \leq |\alpha(s) - \mathbf{a}| \cdot |\alpha''(s)| \leq r \cdot \frac{1}{r} = 1$$

using $(*)$ and $(**)$. This implies that,

$$0 \leq \big(\alpha(s) - \mathbf{a}\big)\cdot\alpha''(s) + 1 \leq 2$$

and hence $f''(s) \geq 0$ for all $s \in \mathbb{R}$.

(ii) For any $x, y \in \mathbb{R}$, using Taylor's theorem with second order approximation we have

$$g(x) = g(y) + g'(y)(x - y) + \frac{1}{2!} g''(c)(x - y)^2$$

for some $c$ in between $x$ and $y$, where $c$ depends on $x$ and $y$. Using our hypothesis, we have

$$g(x) \geq g(y) + g'(y)(x - y).$$

for any $x, y \in \mathbb{R}$.

Now suppose, $g$ is not constant. Then there exists $y \in \mathbb{R}$ such that $g'(y) > 0$ or $g'(y) < 0$. In the first case, keeping $y$ fixed, we have $g(x) \to +\infty$ as $x \to +\infty$. In the second case, keeping $y$ fixed, we have $g(x) \to -\infty$ as $x \to +\infty$. In either case, it contradicts the fact that $g$ is bounded. Hence $g$ must be a constant map.

(iii) The function $f$ in (i) satisfies the property stated in (ii), and hence is a constant function. Using $(*)$ we have

$$|\alpha(s) - \mathbf{a}| = \lambda \quad (s \in \mathbb{R}) \qquad (* * *)$$

for some constant $0 \leq \lambda \leq r$. If $\lambda = 0$, then $\alpha$ itself is a constant map, which contradicts $|\alpha'(s)| = 1$ for all $s \in \mathbb{R}$. This implies that $0 < \lambda \leq r$.

Now assume that $\lambda < r$. Then we can write

$$\alpha(s) = (x(s), y(s)) = \big(a_1 + \lambda \cos\theta(s), a_2 + \lambda \sin\theta(s)\big) \quad (s \in \mathbb{R}),$$

for some $\theta : \mathbb{R} \to \mathbb{R}$. Now notice that $\alpha$ is not even locally constant, since $|\alpha'(s)| = 1$ for all $s \in \mathbb{R}$. This implies that, except possibly for the points where either the cosine or the sine functions vanish, $\theta$ is differentiable. For any such point $s \in \mathbb{R}$, we have

$$\alpha'(s) = \big( - \lambda \sin\theta(s)\theta'(s), \lambda \cos\theta(s)\theta'(s)\big).$$

Now since $|\alpha'(s)| = 1$, we have $|\theta'(s)| = 1/\lambda$, and hence $\theta'(s) = \pm 1/\lambda$. From this we have

$$\begin{aligned}
\alpha'(s) &= \big(\mp \sin\theta(s), \pm \cos\theta(s)\big), \\
\alpha''(s) &= \big(\mp \cos\theta(s)\theta'(s), \mp \sin\theta(s)\theta'(s)\big) = \Big( - \frac{1}{\lambda}\cos\theta(s), -\frac{1}{\lambda}\sin\theta(s)\Big)
\end{aligned}$$

Now using the Frenet's equations we have $|k(s)| = 1/\lambda$. However, using equation $(**)$, we have $1/\lambda^2 \leq 1/r^2$, which contradict $\lambda < r$. Denoting by $C[\mathbf{a}; r]$, the set of points on the circle with center $\mathbf{a}$ and radius $r$, we have $\alpha(s) \in C[\mathbf{a}; r]$ for every $s \in \mathbb{R}$, where $\theta(s)$ is differentiable.

Now if $\theta$ is differentiable at $s$, we have $\theta(s) = \pm\frac{1}{r}s+\mu$, for some constant $\mu \in \mathbb{R}$, where the sign would depend on the point $s$. This implies that

$$\alpha(s) = \Big(a_1 + r\cos\big((\pm 1/r)s + \mu\big), a_2 + r\sin\big((\pm 1/r)s + \mu\big)\Big).$$

Finally, since $\alpha$ is continuous, we have the above equation hold for every $s \in \mathbb{R}$, where the sign is either $+$ or $-$ for all $s \in \mathbb{R}$.

**Question 2.** Let $\alpha$ and $\beta$ be two regular plane curves defined, respectively, on two open intervals of $\mathbb{R}$ containing the origin. Suppose that $\alpha(0) = \beta(0) = p$ and that $\alpha'(0) = \beta'(0)$.

We say that $\alpha$ **is over** $\beta$ **at** $p$ if there is a neighbourhood of 0 in $\mathbb{R}$ where $\langle \alpha - \alpha(0), N_\alpha(0)\rangle \geq \langle \beta - \beta(0), N_\beta(0)\rangle$. Prove that:

(i) If $\alpha$ is over $\beta$ at $p$, then $k_\alpha(0) \geq k_\beta(0)$.

(ii) If $k_\alpha(0) > k_\beta(0)$, then $\alpha$ is over $\beta$ at $p$.

**Solution.** Let $\alpha$ and $\beta$ be defined on the open intervals $I_1$ and $I_2$ and we consider the open intervals $I := I_1 \cap I_2$. By hypothesis, we have

$$T_\alpha(0) = \frac{\alpha'(0)}{|\alpha'(0)|} = \frac{\beta'(0)}{|\beta'(0)|} = T_\beta(0)$$

and consequently,

$$N_\alpha(0) = JT_\alpha(0) = JT_\beta(0) = N_\beta(0).$$

We denote this common normal at 0 by $u$ (and we have $|u| = 1$). Now define $f, g : I \to \mathbb{R}$ by

$$f(s) := \langle \alpha(s) - p, u\rangle, \quad g(s) := \langle \beta(s) - p, u\rangle \qquad (s \in I).$$

(i) By hypothesis, there exists an open interval $I_0 \subseteq I$ containing 0 such that $f(s) \geq g(s)$ for all $s \in I_0$. Define $\varphi : I \to \mathbb{R}$ by

$$\varphi(s) := f(s) - g(s) \ (s \in I).$$

Since $\alpha, \beta$ are $C^\infty$-functions, so is $\varphi$. Now, $f(0) = g(0)$ and hence $\varphi$ has a local minima at $s = 0$ (within $I_0$). Now if $\varphi$ is locally constant at 0, say within a neighbourhood $0 \in I_0' \subseteq I_0$, then $\varphi(s) = 0$ for all $s \in I_0'$. In that case,

$$\langle \alpha(s) - \beta(s), u\rangle = 0$$

for all $s \in I_0'$ and consequently, $\alpha = \beta$ on $I_0'$. In this case $k_\alpha(0) = k_\beta(0)$. So assume that $\varphi$ is not locally constant at 0. Then $\varphi'(0) = 0$ and $\varphi''(0) > 0$. Note that

$$\varphi''(0) = f''(0) - g''(0) = \langle \alpha''(0) - \beta''(0), u\rangle > 0.$$

**Step 1.** We deduce

$$\frac{d}{ds}\Big|_{s=0}|\alpha'(s)| \;=\; \frac{1}{|\alpha'(0)|}\langle \alpha''(0), \alpha'(0)\rangle.$$

If we write $\alpha(s) = (x_\alpha(s), y_\alpha(s))$. Then $|\alpha'(s)|^2 = x'_\alpha(s)^2 + y'_\alpha(s)^2$. Differentiating this with respect to $s$ we have

$$2|\alpha'(s)|\;\;\frac{d}{ds}|\alpha'(s)| = 2\Big(x'_\alpha(s)x''_\alpha(s) + y'_\alpha(s)y''_\alpha(s)\Big) = 2\langle \alpha''(s), \alpha'(s)\rangle,$$

which implies that

$$\frac{d}{ds}|\alpha'(s)| = \frac{1}{|\alpha'(s)|}\;\langle \alpha''(s), \alpha'(s)\rangle,$$

and then the expression follows by setting $s = 0$ to both sides of the equation.

**Step 2.** Now we deduce

$$T'_\alpha(0) = \frac{\alpha''(0)}{|\alpha'(0)|} - \frac{\langle \alpha''(0), \alpha'(0)\rangle}{|\alpha'(0)|^3}\alpha'(0).$$

Differentiating $T_\alpha(s) = \alpha'(s)/|\alpha'(s)|$, we have

$$T'_\alpha(s) = \frac{\alpha''(s)|\alpha'(s)| - \alpha'(s)\frac{d}{ds}|\alpha'(s)|}{|\alpha'(s)|^2} = \frac{\alpha''(s)}{|\alpha'(s)|} - \frac{\alpha'(s)}{|\alpha'(s)|^2}\frac{d}{ds}|\alpha'(s)|.$$

Now setting $s = 0$ in above equation and using Step 1, we obtain

$$T'_\alpha(0) = \frac{\alpha''(0)}{|\alpha'(0)|} - \frac{\alpha'(0)}{|\alpha'(0)|^2}\;\frac{d}{ds}\Big|_{s=0}|\alpha'(s)| = \frac{\alpha''(0)}{|\alpha'(0)|} - \frac{\alpha'(0)}{|\alpha'(0)|^3}\langle \alpha''(0), \alpha'(0)\rangle$$

**Step 3.** We deduce

$$k_\alpha(0) = \frac{1}{|\alpha'(0)|}\langle \alpha''(0), u\rangle, \qquad k_\beta(0) = \frac{1}{|\beta'(0)|}\langle \beta''(0), u\rangle$$

From definition of curvature and using $u = N_\alpha(0)$, we have

$$k_\alpha(0) = \langle T'_\alpha(0), u\rangle = \frac{1}{|\alpha'(0)|}\langle \alpha''(0), u\rangle - \frac{\langle \alpha''(0), \alpha'(0)\rangle}{|\alpha'(0)|^3}\langle \alpha'(0), u\rangle.$$

Now since $\alpha'(0)$ is orthogonal to $u$, the first statement follows. The second statement can be similarly obtained by replacing $\alpha$ by $\beta$ in the Steps 1, 2 and as arguing as above.

**Final Step.**

Using the fact $\alpha'(0) = \beta'(0)$ and Step 3, we have

$$k_\alpha(0) - k_\beta(0) = \frac{1}{|\alpha'(0)|}\langle \alpha''(0) - \beta''(0), u\rangle = \frac{1}{|\alpha'(0)|}\varphi''(0) > 0.$$

(ii) In the above steps, we have not used the definition of $\alpha$ is over $\beta$ at $p$. So, repeating these arguments it follows that, if $k_\alpha(0) - k_\beta(0) > 0$, then $\varphi'' > 0$. Now from hypothesis, we have $\varphi'(0) = \langle \alpha'(0) - \beta'(0), u \rangle = 0$. Hence $\varphi$ has a local minima at $s = 0$. The statement is now immediate from this.

**Question 3.** Let $O_1, O_2 \subseteq \mathbb{R}^3$ be Euclidean open sets and let $\phi : O_1 \to O_2$ be a diffeomorphism. If $S \subseteq O_1$ is a surface, show that $\phi(S)$ is a surface. Prove that the restriction map $\phi : S \to \phi(S)$ is a diffeomorphism.

**Solution.** Let $q \in \phi(S)$ be any arbitrary point. Let $p \in S$ such that $\phi(p) = q$. Since $S$ is a surface, there exists an open set $U \subseteq \mathbb{R}^2$, an open neighbourhood $V$ of $p$ in $S$, and a differentiable map $X : U \to \mathbb{R}^3$ such that:

(i) $X(U) = V$,

(ii) $X : U \to V$ is a homeomorphism,

(iii) $(dX)_u : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $u \in U$.

Since a diffeomorphism is a homeomorphism, we have $\phi(V)$ is an open neighbourhood of $q = \phi(p)$ in $\phi(S)$, and $\phi \circ X : U \to \phi(V)$ is a homeomorphism. Next, using chain rule, we have for any $u \in U$,

$$d(\phi \circ X)_u = d\phi_{X(u)} \circ (dX)_u,$$

where $d\phi_{X(u)} : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear isomorphism. This implies that $d(\phi \circ X)_u : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $u \in U$. This proves that $\phi(S)$ is a surface.

We use the notation $\psi : S \to \phi(S)$ to denote this restriction map. If $i : \phi(S) \to \mathbb{R}^3$ denote the inclusion map, then $i \circ \psi = \phi$. Now let $X : U \to \mathbb{R}^3$ be an arbitrary parametrization of $S$ (and we can assume the properties (i)-(iii) listed as above for this $X$). Since $\phi : O_1 \to \mathbb{R}^3$ is differentiable, we have $\phi \circ X : U \to \mathbb{R}^3$ is differentiable, and hence $(i \circ \psi) \circ X = \phi \circ X$ is differentiable function. Hence, from definition, we have $\psi$ is a differentiable map in sense of definition 2.27 **A.** (Page 40, Montiel & Ros). Reversing the roles of $O_1$ and $O_2$, we can similarly show that $\psi^{-1} : \phi(S) \to S$ is also differentiable. Hence $\psi$ is a diffeomorphism.

**Question 4.** Let $S$ be the right cylinder of radius $r > 0$, whose axis is the line passing through the origin with direction $\mathbf{a}$ (with $|\mathbf{a}| = 1$), given by

$$S := \Big\{ p \in \mathbb{R}^3 \ : \ |p|^2 - \langle p, \mathbf{a} \rangle^2 = r^2 \Big\}.$$

Prove that

$$T_pS = \Big\{ v \in \mathbb{R}^3 \ : \ \langle p, v\rangle - \langle p, a\rangle\langle a, v\rangle = 0 \Big\}.$$

Conclude that all the normal lines of $S$ cut the axis perpendicularly.

**Solution.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ denote the function defined by

$$f(p) := \langle p, p\rangle - \langle p, \mathbf{a}\rangle^2 \quad (p \in \mathbb{R}^3).$$

Denoting $\mathbf{a} = (a_1, a_2, a_3)$, we have $a_1^2 + a_2^2 + a_3^2 = 1$ and we can write this function explicitly as

$$f(x, y, z) = (x^2 + y^2 + z^2) - (a_1x + a_2y + a_3z)^2 \quad \Big((x, y, z) \in \mathbb{R}^3\Big).$$

If $M_{(x,y,z)}$ denote the matrix of the linear map $(df)_{(x,y,z)}$ with respect to the canonical basis of $\mathbb{R}^3$, then

$$M_{(x,y,z)} = \Big(2x-2(a_1x+a_2y+a_3z)a_1,\ 2y-(a_1x+a_2y+a_3z)a_2,\ 2z-2(a_1x+a_2y+a_3z)a_3\Big)$$

Now, if $(df)_{(x,y,z)} = (0, 0, 0)$, then $(x, y, z) \in \mathbb{R}^3$ satisfies the equations:

$$\begin{aligned} x &= a_1(a_1x + a_2y + a_3z), \\ y &= a_2(a_1x + a_2y + a_3z), \\ z &= a_3(a_1x + a_2y + a_3z). \end{aligned}$$

Now if $f(x, y, z) = r^2$, then

$$\begin{aligned} r^2 &= \Big(a_1(a_1x + a_2y + a_3z)\Big)^2 + \Big(a_2(a_1x + a_2y + a_3z)\Big)^2 + \Big(a_3(a_1x + a_2y + a_3z)\Big)^2 \\ &\quad -(a_1x + a_2y + a_3z)^2 = 0 \end{aligned}$$

since $a_1^2 + a_2^2 + a_3^2 = 1$. This is absurd, since $r > 0$. This shows that $r^2$ is a regular value of the function $f$ and the cylinder $S$ is given as $S = f^{-1}(\{r^2\})$.

Recall from Example 2.51, we have $T_{(x,y,z)}S = \ker\big((df)_{(x,y,z)} : \mathbb{R}^3 \to \mathbb{R}\big)$. The linear map $(df)_{(x,y,z)}$ is given by

$$\begin{aligned} (df)_{(x,y,z)} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= M_{(x,y,z)} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= \Big(2x - 2(a_1x + a_2y + a_3z)a_1\Big)v_1 + \Big(2y - (a_1x + a_2y + a_3z)a_2\Big)v_2 \\ &\quad + \Big(2z - 2(a_1x + a_2y + a_3z)a_3\Big)v_3 \end{aligned}$$

Now setting $p = (x, y, z)$ and $v = (v_1, v_2, v_3)$ we have

$$(df)_p(v) = 2\langle p, v\rangle - 2\langle p, a\rangle\langle a, v\rangle \quad (v \in \mathbb{R}^3).$$

Then the required description of $T_pS$ follows immediately.

Next, modifying the description of $T_pS$, we can write

$$\begin{aligned} T_pS &= \Big\{v \in \mathbb{R}^3 \ : \ \langle v, p\rangle - \langle p, a\rangle\langle v, a\rangle = 0\Big\} \\ &= \Big\{v \in \mathbb{R}^3 \ : \ \langle v, p - \langle p, a\rangle a\rangle = 0\Big\} \end{aligned}$$

which is the equation of the plane that passes through origin and is normal to the vector $p - \langle p, a\rangle a$. Now, for any $p \in S$, we have

$$\langle p - \langle p, a\rangle a, a\rangle = \langle p, a\rangle - \langle p, a\rangle\langle a, a\rangle = 0,$$

since $\langle a, a\rangle = |a|^2 = 1$. This proves that for every $p \in S$, the normal line $\mathbb{R}(p - \langle p, a\rangle a)$ at $p$ is orthogonal to the axis $\mathbb{R}a$ of $S$.