MTH 406 : Differential Geometry of Curves and Surfaces

Homework 2 9th February, 2024

Deadline of submission. 14th February, 2024 prior to the class time.

Question 1. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve and let R be a straight line in $\mathbb{R}^2$. If one can find a number $t_0 \in I$ such that the distance from $\alpha(t)$ to R is greater than or equal to the distance from $\alpha(t_0)$ to R, for all $t \in I$, and such that $\alpha(t_0) \notin R$, then show that the tangent line of α at t_0 is parallel to R.

Question 2. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve. Let $[a, b] \subset I$ be such that $\alpha(a) \neq \alpha(b)$. Prove that there exists some $t_0 \in (a, b)$ such that the tangent line of α at t_0 is parallel to the segment of the straight line joining $\alpha(a)$ with $\alpha(b)$. (This is a generalization of Rolle's theorem of elementary calculus.)

Question 3. Prove that a curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l. is a segment of a straight line if and only if all its tangent lines are concurrent.

Question 4. Prove that a curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l. is an arc of a circle if and only if all its normal lines pass through a given point.

Question 5. Prove that a curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l. is a segment of a straight line or an arc of a circle if and only if all its tangent lines are equidistant from a given point.

Question 6. Given a curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l., prove that all the normal lines of α are equidistant from a point if and only if there are $a, b \in \mathbb{R}$ such that

$$k(s) = \pm \frac{1}{\sqrt{as + b}}$$

for each $s \in I$.

Solutions.

Question 1. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve and let R be a straight line in $\mathbb{R}^2$. If one can find a number $t_0 \in I$ such that the distance from $\alpha(t)$ to R is greater than or equal to the distance from $\alpha(t_0)$ to R, for all $t \in I$, and such that $\alpha(t_0) \notin R$, then show that the tangent line of α at t_0 is parallel to R.

Solution. Choose any point a from the straight line R and fix it. Let u denote the unit normal vector to the straight line R. Define $f : I \to \mathbb{R}$ by

$$f(t) := \langle \alpha(t) - a, u\rangle \quad (t \in I).$$

Then the function f determine the oriented distance from the $\alpha(t)$ to the line R (see the figure below).

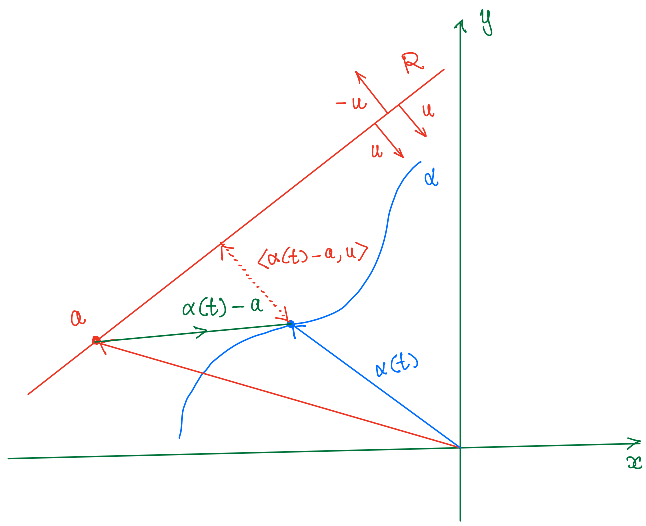

FIGURE 1

Since α is differentiable, so is f on I. Next, $\alpha(t_0) \notin R$, implies that $f(t_0) \neq 0$. Using continuity of f, there is an open neighbourhood $J \subseteq I$ of t_0 such that $f(t) \neq 0$ for all $t \in J$. This means that either $f(t) > 0$ for all $t \in J$, or $f(t) < 0$ for all $t \in J$. Replacing f by $-f$ if necessary (which amounts to replace u by $-u$), we may assume that $f(t) > 0$ for all $t \in J$, which means that the modified function f now measures the distance of $\alpha(t)$ from R at each $t \in J$. From the given hypothesis, it follows that f has a local minima at t_0 and hence

$$\langle \alpha'(t_0), u\rangle = f'(t_0) = 0.$$

Let T_{t_0} denote the (affine) tangent line of α at t_0. Then T_{t_0} is parallel to the line represented by the subspace $\mathbb{R}\alpha'(t_0)$, and from above equation it follows that $\mathbb{R}\alpha'(t_0)$ is orthogonal to the normal direction u of R. Hence $\mathbb{R}\alpha'(t_0)$ (and hence T_{t_0}) must be parallel to R.

Question 2. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve. Let $[a, b] \subset I$ be such that $\alpha(a) \neq \alpha(b)$. Prove that there exists some $t_0 \in (a, b)$ such that the tangent line of α at t_0 is parallel to the segment of the straight line joining $\alpha(a)$ with $\alpha(b)$. (This is a generalization of Rolle's theorem of elementary calculus.)

Solution. The segment of the straight line joining $\alpha(a)$ with $\alpha(b)$ is represented by the vector $\alpha(b) - \alpha(a)$ (see the figure below).

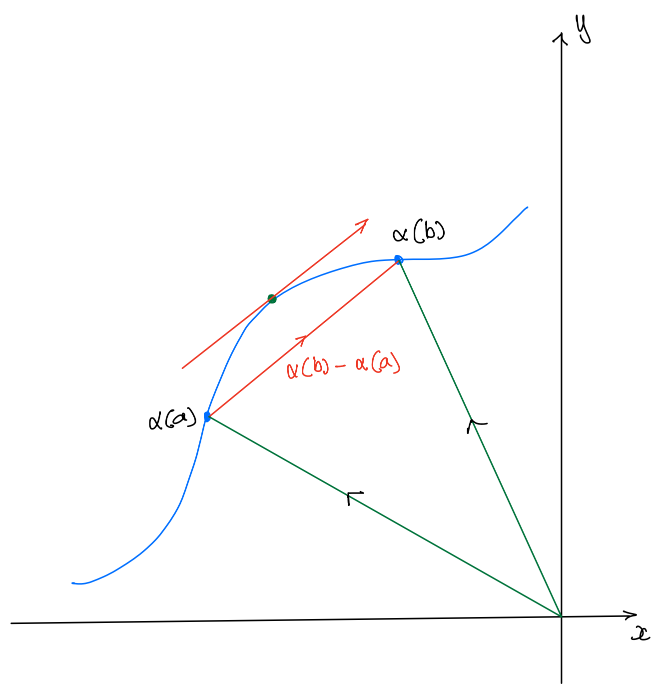

Figure 2

This is parallel to the tangent line of α at t if and only if the vectors $\alpha(b) - \alpha(a)$ and $\alpha'(t)$ are scalar multiple of each other as a member of the 2-dimensional $\mathbb{R}$-linear space $\mathbb{R}^2$. Now we consider the map $f : [a, b] \to \mathbb{R}$ defined by

$$f(t) := \det\big(\alpha(t), \alpha(a) - \alpha(b)\big) \quad (a \leq t \leq b).$$

Since α is regular and $\alpha(a) \neq \alpha(b)$, the function f is differentiable and doesn't vanish on the entire domain $[a, b]$. To apply Rolle's theorem here, we need to verify $f(a) = f(b)$. We write

$$\alpha(a) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \alpha(b) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \alpha(a) - \alpha(b) = \begin{pmatrix} x \\ y \end{pmatrix},$$

and consequently, $a_1 - b_1 = x, a_2 - b_2 = y$. Then, we have

$$\begin{aligned} f(a) &= \det \begin{pmatrix} a_1 & x \\ a_2 & y \end{pmatrix} = a_1 y - a_2 x = (b_1 + x)y - (b_2 + y)x = b_1 y - b_2 x \\ &= \det \begin{pmatrix} b_1 & x \\ b_2 & y \end{pmatrix} = f(b). \end{aligned}$$

which proves that $f(a) = f(b)$. Now using Rolle's theorem, we have

$$\det\big(\alpha'(t_0), \alpha(a) - \alpha(b)\big) = f'(t_0) = 0$$

for some $t_0 \in (a, b)$. Since $\alpha(a) - \alpha(b)$ is a non-zero vector, this implies that $\alpha(a) - \alpha(b)$ is a scalar multiple of $\alpha'(t_0)$ (as elements of the 2-dimensional $\mathbb{R}$-linear space $\mathbb{R}^2$), and consequently, are parallel to each other.

Question 3. Prove that a curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l. is a segment of a straight line if and only if all its tangent lines are concurrent.

Solution. If the α is a segment of a straight line, then all its tangent lines are concurrent. So we assume that $\alpha'(t)$ $(t \in I)$ are concurrent; i.e., they all pass through a single point, say $\mathbf{a} \in \mathbb{R}^2$.

Then for each $t \in I$, we have the vector $\alpha(t) - \mathbf{a}$ is parallel to $\alpha'(t)$. We define $g : I \to \mathbb{R}$ to be a map so that

$$\alpha(t) = g(t)\alpha'(t) + \mathbf{a}, \qquad (t \in I) \qquad\qquad (*)$$

Since α is p.b.a.l., we have $\alpha'(t) \neq 0$ for all $t \in I$, and consequently, g is a differentiable map.

We claim that $\alpha''(t) = 0$ for all $t \in I$. From this claim, using primitive function theorem of calculus applied to each component of α, it will follow that α is a straight line.

Suppose $\alpha''(t_0) \neq 0$ for some $t_0 \in I$. Then the set $\{t \in I \ : \ \alpha''(t) \neq 0\}$ is an open subset of I, using the fact that α'' is a differentiable (and hence continuous) map. Then there exists an open interval $J \subseteq I$ containing t_0 such that $\alpha''(t) \neq 0$ for all $t \in J$.

Now differentiating both sides of the equation in $(*)$ we have

$$\big(-1 + g'(t)\big)\alpha'(t) + g(t)\alpha''(t) = 0 \qquad (t \in J).$$

For any $t \in J$, we have $\alpha''(t) \neq 0$, and hence $\alpha''(t)$ is parallel to the normal to α at t. This implies that $\{\alpha'(t), \alpha''(t)\}$ is a linearly independent subset of $\mathbb{R}$. From above equation, it follows that the restriction $g|_J \colon J \to \mathbb{R}$ is the zero map, whereas $g'(t) = 1$ for all $t \in J$, which is absurd. This proves our claim.

Question 4. Prove that a curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l. is an arc of a circle if and only if all its normal lines pass through a given point.

Solution. If α is an arc of a circle, then all its normal lines are concurrent and pass through the center of the corresponding circle. So we assume that all normal lines of α pass through a single point $\mathbf{a} \in \mathbb{R}^2$. Define the map $f : I \to \mathbb{R}$ by

$$f(t) := \langle \alpha(t) - \mathbf{a}, \alpha(t) - \mathbf{a} \rangle, \qquad (t \in I).$$

Since α is regular, f is differentiable and

$$f'(t) = 2\langle \alpha(t) - \mathbf{a}, \alpha'(t) \rangle \qquad (t \in I).$$

As α is p.b.a.l., we have $\alpha'(t)$ is equal to the unit tangent to α at t, and by hypothesis, $\alpha(t) - \mathbf{a}$ is parallel to the normal line to α at t. This implies that $f'(t) = 0$ for all $t \in I$. This implies that f is constant, say $f(t) = r^2 \geq 0$ for all $t \in I$. If $r = 0$, then α is identically equal to $\mathbf{a}$ on I, contradicting that $\alpha'(t) \neq 0$ for all $t \in I$ (since α is p.b.a.l.) Hence α represents an arc of the circle with radius $r > 0$ and center $\mathbf{a}$.

Question 5. Prove that a curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l. is a segment of a straight line or an arc of a circle if and only if all its tangent lines are equidistant from a given point.

Solution. "$\Leftarrow$" In case α is a segment of a straight line, the tangents at the points of α at various points coincide with the curve itself, and hence for any given point $\mathbf{a} \in \mathbb{R}^2$, the distance between $\alpha'(t) = T_\alpha(t)$ and $\mathbf{a}$ is constant. In case α is an arc of a circle, the same statement hold by choosing $\mathbf{a}$ as the center of the corresponding circle.
"$\Rightarrow$" We assume that there exists $\mathbf{a} \in \mathbb{R}^2$ such that $\mathrm{dist}(\mathbf{a}, \alpha'(t))$ is constant for all $t \in I$. This implies that

$$\langle \alpha(t) - \mathbf{a}, N(t) \rangle = \pm c$$

for some fixed real number $c \geq 0$. In case $c > 0$, using the fact that the left side of the above equation represents a continuous function, there exists an open subinterval of I, where the function doesn't change its sign. So, by replacing $N(t)$ by $-N(t)$ if necessary, we may assume that

$$\langle \alpha(t) - \mathbf{a}, N(t) \rangle = c$$

for all $t \in I$. Differentiating this w.r.t. t we have

$$\langle \alpha'(t), N(t) \rangle + \langle \alpha(t) - \mathbf{a}, N'(t) \rangle = 0.$$

Since $T(t) = \alpha'(t)$ (as α is p.b.a.l.) we have $\langle \alpha'(t), N(t) \rangle = 0$, and hence using the second Frenet's equation $N'(t) = -k(t)T(t)$, we have

$$-k(t)\langle \alpha(t) - \mathbf{a}, T(t) \rangle = 0 \qquad (t \in I).$$

Now if $k(t) \neq 0$, then $\langle \alpha(t) - \mathbf{a}, T(t) \rangle = 0$. Differentiating this equation (notice that $k(s) \neq 0$ on a neighbourhood of t) w.r.t. t we have

$$1 + \langle \alpha(t) - \mathbf{a}, T'(t) \rangle = \langle \alpha'(t), T(t) \rangle + \langle \alpha(t) - \mathbf{a}, T'(t) \rangle = 0.$$

Now using the first Frenet's equation and our hypothesis, we have $1 + k(t)c = 0$. This implies that for all $t \in I$, either $k(t) = 0$, or else $c > 0$ and $k(t) = -1/c$. Since curvature is a continuous function, we have either $k \equiv 0$ on I, or else $k \equiv -1/c$ on I. Now using the calculations made in Exercise 1.19 (not just the statement there), the statement follows.

Question 6. Given a curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l., prove that all the normal lines of α are equidistant from a point if and only if there are $a, b \in \mathbb{R}$ such that

$$k(s) = \pm \frac{1}{\sqrt{as + b}}$$

for each $s \in I$.

Solution. "$\Rightarrow$" The normal line of α at t passes through $\alpha(t)$ and is perpendicular to the unit tangent vector $T(t) = \alpha'(t)$. The distance of the normal line of α at t from a point $\mathbf{a} \in \mathbb{R}^2$ is the square root of the absolute value of $\langle \alpha(t) - \mathbf{a}, T(t) \rangle$, which is the oriented distance of $N(t)$ to $\mathbf{a}$ (see the figure below).

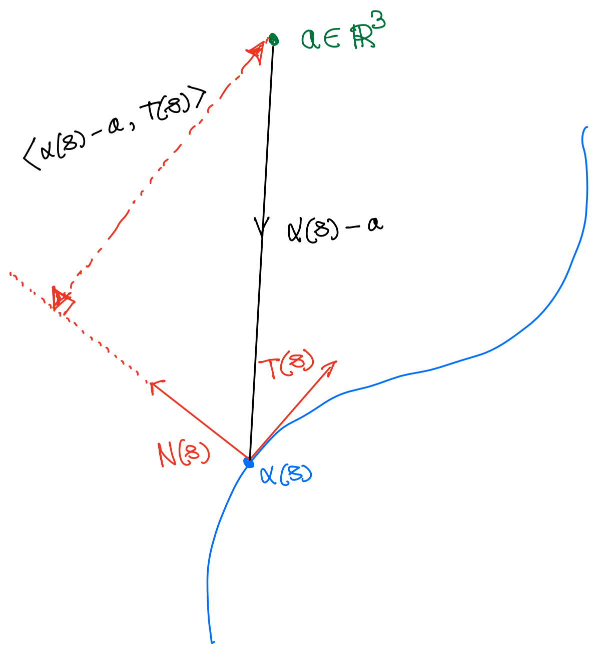

FIGURE 3

Hence, there exists a constant $c \in \mathbb{R}$ such that

$$\langle \alpha(t) - \mathbf{a}, T(t) \rangle = c \qquad (t \in I) \qquad\qquad (*)^I.$$

Differentiating this w.r.t. t and using first Frenet's equation (as in the previous exercise) we have

$$1 + k(t)\langle \alpha(t) - \mathbf{a}, N(t) \rangle = 0 \qquad (t \in I) \qquad\qquad (*)^{II}.$$

Differentiating this again w.r.t. t we have

$$k'(t)\langle \alpha(t) - \mathbf{a}, N(t) \rangle + k(t)\Big(\langle \alpha'(t), N(t) \rangle + \langle \alpha(t) - \mathbf{a}, N'(t) \rangle \Big) = 0 \quad (t \in I).$$

The first term inside the bracket as above is zero, and using the other Frenet's equation we obtain

$$k'(t)\langle \alpha(t) - \mathbf{a}, N(t) \rangle - k(t)^2 \langle \alpha(t) - \mathbf{a}, T(t) \rangle = 0 \quad (t \in I).$$

Multiplying by $k(t)$ to both sides and using the equations $(*)^I, (*)^{II}$, we have

$$k'(t) + ck(t)^3 = 0 \qquad (t \in I).$$

From equation $(*)^I$, it follows that $k(t) \neq 0$ for all $t \in I$. Then a straightforward calculation using the derivative formulas and the above equation we obtain $(1/k(t)^2)'' = 0$, which means that $1/k(t)^2$ is the equation of a straight line, and the assertion follows.

"$\Rightarrow$" Conversely, suppose there exists $a, b \in \mathbb{R}$ such that

$$k(t) = \pm\frac{1}{\sqrt{at+b}} \qquad (t \in I).$$

Then $k(t) \neq 0$ for any $t \in I$. Now,

$$a = \Big(\frac{1}{k(t)^2}\Big)' = 2\Big(\frac{1}{k(t)}\Big)\Big(\frac{1}{k(t)}\Big)' \qquad (*)^{III}$$

Now define a function $f : I \to \mathbb{R}^2$ by

$$f(t) := \alpha(t) + \frac{1}{k(t)}N(t) - \frac{a}{2}T(t) \qquad (t \in I).$$

Then we have

$$\begin{aligned} f'(t) &= \alpha'(t) + \Big(\frac{1}{k(t)}\Big)'N(t) + \frac{1}{k(t)}N'(t) - \frac{a}{2}T'(t) \\ &= \alpha'(t) + \frac{a}{2}k(t)N(t) + \frac{1}{k(t)}N'(t) - \frac{a}{2}k(t)N(t) \qquad ((*)^{III} \text{ and Frenet's eqn}) \\ &= \alpha'(t) + \frac{1}{k(t)}N'(t) \\ &= \alpha'(t) - T(t) = 0 \qquad (\text{Frenet's eqn}) \end{aligned}$$

This implies that, there exists $\mathbf{a} \in \mathbb{R}^2$ such that $f(t) = \mathbf{a}$ for all $t \in I$, and hence

$$\alpha(t) - \mathbf{a} = \frac{a}{2}T(t) - \frac{1}{k(t)}N(t) \qquad (t \in I).$$

Then for any $t \in I$, we have $\langle \alpha(t) - \mathbf{a}, T(t)\rangle = \frac{a}{2}$, which proves our assertion.