MTH 406 : Differential Geometry of Curves and Surfaces

Homework 1 25th January, 2024

In the following exercises $I \subset \mathbb{R}$ always denotes an open interval, which could possibly be unbounded as well. All curves are assumed to be C^∞ unless otherwise mentioned.

Deadline of submission. January 31st, 2024 prior to the class time.

Question 1. Let $\alpha : I \to \mathbb{R}^3$ be a curve and $M : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Prove that $L_a^b(\alpha) = L_a^b(M \circ \alpha)$ for any $[a, b] \subset I$; i.e., rigid motions preserve the length of a curve.

Question 2. Let $\alpha : I \to \mathbb{R}^3$ be a curve. Show that

$$L_a^b(\alpha) = \sup \left\{ L_a^b(\alpha, P) \ : \ P \text{ is a partition of } [a, b] \right\}.$$

Question 3. Let $\phi : J \to I$ be a diffeomorphism and let $\alpha : I \to \mathbb{R}^3$ be a curve. Given $[a, b] \subset J$ with $\phi([a, b]) = [c, d]$, prove that $L_a^b(\alpha \circ \phi) = L_c^d(\alpha)$.

Question 4. Consider the logarithmic spiral $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t) \quad (t \in \mathbb{R})$$

with $a > 0, b < 0$. Compute the arc length function $S : \mathbb{R} \to \mathbb{R}$ for $t_0 \in \mathbb{R}$. Reparametrize this curve by arc length and study its trace.

Question 5. (Curvature for non p.b.a.l. curves) Let $\alpha : I \to \mathbb{R}^2$ be a regular curve, not necessarily p.b.a.l. Recall that there exists an interval $J \subset \mathbb{R}$ and a diffeomorphism $f : J \to I$ such that $\beta := \alpha \circ f$ is a curve p.b.a.l. having the same trace as α. We define the curvature of α at time $t \in I$ by

$$k_\alpha(t) := k_\beta(f^{-1}(t)) \quad (t \in I).$$

Prove that

$$k_\alpha(t) = \frac{1}{|\alpha'(t)|^3} \det(\alpha'(t), \alpha''(t)) = \det(T(t), T'(t)) \quad (t \in I),$$

where $T(t) = \alpha'(t)/|\alpha'(t)|$ is the unit vector tangent of α at time t. (Note: This proves that the curvature of α does not depend on the choice of the diffeomorphism).

Question 6. Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l. and let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a rigid motion. Define $\beta := M \circ \alpha$ and prove that

$$k_\beta(s) = \begin{cases} k_\alpha(s) & \text{for all } s \in I, \text{if } M \text{ is direct,} \\ -k_\alpha(s) & \text{for all } s \in I, \text{if } M \text{ is inverse.} \end{cases}$$

Question 7. Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l. Prove that α is a segment of a straight line or an arc of a circle if and only if the curvature of α is constant.

Question 8. Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l. If there is a differentiable function $\theta : I \to \mathbb{R}$ such that $\theta(s)$ is the angle that the tangent line of α at s makes with a fixed direction, show that $\theta'(s) = \pm k(s)$.

Question 9. Let $\alpha, \beta : I \to \mathbb{R}^2$ be two curves p.b.a.l. such that $k_\alpha(s) = -k_\beta(s)$ for every $s \in I$. Show that there is an inverse rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\beta = M \circ \alpha$.

Question 10. If $\alpha : I \to \mathbb{R}^2$ is a curve p.b.a.l. with $0 \in I$ and symmetric around 0, we define another curve $\beta : I \to \mathbb{R}^2$ by $\beta(s) := \alpha(-s)$ for every $s \in I$. Prove that β is p.b.a.l. and that $k_\beta(s) = -k_\alpha(-s)$ for each $s \in I$.

Solutions.

Question 1. Let $\alpha : I \to \mathbb{R}^3$ be a curve and $M : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Prove that $L_a^b(\alpha) = L_a^b(M \circ \alpha)$ for any $[a, b] \subset I$; i.e., rigid motions preserve the length of a curve.

Solution. There exists an $A \in \mathrm{O}(3)$ and $\mathbf{b} \in \mathbb{R}^3$ such that

$$M(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \quad (\mathbf{x} \in \mathbb{R}^3).$$

Since A is a linear map, we have $M'(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ a linear map given by

$$M'(\mathbf{x})(v) = Av \quad (v \in \mathbb{R}^3)$$

for any $\mathbf{x} \in \mathbb{R}^3$ (In other words, $M'(\mathbf{x}) = A$). Now using chain rule, we have

$$(M \circ \alpha)'(t) = M'(\alpha(t))\big(\alpha'(t)\big) = A\alpha'(t)$$

for every $t \in I$. Since $A \in \mathrm{SO}(n)$, we have $|Av| = |v|$ for all $v \in \mathbb{R}^3$. Now using the definition, we have

$$L_a^b(M \circ \alpha) = \int_a^b |(M \circ \alpha)'(t)| dt = \int_a^b |A\alpha'(t)| dt = \int_a^b |\alpha'(t)| dt = L_a^b(\alpha).$$

Question 2. Let $\alpha : I \to \mathbb{R}^3$ be a curve. Show that

$$L_a^b(\alpha) = \sup \big\{ L_a^b(\alpha, P) \ : \ P \text{ is a partition of } [a, b] \big\}.$$

Solution. Let $[u, v] \subseteq I$ be a closed interval. Then, from Schwarz inequality (for integrals) and fundamental theorem of integrals we have

$$|\alpha(v) - \alpha(u)| = \Big| \int_u^v \alpha'(t) dt \Big| \leq \int_u^v |\alpha'(t)| dt = L_u^v(\alpha).$$

Now, let $P = \{a = t_0 < t_1 < \ldots t_{n-1} < t_n = b\}$ be any partition of $[a, b]$. Then, by definition and using above argument we have,

$$L_a^b(\alpha, P) = \sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| \leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt = \int_a^b |\alpha'(t)| dt = L_a^b(\alpha).$$

This proves that

$$\sup \big\{ L_a^b(\alpha, P) \ : \ P \text{ is a partition of } [a, b] \big\} \leq L_a^b(\alpha).$$

To prove the equality, we take any arbitrary $\epsilon > 0$. From Proposition 1.6, there exists $\delta > 0$ such that

$$|P| < \delta \ \Rightarrow \ \big| L_a^b(\alpha, P) - L_a^b(\alpha) \big| < \epsilon \qquad (*)$$

Now let $N \in \mathbb{N}$ such that $d := \frac{b-a}{N} < \delta$ (such an N exists since $\lim_{n\to\infty} \frac{b-a}{n} = 0$), and consider the partition

$$P_0 := \{a = t_0 < t_1 < \ldots t_{N-1} < t_N = b\}$$

where $t_i := a + di$ $(0 \leq i \leq N)$. Then $|P| = d < \delta$ and hence from $(*)$ we have $L_a^b(\alpha) - \epsilon < L_a^b(\alpha, P_0)$. Using the definition of supremum, this proves the desired equality.

Question 3. Let $\phi : J \to I$ be a diffeomorphism and let $\alpha : I \to \mathbb{R}^3$ be a curve. Given $[a, b] \subset J$ with $\phi([a, b]) = [c, d]$, prove that $L_a^b(\alpha \circ \phi) = L_c^d(\alpha)$.

Solution. We first claim that, either $\phi'(t) > 0$ or $\phi'(t) < 0$ for every $t \in [a, b]$. Suppose $t_0 \in [a, b]$ such that $\phi'(t_0) = 0$. In case $t_0 = a$ or $t_0 = b$, we may extend the end points to make sure t_0 is an interior point of $[a, b]$. Now using chain rule we have

$$1 = (\phi^{-1} \circ \phi)'(t_0) = (\phi^{-1})'(\phi(t_0)) \cdot \phi'(t_0) = 0,$$

a contradiction. This proves our claim.

Now since ϕ is a homeomorphism, it maps boundary points to boundary points. Hence either $\phi(a) = c, \phi(b) = d$ or $\phi(a) = d, \phi(b) = c$.

First suppose $\phi'(t) > 0$ for every $t \in [a, b]$. Then $\phi : [a, b] \to [c, d]$ is increasing, and hence $\phi(a) = c, \phi(b) = d$. Then

$$L_a^b(\alpha\circ\phi) = \int_a^b (\alpha\circ\phi)'(t)dt = \int_a^b \big|\alpha'(\phi(t))\phi'(t)\big|dt = \int_a^b \big|\alpha'(\phi(t))\big|\phi'(t)dt.$$

Now we use the change of variables by setting $w := \phi(t)$. Then the above integral is equal to

$$\int_c^d |\alpha'(w)|dw = L_c^d(\alpha).$$

Next assume $\phi'(t) < 0$ for every $t \in [a, b]$. Then $\phi : [a, b] \to [c, d]$ is decreasing, and $\phi(a) = d, \phi(b) = c$. Now, using the same change of variable as above we obtain

$$L_a^b(\alpha\circ\phi) = -\int_a^b \big|\alpha'(\phi(t))\big|\phi'(t)dt = -\int_d^c |\alpha'(w)|dw = \int_c^d |\alpha'(w)|dw = L_c^d(\alpha).$$

Question 4. Consider the logarithmic spiral $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t) \quad (t \in \mathbb{R})$$

with $a > 0, b < 0$. Compute the arc length function $S : \mathbb{R} \to \mathbb{R}$ for $t_0 \in \mathbb{R}$. Reparametrize this curve by arc length and study its trace.

Solution. Here we have

$$\alpha'(t) = \big(ae^{bt}(b\cos t - \sin t), ae^{bt}(b\sin t + \cos t)\big)$$

and hence $|\alpha'(t)| = a\sqrt{b^2+1}e^{bt}$ for all $t \in \mathbb{R}$. The arc length function $S : \mathbb{R} \to \mathbb{R}$ from $t_0 \in \mathbb{R}$ is given by

$$S(t) = \int_{t_0}^{t} |\alpha'(u)|du = \frac{a\sqrt{b^2+1}}{b}\big(e^{bt} - e^{bt_0}\big) \quad (t \in \mathbb{R}).$$

To reparametrize the curve by arc length, we need to find a diffeomorphism $\phi : J \to \mathbb{R}$ with the necessary condition, where $J = S(\mathbb{R})$. Notice that, since $\frac{a\sqrt{b^2+1}}{b}e^{bt} < 0$ for all $t \in \mathbb{R}$, we have $J = S(I) = (-\infty, \frac{a\sqrt{b^2+1}}{-b}e^{bt_0})$. Now the expression of $\phi(s)$ can be found from the equation

$$s = \frac{a\sqrt{b^2+1}}{b}\big(e^{bt} - e^{bt_0}\big)$$

as $t = \phi(s)$. Solving this, we obtain $\phi : J \to \mathbb{R}$ given by

$$\phi(s) = \frac{1}{b}\ln\Big(\frac{b}{a\sqrt{b^2+1}}s + e^{bt_0}\Big).$$

The reparametrization $\beta : (-\infty, \frac{a\sqrt{b^2+1}}{-b}e^{bt_0}) \to \mathbb{R}$, given by $\beta = \alpha \circ \phi$ is given by

$$\begin{aligned}
\beta(s) &= \Big(ae^{b\phi(s)}\cos(\phi(s)), ae^{b\phi(s)}\sin(\phi(s))\Big) \\
&= \Big(\lambda_0(s)\cos\Big\{\frac{1}{b}\ln\Big(\frac{b}{a\sqrt{b^2+1}}s + e^{bt_0}\Big)\Big\}, \lambda_0(s)\sin\Big\{\frac{1}{b}\ln\Big(\frac{b}{a\sqrt{b^2+1}}s + e^{bt_0}\Big)\Big\}\Big)
\end{aligned}$$

where $\lambda_0(s) = ae^{bt_0} + \frac{bs}{\sqrt{b^2+1}}$.

Question 5. (Curvature for non p.b.a.l. curves) Let $\alpha : I \to \mathbb{R}^2$ be a regular curve, not necessarily p.b.a.l. Recall that there exists an interval $J \subset \mathbb{R}$ and a diffeomorphism $f : J \to I$ such that $\beta := \alpha \circ f$ is a curve p.b.a.l. having the same trace as α. We define the curvature of α at time $t \in I$ by

$$k_\alpha(t) := k_\beta(f^{-1}(t)) \quad (t \in I).$$

Prove that

$$k_\alpha(t) = \frac{1}{|\alpha'(t)|^3}\det(\alpha'(t), \alpha''(t)) = \det(T(t), T'(t)) \quad (t \in I),$$

where $T(t) = \alpha'(t)/|\alpha'(t)|$ is the unit vector tangent of α at time t. (Note: This proves that the curvature of α does not depend on the choice of the diffeomorphism).

Solution. We set $s := f^{-1}(t)$ and we write $\alpha(t) = (x(t), y(t))^T$, where $()^T$ denotes the transpose of a row vector. Then, we have

$$\alpha'(t) = (x'(t), y'(t))^T, \quad \alpha''(t) = (x''(t), y''(t))^T \quad (t \in I).$$

Now we have

$$\det(\alpha'(t), \alpha''(t)) = \begin{vmatrix} x'(t) & x''(t) \\ y'(t) & y''(t) \end{vmatrix} = x'(t)y''(t) - x''(t)y'(t) \quad (t \in I).$$

and

$$|\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2} \quad (t \in I).$$

As discussed in section 1.3 (Montiel, Ros. Page 8), we can assume that f is non-decreasing. Using chain rule, we have $1 = (\alpha \circ f)'(s) = \alpha'(f(s)) \cdot f'(s)$ and hence

$$\frac{1}{|\alpha'(f(s))|} = f'(s) > 0 \quad (s \in J) \qquad (**)$$

Now,

$$\begin{aligned} T_\beta(s) = \beta'(s) = (\alpha \circ f)'(s) &= \begin{pmatrix} f'(s)x'(f(s)) \\ f'(s)y'(f(s)) \end{pmatrix} \\ N_\beta(s) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T_\beta(s) = \begin{pmatrix} -f'(s)y'(f(s)) \\ f'(s)x'(f(s)) \end{pmatrix} \\ T'_\beta(s) = \beta''(s) &= \begin{pmatrix} f''(s)x'(f(s)) + f'(s)^2 x''(f(s)) \\ f''(s)y'(f(s)) + f'(s)^2 y''(f(s)) \end{pmatrix} \end{aligned}$$

Then $k_\beta(s) \in \mathbb{R}$ satisfies $T'_\beta(s) = k_\beta(s)N_\beta(s)$, which give the two equations given by

$$\begin{aligned} -k_\beta(s)f'(s)y'(f(s)) &= f''(s)x'(f(s)) + f'(s)^2 x''(f(s)) \\ k_\beta(s)f'(s)x'(f(s)) &= f''(s)y'(f(s)) + f'(s)^2 y''(f(s)) \end{aligned}$$

Multiplying the first equation by $-y'(f(s))$, the second equation by $x'(f(s))$ and adding them we obtain

$$k_\beta(s)f'(s)\Big(y'(f(s))^2 + x'(f(s))^2\Big) = f'(s)^2\Big(y''(f(s))x'(f(s)) - x''(f(s))y'(f(s))\Big)$$

which implies that

$$k_\beta(s)f'(s)|\alpha'(f(s))|^2 = f'(s)^2 \det(\alpha'(f(s)), \alpha''(f(s)))$$

Using $(**)$ in above equation, we have

$$k_\alpha(t) = k_\beta(s) = \frac{1}{|\alpha'(f(s))|^3}\det(\alpha'(f(s)), \alpha''(f(s))) = \frac{1}{|\alpha'(t)|^3}\det(\alpha'(t), \alpha''(t)).$$

To obtain the second expression we set $T(t) := \frac{\alpha'(t)}{|\alpha'(t)|}$ and then we try to express $T(t)$ (resp. $T'(t)$) in terms of $\alpha'(t)$ (resp. $\alpha''(t)$). From $(**)$

we have $T(t) = \alpha'(t)f'(s)$. Differentiating this with respect to t, we have

$$\begin{aligned} T'(t) &= f'(s)\frac{\partial}{\partial t}\alpha'(t) + \alpha'(t)\frac{\partial}{\partial t}f'(s) = \alpha''(t)f'(s) + \alpha'(t)f''(s)\frac{\partial s}{\partial t} \\ &= \alpha''(t)f'(s) + \alpha'(t)f''(s)(f^{-1})'(t) \\ &= \alpha''(t)f'(s) + \alpha'(t)f''(s)\frac{1}{f'(s)} \end{aligned}$$

In the last expression, we notice that the second part of $T'(t)$ is a scalar multiple of the vector $\alpha'(t)$. Hence

$$\begin{aligned} \det\Big(T(t), T'(t)\Big) &= \det\Big(\alpha'(t)f'(s), \alpha''(t)f'(s) + \alpha'(t)f''(s)\frac{1}{f'(s)}\Big) \\ &= \det\Big(\alpha'(t)f'(s), \alpha''(t)f'(s)\Big) \\ &= (f'(s))^2\det\Big(\alpha'(t), \alpha''(t)\Big) = |\alpha'(t)|k_\alpha(t). \end{aligned}$$

Question 6. Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l. and let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a rigid motion. Define $\beta := M \circ \alpha$ and prove that

$$k_\beta(s) = \begin{cases} k_\alpha(s) & \text{for all } s \in I, \text{if } M \text{ is direct,} \\ -k_\alpha(s) & \text{for all } s \in I, \text{if } M \text{ is inverse.} \end{cases}$$

Solution. First suppose M is a direct rigid motion. Then there exists $A \in \mathrm{SO}(2,\mathbb{R}), \mathbf{b} \in \mathbb{R}^2$, such that $M(v) = Av + \mathbf{b}$ $(v \in \mathbb{R}^2)$. Now we have

$$\alpha'(s) = T_\alpha(s), \quad \alpha''(s) = T'_\alpha(s) = k_\alpha(s)N_\alpha(s) = k_\alpha(s)JT_\alpha(s).$$

Since a rigid motion preserve the length, β is also p.b.a.l. and hence

$$T_\beta(s) = \beta'(s) = (M \circ \alpha)'(s) = A\alpha'(s).$$

Now notice that J is a rotation matrix, and hence $J \in \mathrm{SO}(2,\mathbb{R})$, which is an abelian group. Using this we get, $N_\beta(s) = JT_\beta(s) = JA\alpha'(s) = AJ\alpha'(s)$. Now,

$$T'_\beta(s) = A\alpha''(s) = A\big(k_\alpha(s)JT_\alpha(s)\big) = k_\alpha(s)AJ\alpha'(s) = k_\alpha(s)N_\beta(s),$$

which implies that $k_\beta(s) = k_\alpha(s)$.

Next, suppose M be a inverse rigid motion and set

$$Q := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that $\mathrm{O}(2,\mathbb{R}) = \mathrm{SO}(2,\mathbb{R}) \cup Q\ \mathrm{SO}(2,\mathbb{R})$, as an union of two cosets. Hence there exists $A \in \mathrm{SO}(2,\mathbb{R}), \mathbf{b} \in \mathbb{R}^2$, such that

$$M(v) = QAv + \mathbf{b}\ (v \in \mathbb{R}^2).$$

Then we compute

$$\begin{aligned}
T_\beta(s) &= \beta'(s) = (M \circ \alpha)'(s) = QA\alpha'(s) \\
N_\beta(s) &= JT_\beta(s) = JQA\alpha'(s) \\
T'_\beta(s) &= QA\alpha''(s) = QA\big(k_\alpha(s)JT_\alpha(s)\big) = k_\alpha(s)QAJ\alpha'(s) \\
&= k_\alpha(s)QJA\alpha'(s) = k_\alpha(s)(QJQ^{-1})QA\alpha'(s) = k_\alpha(s)(-J)QA\alpha'(s) \\
&= -k_\alpha(s)N_\beta(s)
\end{aligned}$$

Since $QJQ^{-1} = -J$. This proves the second statement.

Question 7. Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l. Prove that α is a segment of a straight line or an arc of a circle if and only if the curvature of α is constant.

Solution. The "only if" part follows from the Examples 1.16 and 1.17 (Page 11, Montiel and Ros). We will prove the "if" part here. We write $\alpha(s) = (x(s), y(s))\ (s \in I)$. Since α is p.b.a.l., we have $\alpha'(s) \in S^1 \subseteq \mathbb{R}^2$. Then we can write

$$T(s) = \alpha'(s) = (\cos\theta(s), \sin\theta(s)) \quad (s \in I) \quad (*)^I$$

for some function $\theta : I \to \mathbb{R}$. Since the map $t \mapsto (\cos t, \sin t)$ from $\mathbb{R} \to S^1$ is locally diffeomorphic, applying a local inverse of this to α, it follows that θ is differentiable. Now we have

$$\begin{aligned}
N(s) &= J\alpha'(s) = (-\sin\theta(s), \cos\theta(s)) \\
T'(s) = \alpha''(s) &= (-\sin\theta(s)\theta'(s), \cos\theta(s)\theta'(s))
\end{aligned}$$

Now, let $k \in \mathbb{R}$ denotes the constant curvature of α. Then using the first Frenet equation $T'(s) = kN(s)\ (s \in I)$, we have $\theta'(s) = k$ for every $s \in I$. This implies that $\theta(s) = ks + d\ (s \in I)$ for some constant $d \in \mathbb{R}$. We now consider the following two cases:

Case I. $k = 0$.

From equation $(*)^I$, we have $\alpha'(s) = (\cos d, \sin d)\ (s \in I)$. Fix a point $s_0 \in I$, and then using the fundamental theorem for integrals we have

$$\begin{aligned}
\alpha(s) &= \Big(\int_{s_0}^s \cos d\ du, \int_{s_0}^s \sin d\ du\Big) = \Big((s - s_0)\cos d, (s - s_0)\sin d\Big) \\
&= (\cos d, \sin d)s + (-s_0\cos d, -s_0\sin d) \quad (s \in I)
\end{aligned}$$

which is the equation for a segment of a straight line.

Case II. $k \neq 0$.
Again fixing an $s_0 \in I$, we have

$$\begin{aligned}
\alpha(s) &= \Big(\int_{s_0}^{s} \cos(ku+d)\ du, \int_{s_0}^{s} \sin(ku+d)\ du\Big)\\
&= \Big(\frac{1}{k}\int_{ks_0+d}^{ks+d} \cos w\ dw, \frac{1}{k}\int_{ks_0+d}^{ks+d} \sin w\ dw\Big)\\
&= \Big(\frac{1}{k}\big\{\sin(ks+d)-\sin(ks_0+d)\big\}, \frac{1}{k}\big\{-\cos(ks+d)+\cos(ks_0+d)\big\}\Big)
\end{aligned}$$

Then the coordinates of $\alpha(s) = (x(s), y(s))$ $(s \in I)$ satisfies the equation

$$\Big(x(s) + \frac{1}{k}\sin(ks_0+d)\Big)^2 + \Big(y(s) - \frac{1}{k}\cos(ks_0+d)\Big)^2 = \frac{1}{k^2}$$

which is the equation for a segment of a circle.

Question 8. Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l. If there is a differentiable function $\theta : I \to \mathbb{R}$ such that $\theta(s)$ is the angle that the tangent line of α at s makes with a fixed direction, show that $\theta'(s) = \pm k(s)$.

Solution. Let $a \in \mathbb{R}$ represents a vector so that $\theta(s)$ is the angle that the tangent line of α at s makes with a. Then

$$\cos\theta(s) = \langle T(s), a\rangle \quad (s \in I).$$

Differentiating this with respect to s and using definitions, we obtain

$$-\sin\theta(s)\ \theta'(s) = \langle T'(s), a\rangle = \langle k(s)N(s), a\rangle = k(s)\langle N(s), a\rangle.$$

Now since $N(s)$ is orthogonal to $T(s)$, we have

$$\langle N(s), a\rangle = \cos(\pm\frac{\pi}{2} + \theta(s)) = \mp\sin\theta(s)$$

Comparing these two equations above, we get $\theta'(s) = \pm k(s)$ for every $s \in I$.

Question 9. Let $\alpha, \beta : I \to \mathbb{R}^2$ be two curves p.b.a.l. such that $k_\alpha(s) = -k_\beta(s)$ for every $s \in I$. Show that there is an inverse rigid motion $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\beta = M \circ \alpha$.

Solution. We write $\alpha(s) = (x(s), y(s))$ $(s \in I)$ and define $\widetilde{\alpha} : I \to \mathbb{R}^2$ defined by

$$\widetilde{\alpha}(s) := (x(s), -y(s)) \quad (s \in I).$$

Then $\widetilde{\alpha}$ is a C^∞-curve since its components are infinitely differentiable. Moreover

$$|\widetilde{\alpha}'(s)| = \sqrt{x(s)^2 + (-y(s))^2} = |\alpha'(s)| = 1$$

for every $s \in I$. Hence $\widetilde{\alpha}$ is p.b.a.l. as well. Now we compute the curvature function of $\widetilde{\alpha}$:

$$\begin{aligned} T_{\widetilde{\alpha}}(s) &= \widetilde{\alpha}'(s) = (x'(s), -y'(s)) \\ N_{\widetilde{\alpha}}(s) &= JT_{\widetilde{\alpha}}(s) = (y'(s), x'(s)) \\ T'_{\widetilde{\alpha}}(s) &= \widetilde{\alpha}''(s) = (x''(s), -y''(s)) \end{aligned}$$

Now using the first Frenet equation corresponding to α

$$(x''(s), y''(s)) = T'_{\alpha}(s) = k_{\alpha}(s)N_{\alpha}(s) = k_{\alpha}(s)(-y'(s), x'(s))$$

we have

$$x''(s) = -k_{\alpha}(s)y'(s), \quad y''(s) = k_{\alpha}(s)x'(s).$$

Using these equations to the expression of $T'_{\widetilde{\alpha}}(s)$ we have

$$T'_{\widetilde{\alpha}}(s) = (-k_{\alpha}(s)y'(s), -k_{\alpha}(s)x'(s)) = -k_{\alpha}(s)N_{\widetilde{\alpha}}(s)$$

for every $s \in I$. This shows that $k_{\beta} = -k_{\alpha} : I \to \mathbb{R}$ is the curvature function for $\widetilde{\alpha}$. From Theorem 1.21 (Page 11, Montiel and Ros) it follows that there exists a direct rigid motion $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\beta = M \circ \widetilde{\alpha}$.

From the definition of $\widetilde{\alpha}$, we have $Q\alpha = \widetilde{\alpha}$, where Q is the matrix defined in Exercise 6. Then

$$\beta = M \circ \widetilde{\alpha} = MQ \circ \alpha = Q(Q^{-1}MQ) \circ \alpha.$$

Since $\mathrm{SO}(2, \mathbb{R})$ is a normal subgroup of $\mathrm{O}(2, \mathbb{R})$, we have $Q^{-1}MQ$ is a direct rigid motion, and hence $M_1 := Q(Q^{-1}MQ)$ is an inverse rigid motion.

Question 10. If $\alpha : I \to \mathbb{R}^2$ is a curve p.b.a.l. with $0 \in I$ and symmetric around 0, we define another curve $\beta : I \to \mathbb{R}^2$ by $\beta(s) := \alpha(-s)$ for every $s \in I$. Prove that β is p.b.a.l. and that $k_{\beta}(s) = -k_{\alpha}(-s)$ for each $s \in I$.

Solution. Setting $\alpha(s) = (x(s), y(s))$ $(s \in I)$ as in the previous exercise, we have the following equations

$$x''(s) = -k_{\alpha}(s)y'(s), \quad y''(s) = k_{\alpha}(s)x'(s) \qquad (*)^{II}$$

Then we have

$$\begin{aligned} \beta(s) &= \big(x(-s), y(-s)\big), \\ \beta'(s) &= \big(-x'(-s), -y'(-s)\big) \end{aligned}$$

and hence $|\beta'(s)| = \sqrt{x'(-s)^2 + y'(-s)^2} = 1$. This proves that β is p.b.a.l. As a consequence we have $T_{\beta}(s) = \beta'(s) = \big(-x'(-s), -y'(-s)\big)$ $(s \in I)$.

Now we compute

$$\begin{aligned} N_\beta(s) &= JT_\beta(s) = \big(y'(-s), -x'(-s)\big) \\ T'_\beta(s) &= \big(x''(-s), y''(-s)\big) \end{aligned}$$

Now using the Frenet equation $T'_\beta(s) = k_\beta(s)N_\beta(s)$ we get the equations

$$x''(-s) = k_\beta(s)y'(-s), \quad y''(-s) = -k_\beta(s)x'(-s).$$

Now comparing the above equation with $(*)^{II}$ as above, the statement follows.